Dependence on Initial Conditions of Oscillator Displacement Modulated by Parametric Noise

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Abstract: Stochastic dynamics involving parametric noise sometimes exhibits transition from an energetic stable state to an unstable state, while dynamics with additive noise only exhibits fluctuation around its average behavior. We investigated the dependence of transient behavior of oscillator on its initial conditions by introducing noise taking discrete values. We found that relaxation time of probability distribution becomes infinity as the intensity of parametric noise approaches a critical value, and above the critical value the difference between the second moments of the oscillator with different initial conditions increases exponentially.

Key–Words: Parametric noise, oscillator, sensitive dependence on initial conditions, second moment, relaxation time

1 Introduction

Stochastic dynamics involving both parametric and additive noise have been exhaustively studied for decades. These studies focus on practical phenomena such as stock market prices, dye-lasers, random velocity fields [1, 2, 3, 4, 5], and so on. The most interesting phenomenon found in these systems is probably noise-induced transition from a Gaussian-distribution to a non-Gaussian distribution with an inverse power law tail. This transition is related to the stability of the system directly, because this transition accompanies divergence of the moments [6]. General understanding of the mechanisms causing power law is also important in other research fields such as self-organized criticality and aggregation systems.

From a theoretical viewpoint, exact calculations for predicting the transition have been successful by introducing certain stochastic models. For example, exact second moment with critical point was obtained for equilibrium behavior of oscillator displacement with dichotomic [7] or delta-correlated parametric noise [8]. For a first-order differential equation with additive and parametric noise, n-th order moments and power law exponents in equilibrium condition were obtained[9, 10]. However, most results obtained by theoretical calculation are limited in equilibrium situations.

In our early works[11], we have pointed out that it is possible to treat this problem easily and sometimes exactly by introducing noise taking discrete values. This idea is based on the fact that we can solve differential equations analytically for each sample function of fluctuating coefficients as long as the fluctuating coefficients change their values at discrete times. This idea enables us to calculate exact second moments of oscillator displacement with parametric noise in a particular case, and to simulate concrete behavior of displacement. However, the process of probability distributions to the equilibrium state has been discussed only briefly in these works.

The aim of the present paper is to investigate the dependence on the initial conditions of the oscillator by exact calculations of oscillator displacement. The difference of the probability distribution caused by the difference of the initial conditions is expected to disappear as fast as the solution of the original deterministic equation. But the energetic instability can change the situation. Sensitive dependence on initial conditions has not been studied sufficiently up to now, while exhaustively studied in chaotic systems.

For convenience, theory definitions and principal formulae are summarized in Section 2 and Section 3, which were already introduced in early works. It should be noted that the time correlation with the discretized parametric noise is newly considered here, which is equivalent to dichotomic noise defined in continues time by taking a limiting procedure. In Section 4, dependence on the initial conditions of the second moment of the oscillator displacement is calculated for two different types of parametric noise.

2 Models

This paper deals with the following oscillator equation with a random coefficient $\omega^2(t)$ and a random driving force $f(t)$,

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega^2(t)x = f(t). \qquad (1)$$

As mentioned in the previous section, the noises $\omega^2(t)$ and $f(t)$ considered here are discreteized for each constant time interval Δt_f and Δt_ω as

$$f(t) = f_j \quad \text{for} \quad \Delta t_f(j-1) \le t < \Delta t_f j \qquad (2)$$

$$\omega(t) = \omega_n \quad \text{for} \quad \Delta t_\omega(n-1) \le t < \Delta t_\omega n, \qquad (3)$$

where j and n are an integer. Suppose that average of $\omega^2(t)$ is ω_0^2, and that driving forces in each time interval are independent and identical random variables with average 0. If we write

$$\omega_n{}^2 = \omega_0{}^2(1+\epsilon_n), \qquad (4)$$

then

$$\langle f_j \rangle = 0, \quad \text{for all} \quad j, \qquad (5)$$

$$\langle \epsilon_n \rangle = 0, \quad \text{for all} \quad n, \qquad (6)$$

$$\langle f_i f_j \rangle = \frac{f^2\delta_{ij}}{\Delta t_f}. \qquad (7)$$

where the bracketed terms denote ensemble averages, and i and j are integers.

As long as Δt_f is sufficiently small, the discretized driving force results in an effect equivalent to the ordinal white noise, because $f(t)$ affects on the statistics of x always through the time integral of $f(t)$ but not through the infinitesimal behavior of $f(t)$. Advantages to employ the discretaized form of $f(t)$ is that the oscillator displacement x and the velocity dx/dt can be calculated at any instant times. Furthermore, discretization of random coefficient enables us to avoid a requirement of advanced mathematical theory such as Ito or Stratonovich integrals, and to carry out simple integration of (1) for each sample function of $\omega^2(t)$.

The following sections consider two different types of time correlation of ω_n^2, which we call model 1 and model 2. For brevity, we consider two-valued stochastic models in both cases.

- model 1 is two-valued independent random variables. Therefore,

$$\left\langle \omega_n^2\omega_l^2 \right\rangle = \omega_0{}^4(1 + \left\langle \epsilon^2 \right\rangle \delta_{nl}) \qquad (8)$$

is hold. This model is contrived as a means of easy calculation of stationary value of second moment of x.

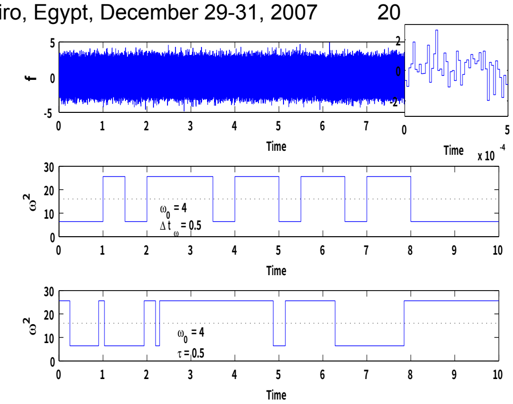

Figure 1: Sample functions of the noises. The top one is a driving force $f(t)$ where $\Delta t_f = 0.00001$. The middle one is a parametric noise of model 1 where $\Delta t_\omega = 0.5$. The value can change at integral multiples of Δt_ω. The bottom one is a parametric noise of model 2 where $\Delta t_\omega = 0.001$ and $\tau = 0.5$.

- model 2 is a two-valued Markov process. Probability of ω_n^2 to take $\omega_0^2(1+\epsilon)$ or $\omega_0^2(1-\epsilon)$ depends only on its previous value ω_{n-1}^2. Assume that ω_n^2 takes the same value of ω_{n-1}^2 with probability β, and takes different value of ω_{n-1}^2 with probability $1-\beta$. Then the time correlation of ω_n^2 can be calculated as

$$\left\langle \omega_n^2\omega_l^2 \right\rangle = \omega_0{}^4 \exp\left(-\frac{|n-l|\Delta t_\omega}{\tau}\right), \qquad (9)$$

where

$$\tau = -\frac{\Delta t_\omega}{\log(2\beta - 1)}. \qquad (10)$$

Correlation time length of $\omega^2(t)$ is considered as τ for model 2, and as Δt_ω for model 1. In the present calculation, parameters are adjusted as $\tau \gg \Delta t_\omega \gg \Delta t_f$. Under this condition the driving force can be considered as white noise and the parametric noise $\omega^2(t)$ can be considered as dichotomic noise (see Figure 1).

In our early works, stationary value of second moment of x for model 1 was calculated, but the transient behavior was only briefly discussed. Our goal is to investigate in detail the transient behavior of second moment of x not only for model 1 but for model 2.

3 State Space Representation

In this section, formulae describing time evolution of state vector $\mathbf{X}_n$ are summarized. These formulae can be derived under the assumption that Δt_ω is integral multiple of Δt_f, that is, $\Delta t_\omega = m\Delta t_f$ where m is an integer. Let the state vector $\mathbf{X}_n$ be $(x, \frac{dx}{dt})$ at time $n\Delta t_\omega = nm\Delta t_f$. Then the time evolution of the state vector can be described as

$$\mathbf{X}_n = L_n \mathbf{X}_{n-1} + \tilde{\mathbf{F}}_n \tag{11}$$

where the matrix L_n and vector $\tilde{\mathbf{F}}_n$ are defined as

$$L_n \equiv \begin{bmatrix} \frac{(\alpha+k_n)\Lambda_n^{(+)}}{2k_n} + \frac{(k_n-\alpha)\Lambda_n^{(-)}}{2k_n} & \frac{\Lambda_n^{(+)}}{k_n} - \frac{\Lambda_n^{(-)}}{k_n} \\ -\frac{\omega_n^2\Lambda_n^{(+)}}{k_n} + \frac{\omega_n^2\Lambda_n^{(-)}}{k_n} & \frac{(k_n-\alpha)\Lambda_n^{(+)}}{2k_n} + \frac{(\alpha+k_n)\Lambda_n^{(-)}}{2k_n} \end{bmatrix} \tag{12}$$

$$\tilde{\mathbf{F}}_n \equiv \begin{bmatrix} 1 & 1 \\ \frac{k_n-\alpha}{2} & \frac{-k_n-\alpha}{2} \end{bmatrix} \mathbf{F}_n, \tag{13}$$

$$\mathbf{F}_n = \begin{bmatrix} F_n^{(1)} \\ F_n^{(2)} \end{bmatrix} = \begin{bmatrix} u_n^{(1)} \sum_{j=1}^m (\lambda_n^{(+)})^{m-j} f_{(n\times m)+j} \\ u_n^{(2)} \sum_{j=1}^m (\lambda_n^{(-)})^{m-j} f_{(n\times m)+j}, \end{bmatrix} \tag{14}$$

$$\mathbf{u}_n = \begin{bmatrix} u_n^{(1)} \\ u_n^{(2)} \end{bmatrix} = \frac{1}{2k_n\omega_n^2} \begin{bmatrix} (-k_n-\alpha)(\exp((k_n-\alpha)\Delta t_f/2)-1) \\ (-k_n+\alpha)(\exp((-k_n-\alpha)\Delta t_f/2)-1) \end{bmatrix}, \tag{15}$$

$$\lambda_n^{(\pm)} = \exp((-\alpha \pm k_n)\Delta t_f/2), \tag{16}$$

$$\Lambda_n^{(\pm)} = (\lambda_n^{(\pm)})^m = \exp((-\alpha \pm k_n)\Delta t_\omega/2), \tag{17}$$

$$k_n = \sqrt{\alpha^2 - 4\omega_n^2}. \tag{18}$$

In these formulae, L_n is a random matrix including the random variable ω_n^2, and $\tilde{\mathbf{F}}_n$ is a random vector including both noises of f_j's and ω_n^2.

The conditional probability of elements of $\tilde{\mathbf{F}}_n$ for given ω_n^2 is a Gaussian distribution with average zero and the variance given in later equations (25) to (27), because the elements of $\tilde{\mathbf{F}}_n$ are merely a sum of independent random variables f_j's. Furthermore, from the statistical independence of the driving force and the parametric noise,

$$\left\langle \tilde{\mathbf{F}}_n \right\rangle = 0, \tag{19}$$

$$\left\langle L_n^{ij} \tilde{\mathbf{F}}_n \right\rangle = 0, \tag{20}$$

where L_n^{ij} is an element of the matrix L_n.

For example, the time evolution of the first and the second moments of x for model 1 is governed by the next formulae,

$$\langle \mathbf{X}_n \rangle = \langle L_n \rangle \langle \mathbf{X}_{n-1} \rangle, \tag{21}$$

and

$$\mathcal{X}_n = \mathcal{L}_n \mathcal{X}_{n-1} + \mathcal{F}_n, \tag{22}$$

where

$$\mathcal{X}_n = \begin{bmatrix} \langle X_n^2 \rangle \\ \left\langle X_n \dot{X}_n \right\rangle \\ \left\langle \dot{X}_n^2 \right\rangle \end{bmatrix} \tag{23}$$

$$\mathcal{L}_n = \begin{bmatrix} \left\langle (L_n^{(11)})^2 \right\rangle & 2\left\langle L_n^{(11)} L_n^{(12)} \right\rangle & \left\langle (L_n^{(12)})^2 \right\rangle \\ \left\langle L_n^{(11)} L_n^{(21)} \right\rangle & \left\langle L_n^{(12)} L_n^{(21)} + L_n^{(11)} L_n^{(22)} \right\rangle & \left\langle L_n^{(12)} L_n^{(22)} \right\rangle \\ \left\langle (L_n^{(21)})^2 \right\rangle & 2\left\langle L_n^{(21)} L_n^{(22)} \right\rangle & \left\langle (L_n^{(22)})^2 \right\rangle \end{bmatrix}. \tag{24}$$

The elements of $\mathcal{F}_n$ are calculated as follows by taking a limiting procedure that $\Delta t_f \to 0$.

$$\begin{aligned} \mathcal{F}^{(1)} &= f^2 \left\langle (k_n+\alpha)(1-(\Lambda_n^{(1)})^2)/4k_n^2\omega_n^2 \right\rangle \\ &- 2f^2 \left\langle (1-\Lambda_n^{(1)}\Lambda^{(2)})/k_n^2\alpha \right\rangle \\ &+ f^2 \left\langle (-k_n+\alpha)(1-(\Lambda^{(2)})^2)/4k^2\omega_n^2 \right\rangle, \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{F}^{(2)} &= -f^2 \left\langle (1-(\Lambda_n^{(1)})^2)/2k_n^2 \right\rangle \\ &+ f^2 \left\langle (1-\Lambda_n^{(1)}\Lambda^{(2)})/k_n^2 \right\rangle \\ &- f^2 \left\langle (1-(\Lambda^{(2)})^2)/2k^2 \right\rangle, \end{aligned} \tag{26}$$

$$\begin{aligned} \mathcal{F}^{(3)} &= f^2 \left\langle (\alpha-k_n)(1-(\Lambda_n^{(1)})^2)/4k_n^2 \right\rangle \\ &- 2f^2 \left\langle \omega_n^2(1-\Lambda_n^{(1)}\Lambda^{(2)})/k_n^2\alpha \right\rangle \\ &+ f^2 \left\langle (k_n+\alpha)(1-(\Lambda^{(2)})^2)/4k^2 \right\rangle. \end{aligned} \tag{27}$$

4 Calculation

In order to investigate the dependence of the oscillator displacement on its initial conditions, let us consider the time dependency of moment difference between different initial conditions $\mathbf{Y}_n = \mathcal{X}_{n|\boldsymbol{x}_0} - \mathcal{X}_{n|\mathbf{0}}$, where $\mathcal{X}_n$ is defined in (23), and the subscript $\boldsymbol{x}_0$ indicates an initial condition where $(x(0), \frac{dx(0)}{dt}) = (x_0, \dot{x}_0)$, and the subscript **0** indicates an initial condition where $(x(0), \frac{dx(0)}{dt}) = (0, 0)$. This vector $\mathbf{Y}_n$ evaluates the difference of probability distributions with different initial conditions through the difference of their second moments. Note that $\mathbf{Y}_n - \langle \mathbf{X}_n \rangle^2_{\boldsymbol{x}_0}$ should be also considered if we are interested in the difference of variance.

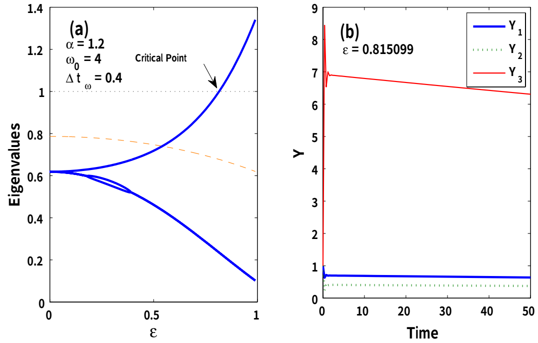

Figure 2: (a)A typical behavior of absolute values of the eigen values with respect to the intensity of the parametric noise ϵ when $\alpha = 1.2$, $\omega_0 = 4$ and $\Delta t_\omega = 0.4$ for model 1. The broken curve represents absolute values of complex conjugate eigen values of $\langle L_n \rangle$ indicating fast decrease of the first moment. (b)Time dependence of the moment difference $\mathbf{Y}_n$ when ϵ is close to the critical value ϵ_c. The $\mathbf{Y}_n$ has a value proportional to the eigenvector calculated as $(0.1004, 0.0589, 0.9932)$.

4.1 Results for Model 1

By iteration of (22) we can deduce the equation

$$\mathcal{X}_n = \mathcal{L}_n \mathcal{L}_{n-1} \cdots \mathcal{L}_1 \mathcal{X}_0 + \text{other terms including } \mathcal{F}_n\text{'s}. \quad (28)$$

Although terms in (28) except the first term are common to all initial conditions, the first term is vanished when the initial state $\mathbf{X}_0$ is a zero vector. Therefore the moment difference $\mathbf{Y}_n$ is determined by the first term $\mathcal{L}_n \mathcal{L}_{n-1} \cdots \mathcal{L}_1 \mathcal{X}_0$. This term is only a product of one constant matrix $\mathcal{L}_n$, because $\mathcal{L}_n$ is independent of n. So it is sufficient only to know the property of the eigen values of the matrix $\mathcal{L}_n$ for further investigations.

Figure 2 shows a typical behavior of absolute values of eigen values with respect to the intensity of the parametric noise ϵ. When two of three eigen values are complex conjugate, the sequence of the eigen values is split into two branches. When all three eigen values are real, the sequence of the eigen values are split into three branches.

The most important eigen value is the largest one that can become to larger than 1 when ϵ is beyond a critical value ϵ_c. Although the moment difference $\mathbf{Y}_n$ disappears exponentially with time as long as ϵ is smaller than ϵ_c, the moment difference $\mathbf{Y}_n$ will remain permanently at the critical point $\epsilon = \epsilon_c$. Further increase of ϵ leads to an exponential increase of the moment difference $\mathbf{Y}_n$ with respect to time. The value of $\mathbf{Y}_n$ can be described by the eigenvector belonging to the largest eigen value, because other modes corresponding to other eigen values disappear faster than the mode corresponding to the largest eigen value. Especially when ϵ is close to the critical point, $\mathbf{Y}_n$ can be remain for a long time with a value proportional to the eigenvector belonging to the largest eigen value (see Figure 2(b)).

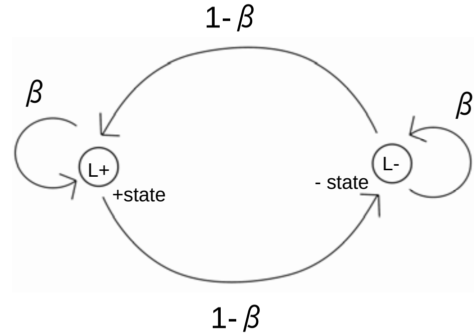

Figure 3: Calculation of a mean value using a Markov chain. There are two states that remain the same state with probability β and transit to the other state with probability $1 - \beta$. Which matrix L^+ or L^- is multiplied is determined by this process. Initially, a vector $\mathbf{Y}_0$ has deterministic elements. In the first step, $\mathbf{Y}_1$ is an average of two vectors $\mathcal{L}^- \mathbf{Y}_0$ and $\mathcal{L}^+ \mathbf{Y}_0$ with a probability $\frac{1}{2}$. In the second step, $\mathbf{Y}_2$ is an average of four vectors $\mathcal{L}^- \mathcal{L}^- \mathbf{Y}_0$, $\mathcal{L}^+ \mathcal{L}^- \mathbf{Y}_0$, $\mathcal{L}^+ \mathcal{L}^+ \mathbf{Y}_0$ and $\mathcal{L}^- \mathcal{L}^+ \mathbf{Y}_0$ with a probability $\frac{1}{2}\beta$, $\frac{1}{2}(1-\beta)$, $\frac{1}{2}\beta$ and $\frac{1}{2}(1-\beta)$, respectively. In further steps, $\mathcal{L}^+$ or $\mathcal{L}^-$ is multiplied with probabilities determined by the Markov chain.

4.2 Results for Model 2

For model 2, calculations for the second moment becomes complicated by the fact that the second term of (22) becomes a worng term when we consider model 2, because $\mathbf{X}_{n-1}$ depends on ω_{n-1}^2. However, iterative relation to obtain moment difference $\mathbf{Y}_n$ can be derived by introducing a conditional average value of moment difference $\mathbf{Y}_n^{\pm}$ under the condition that the value of ω_n^2 is given as follows.

Note that L_{n+1} can takes two forms, L^+ corresponding to $\omega_{n+1}^2 = \omega_0^2(1+\epsilon)$ and L^- corresponding to $\omega_{n+1}^2 = \omega_0^2(1-\epsilon)$, with probabilities dependent on the previous value of ω_n^2. A graphical representation to calculate $\mathbf{Y}_n$ is illustrated in Figure 3. Let the $\mathcal{L}^+ \mathbf{Y}_0$ and $\mathcal{L}^- \mathbf{Y}_0$ be $\mathbf{Y}_1^+$ and $\mathbf{Y}_1^-$, respectively.

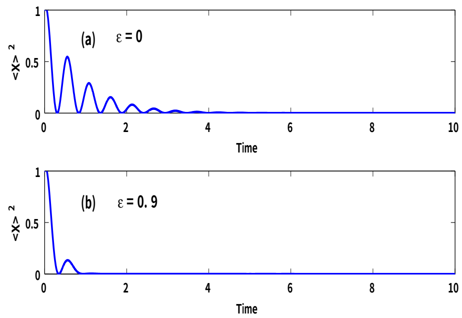

Figure 4: Examples of time evolution of $\langle X_n \rangle^2_{\mathbf{x_0}}$. The parameters are $\alpha = 1.2$ and $\omega_0 = 6$. (a)A case for only existing driving force. (b)The intensity of parametric noise ϵ is over the critical point (compare to Figure 5(d)

According to the rules described in Figure 3, $\mathbf{Y}_2$ is obtained as

$$\mathbf{Y}_2 = \frac{1}{2}\{\beta\mathcal{L}^-\mathbf{Y}_1^- + (1-\beta)\mathcal{L}^-\mathbf{Y}_1^+\} + \frac{1}{2}\{(1-\beta)\mathcal{L}^+\mathbf{Y}_1^- + \beta\mathcal{L}^+\mathbf{Y}_1^+\} \equiv \frac{\mathbf{Y}_2^- + \mathbf{Y}_2^+}{2}. \quad (29)$$

In a similar way,

$$\begin{aligned}\mathbf{Y}_3 = & \frac{1}{2}\beta\mathcal{L}^-\{\beta\mathcal{L}^-\mathbf{Y}_1^- + (1-\beta)\mathcal{L}^-\mathbf{Y}_1^+\} \\ & + \frac{1}{2}(1-\beta)\mathcal{L}^-\{(1-\beta)\mathcal{L}^+\mathbf{Y}_1^- + \beta\mathcal{L}^+\mathbf{Y}_1^+\}\} \\ & + \frac{1}{2}\{(1-\beta)\mathcal{L}^+\{\beta\mathcal{L}^-\mathbf{Y}_1^- + (1-\beta)\mathcal{L}^-\mathbf{Y}_1^+\} + \\ & \frac{1}{2}\beta\mathcal{L}^+\{(1-\beta)\mathcal{L}^+\mathbf{Y}_1^- + \beta\mathcal{L}^+\mathbf{Y}_1^+\}\} \\ = & \frac{1}{2}\{\beta\mathcal{L}^-\mathbf{Y}_2^- + (1-\beta)\mathcal{L}^-\mathbf{Y}_2^+\} \\ & + \frac{1}{2}\{(1-\beta)\mathcal{L}^+\mathbf{Y}_2^- + \beta\mathcal{L}^+\mathbf{Y}_2^+\} \\ \equiv & \frac{\mathbf{Y}_3^- + \mathbf{Y}_3^+}{2}. \quad (30)\end{aligned}$$

We obtain in general

$$\begin{aligned}\mathbf{Y}_n^+ &= \beta\mathcal{L}^+\mathbf{Y}_{n-1}^+ + (1-\beta)\mathcal{L}^+\mathbf{Y}_{n-1}^-, \\ \mathbf{Y}_n^- &= (\beta-1)\mathcal{L}^-\mathbf{Y}_{n-1}^+ + \beta\mathcal{L}^-\mathbf{Y}_{n-1}^-. \quad (31)\end{aligned}$$

and

$$\mathbf{Y}_n = \frac{\mathbf{Y}_n^+ + \mathbf{Y}_n^-}{2}, \quad (32)$$

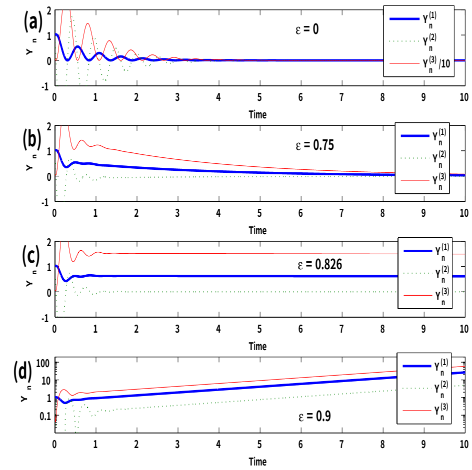

Figure 5: Examples of time evolution of $\mathbf{Y}_n$. The parameters are $\alpha = 1.2$, $\omega_0 = 6$ and $\tau = 0.2$. (a)A case for only existing driving force. (b)At ϵ near the critical point. (c) At ϵ very close to the critical point. (d) At ϵ over the critical point. Notice that this is semi-logarithmic graph.

Mean value $\langle \mathbf{X}_n \rangle_{\mathbf{x_0}}$ can also be calculated by the same method. Figure 4 and Figure 5 shows examples of behavior of $\langle \mathbf{X}_n \rangle^2_{\mathbf{x_0}}$ and $\mathbf{Y}_n$, respectively.

As expected from the results of model 1, the squared value of $\langle \mathbf{X}_n \rangle_{\mathbf{x_0}}$ decays rapidly regardless of the value of ϵ. The difference $\mathbf{Y}_n$ also decays with time when ϵ is sufficiently small. By comparing Figure 4(a) Figure 5(a), it is found that the moment difference $\mathbf{Y}_n$ near the initial time is caused by $\langle X_n \rangle^2_{\mathbf{x_0}}$. In other words, dependence on initial condition of the variance is not observed in this situation. On the other hand, sufficiently large ϵ produces different results. Figure 5(c) and (d) present the behavior of the moment difference $\mathbf{Y}_n$ at ϵ close to the critical point and above the critical point, respectively. Figure 5(c) shows that elements of $\mathbf{Y}_n$ except $\langle x\dot{x} \rangle$ remains for a long time. Considering there are many other terms with similar effects on the moments as shown in (28), second moments except $\langle x\dot{x} \rangle$ must be infinity. Above the critical point, the moment difference $\mathbf{Y}_n$ increases with time exponentially.

We can make a phase diagram systematically by judging whether $\mathbf{Y}_n$ increases exponentially or not. Figure 6 shows examples of phase diagram obtained by this method. It was found that increase of mean value of parametric noise ω_0^2 facilitates transition caused by the increase of ϵ and reduction of resonance correlation time τ_r indicating the peak point of

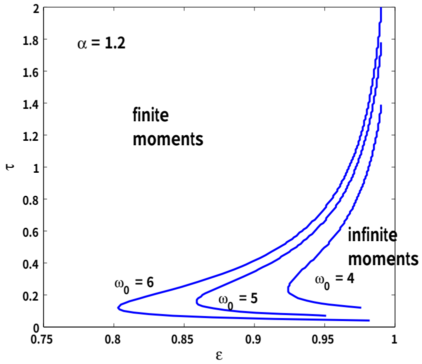

Figure 6: Phase diagram obtained by calculating of $\mathbf{Y}_n$ for model 2. Calculations are performed for $\alpha = 1.2$, $\Delta t_\omega = 0.001$, $\Delta t_f = 0.00001$ and different mean parametric noises $\omega_0^2 = 16, 25$ and 36.

the curve.

5 Conclusions

We could evaluate dependence on initial conditions of an oscillator displacement modulated by a parametric noise, especially difference of second moments with different initial conditions, by introducing discretized noises. Our method enables us not only to find the divergence point of the second moment but also to calculate the time dependence of the second moment with different initial conditions. It is interesting that the sensitive dependence of probability distribution on initial conditions can be detected for a stochastic dynamics with parametric noise. Discretized noise does not always describe a specialized situation, for the influence of driving force on the statistics is identical with ordinal white noise when Δt_f is sufficiently small.

We considered two types of two-valued parametric noise. One is a sequence of independent random variables ω_n^2's defined on each time intervals with length Δt_ω. This type provides easy treatment of random variables and eigenvalue problem including the essential properties of this problem. For example, the divergence point of the second moment and residual moments at the critical point after passing a long time can be evaluated by the method. The other type of parametric noise is a Markov chain ω_n^2's, which can perform a behavior similar to dichotomic noise when $\tau \gg \Delta t_\omega \gg \Delta t_f$. As a result, the influence of initial conditions remains very long time when the intensity of parametric noise ϵ is just below the critical value ϵ_c. When $\epsilon > \epsilon_c$, the difference of the moment with different initial conditions $\mathbf{Y}_n$ increases with time exponentially. We can make phase diagrams according to whether the difference $\mathbf{Y}_n$ becomes zero or infinity. Our method does not include advanced mathematical theories, so similar calculations may be necessary using other equations.

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