Similarity solutions of a MHD boundary–layer flow of a non-Newtonian fluid past a continuous moving surface

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Abstract: The present paper deals with a theoretical and numerical analysis of similarity solutions of the two–dimensional boundary–layer flow of a power-law non-Newtonian fluid past a permeable surface in the presence of a magnitic field $B(x)$ applied perpendiculaire to the surface. The magnetic field B is assumed to be proportional to $x^{\frac{m-1}{2}}$, where x is the coordinate along the plate measured from the leading edge and m is a constant. The problem depends on the power law exponent m, the power-law index,n, and the magnetic parameter M or the Stewart number. It is shown, under certain circumstance, that the problem has an infinite number of solutions.

1. Introduction

The prototype of the problem under investigation is

$$\alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right)+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial xy} \\ +u_e\frac{\partial u_e}{\partial x}-\sigma B^2\left(\frac{\partial\psi}{\partial y}-u_e\right)=0, \tag{1.1}$$

with the boundary conditions

$$\frac{\partial\psi}{\partial y}(x,0)=u_w(x),\quad \frac{\partial\psi}{\partial x}(x,0)=v_w(x), \tag{1.2}$$

and

$$\lim_{y\to\infty}\frac{\partial\psi}{\partial y}(x,0)=u_e(x), \tag{1.3}$$

where the unknown function is the streamfunction ψ, u_e is the free stream velocity, k,ρ,σ and n are permeability, fluid density, electric conductivity and power-law index, respectively. The above problem is a model for the first approximation to two-dimensional laminar incompressible flow of an electrically conducting non-Newtonian power-law fluid pat a moving plate surface. Here the $x\geq 0$ and $y\geq 0$ are the Cartesian coordinates along and normal to the plate with $y=0$ is the plate, the plate origin located at $x=y=0$. The magnetic field is given by $B(x)=B_0x^{\frac{m-1}{2}}$, $B_0>0$, and is assumed to be applied normally to the surface.
Problem (1.1)–(1.3) is deduced from the boundary-layer approximation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \\ =\alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right)+u_e\frac{\partial u_e}{\partial x}-\sigma B^2(u-u_e), \tag{1.5}$$

and

$$u(x,0)=u_w(x),\quad v(x,0)=v_w(x), \\ \lim_{y\to\infty}u(x,y)=u_e(x), \tag{1.6}$$

according to $u=\frac{\partial\psi}{\partial y}$, and $v=-\frac{\partial\psi}{\partial x}$, where u and v represent the components of the fluid velocity in the direction of increasing x and y. Here, it is assumed that the flow behavior of the non-Newtonian fluid is described by the Ostwald-de Waele power law model, where the shear stress is related to the strain rate $\partial u/\partial y$ by the expression [7], [13], [20],

$$\tau = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y},$$

where K is a positive constant and $n > 0$ is called the power-law index. The case $n < 1$ is referred to as pseudo-plastic fluids (or shear-thinning fluids), the case $n > 1$ is known as dilatant or shear-thickening fluids. The Newtonian fluid is, of course, a special case where the power-law index n is one. The stretching, suction/injection velocities and the free stream velocity are assumed to be of the form

$$u_w(x) = u_w x^m, \quad v_w(x) = -v_s x^{\frac{m(2n-1)-n}{n+1}}, \quad u_e(x) = u_\infty x^m, \tag{1.7}$$

where u_w and u_∞ are positive constants and v_s is a real number with $v_s < 0$ for injection and $v_s > 0$ for suction.

The magnetohydrodynamic (MHD) flow problems find applications in many physical, geophysical and industrial fields. Pavlov [17] was the first who examined the MHD flow over a stretching wall in an electrically conducting fluid, with an uniform magnetic field. Further studies in this direction are those of Chakrabarti and Gupta [8], Vajravelu [26], Takhar et al. [25, 22], Kumari et al. [14], Andersson et al. [3] and Watanabe and Pop [27]. The possibility of obtaining similarity solutions for the MHD flow over a stretching permeable surface subject to suction or injection was considered by [8], [26] for some values of the mass transfer parameter, say, f_w and by Pop and Na [18], for large values of f_w and where the stretching velocity varies linearly with the distance and where the suction/injection velocity is constant. The MHD flow over a stretching permeable surface with variable suction/injection velocity can be found in [9] A complet physical interpretation of the problem can be found in [8], [19], [21], [24].

In the present paper, we will examine semilarity solutions to (1.1)–(1.3) in the usual form

$$\psi(x, y) = \lambda x^s f(\eta), \quad \eta = \gamma\frac{y}{x^r}, \tag{1.8}$$

where s and r are real numbers, $\lambda > 0$ and $\gamma > 0$ are such that

$$\lambda\gamma = u_\infty, \quad \alpha\lambda^{n-2}\gamma^{2(n-1)} = 1.$$

Using (1.1) and (1.8) we find that the profile function satisfies

$$\left(|f''|^{n-1}f''\right)' + sff'' + m\left(1 - f'^2\right) + M\left(1 - f'\right) = 0, \tag{1.9}$$

if and only if

$$m = s - r, \quad s(2-n) + r(2n-1) = 1,$$

which leads to

$$s = \frac{1 + m(2n-1)}{1+n}.$$

In equation (1.9) the primes denote differentiations with respect to the similarity variable $\eta \in (0, \infty)$ and the unknown function f denotes the similar stream function and its derivative, after suitable normalisation, represents the velocity parallel to the surface. The parameter $M = \frac{\sigma B_0^2}{u_\infty \rho}$ is the magnetic parameter. Equation (1.9) will be solved subject to the boundary conditions

$$f(0) = a, \ f'(0) = b, \tag{1.10}$$

and

$$f'(\infty) = \lim_{\eta\to\infty} f'(\eta) = 1. \tag{1.11}$$

The parameters a and b are given by where $a = (n+1)v_s\left(\alpha u_\infty^{2n-1}\right)^{-1/(n+1)}$ and $b = \frac{u_w}{u_\infty}$. For the Newtonian fluid ($n = 1$) The ODE reads

$$f''' + sff'' + m\left(1 - f'^2\right) + M\left(1 - f'\right) = 0, \quad s = \frac{m+1}{2}. \tag{1.12}$$

Numerical and analytical solutions to (1.12), in the absence of the free stream function ($f'(\infty) = 0$) were obtained in [9], [11], [18], [23]. Numerical solutions, in the presence of the free stream velocity can be found in [4], [19], [24], for both momentum and heat tranfers.

In a physical different but mathematically identical context, equation (1.12), with $M = -m$, which reads (by a scaling)

$$f''' + (1+m)ff'' + 2mf'(1 - f') = 0, \tag{1.13}$$

has been investigated by Aly et al. [2], Brighi et al. [5], Brighi and Hoernel [6], Guedda [12], Magyari and Aly [15] and Nazar et al. [16]. This equation with the boundary condition ($a = 0, b = 1 + \varepsilon$)

$$f(0) = 0, \quad f'(0) = 1 + \varepsilon, \quad f'(\infty) = 1, \tag{1.14}$$

arises in the modeling the mixed convection boundary–layer flow in a porous medium. In [2] it is found that if m is positive and ε takes place in the rang $[\varepsilon_0, \infty)$, for some negative ε_0, there are two numerical solutions. The case $-1 \le m \le 0$ is also considered in [2]. The authors studied the problem for $\varepsilon_c \le \varepsilon \le 0.5$, for some $\varepsilon_c < 0$. It is shown that there exists ε_t such that the problem has two numerical solutions for $\varepsilon_c \le \varepsilon \le \varepsilon_t$. In [12] Guedda has investigated the theoritical analysis of (1.13), (1.14). It was shown that, if $-1 < m < 0$ and $-1 < \varepsilon < 1/2$, there is an infinite number of solutions, which indeed motivated the present work. Some new interesting results on the uniqueness of concave and convex solutions to (1.13) (1.14), for $m > 0$ and $\varepsilon > -1$ were reported in [6].

Most recently Aly et al. [1] have investigated the numerical and theoritical analysis of the existence, the uniqueness and non–uniqueness of solutions to (1.13), (1.14). It is shown that the problem has a unique concave solution and a unique convex solution for any $m > 0$ and $M \ge 0$. The case where the free stream is being retarded (increasing pressure) is also considered. The authors proved that, for any $-\frac{1}{3} < m < -M < 0$ and any real number a, the problem (Newtonian case) has an infinite number of solutions. The multiplicity of solutions is also examined for $-\frac{1}{2} < m < -M < 0$ provided $b > \frac{M}{m+1}$ and $a \ge \frac{b}{\sqrt{(m+1)b-M}}$.

The purpose of the this note is to examine problem (1.9)-(1.11) for $< m < -M < 0$.

2. Existence of infinitely many solutions

The interest in this section will be in the existence question of multiple solutions of problem (1.9)–(1.11), where $-1 < m(2n-1)$, $m < 0$ and $m + M < 0$. The existence result will be established by means of a shooting method. Hence, the boundary condition at infinity is replaced by the condition

$$f''(0) = \tau, \tag{2.1}$$

where γ is the shooting parameter which has to be determined. Local in η solution to (1.9), (1.10), (2.1) exists for every $\gamma \in \mathbb{R}$, and it is unique. Denote this solution by f_τ. Let us describe what conditions will be imposed for f_τ to be global and satisfies (1.11). Note that the real number τ has a physical meaning. This parameter originates from the local skin friction coefficient, c_f, and the local Reynolds numbers, Re_x,

$$\frac{1}{2}c_f Re_x^{1/n+1} = \left[\frac{m(2n-1)+1}{n(n+1)}\right]^{n/(n+1)} |f_\tau''(0)|^{n-1} f_\tau''(0),$$

where $Re_x = \frac{u_w(x)^{2-n}x^n}{\alpha K}$.

Returning to the initial value problem (1.9), (1.10), (2.1), our purpose is to derive favorable conditions on m, a and b such that f_τ is global and satisfies $f_\tau'(\infty) = 1$. We shall impose the condition $m \in (-1, 0)$. The local solution f_τ satisfies the following equality that will be useful later on:

$$\begin{aligned} |f_\tau''(\eta)|^{n-1} f_\tau''(\eta) + s f_\tau'(\eta) f_\tau(\eta) - M f_\tau(\eta) = \\ |\tau|^{n-1}\tau + sab - Ma - (M+m)t \\ + \tfrac{1+3nm}{n+1}\int_0^\eta f_\tau'(s)^2 ds, \end{aligned} \tag{2.2}$$

for all $0 \le \eta < \eta_\tau$, where $(0, \eta_\tau)$ is the maximal interval of existence. Let us note that if η_τ is finite the function f_τ is unbounded on $(0, \eta_\tau)$ [1], [10]. Define

$$\Gamma = -\frac{3M}{4m}\left[1 + \sqrt{1 + \frac{16}{3}\frac{m}{M^2}(M+m)}\right] > 1,$$

where $M > 0$ and $m + M < 0$. Our main result is the following:

THEOREM 2.1. *Let $M > 0, -1 < m(2n-1)$ and $m < -M$. Assume $a \ge 0$ and $b \in (0, \Gamma)$. For any $\tau \in \mathbb{R}$ such that*

$$\tau^{n+1} \le (n+1)\left[\frac{1}{3}mb^3 + \frac{1}{2}Mb^2 - (M+m)b\right], \tag{2.3}$$

f_τ is global and satisfies (1.11).

Note that, since τ is arbitrary, problem (1.9)–(1.11) has an infinite number of solutions. To prove Theorem 2.1 we use an idea given in [12]. First we have the following result.

LEMMA 2.1. *For any $a \ge 0$, $0 < b < \Gamma$ and τ satisfying condition* (2.3), *the function f_τ is positive, monotonic increasing on $(0, \eta_\tau)$ and global. Moreover $f_\tau(\eta)$ tends to infinity with η and $\lim_{\eta\to\infty} f_\tau''(\eta) = 0$.*

PROOF. From equation (1.9) one sees

$$E' = -s f_\tau f_\tau''^2,$$

on $(0, \eta_\tau)$, where E is the "Lyapunov " function for f_τ defined by

$$E = \frac{1}{n+1}|f_\tau''|^{n+1} - \frac{m}{3}f_\tau'^3 - \frac{M}{2}f_\tau'^2 + (M+m)f_\tau'.$$

On the other hand, since $a \ge 0$ and $b > 0$ we may assume $f_\tau, f_\tau' > 0$ on some $(0, \eta_0), 0 < \eta_0 < \eta_\tau$.

Hence, the function E is monotonic decreasing on $(0,\eta_0)$. This implies

$$E(\eta_0) \leq E(0), \tag{2.4}$$

which shows that $E(\eta_0) \leq 0$, tanks to (2.3). If $f'_\tau(\eta_0) = 0$, we get $E(\eta_0) = E(0) = 0$, and then $E(\eta) = 0$ for all $0 \leq \eta \leq \eta_0$. Therefore $f''_\tau \equiv 0$ on $(0,\eta_0)$, and this implies $\tau = 0$ and $b = 0$ or $b = \Gamma$, a contradiction. Hence f_τ is monotonic stricty increasing.

To show that f_τ is global, we use again the function E to deduce

$$\begin{aligned}&\tfrac{1}{n+1}|f''_\tau|^{n+1} - \tfrac{m}{3}{f'_\tau}^3 - \tfrac{M}{2}{f'_\tau}^2 + (M+m)f'_\tau \\ &\quad\leq \tfrac{1}{n+1}|\tau|^{n+1} - \tfrac{m}{3}b^3 - \tfrac{M}{2}b^2 + (M+m)b.\end{aligned} \tag{2.5}$$

Therefore f''_τ and f'_τ are bounded. Hence, f_τ is bounded on $(0,\eta_\tau)$, if η_τ is finite, which is absurd. Consequently $\eta_\tau = \infty$; that is f_τ is global. Moreover, f_τ has a limit, say $L \in (0,\infty]$, at infinity, since f'_τ is positive. To demonstrate that L is infinite, we assume for the sake of contradiction that $L < \infty$. Hence, there exists a sequence (η_r) converging to infinity with r such that $f'_\tau(\eta_r)$ tends to 0 as n tends to infinity. Clearly,

$$-\frac{m}{3}f'_\tau(\eta_r)^3 - \frac{M}{2}f'_\tau(\eta_r)^2 + (M+m)f'_\tau(\eta_r)$$

$$\leq E(\eta_r) \leq E(0), \quad \forall\, n \in \mathbb{N},$$

which implies $0 \leq E(\infty) \leq E(0)$. As above, we get a contradiction. It remains to show that the second derivative of f_τ tends to 0 at infinity, which is the case if f''_τ is monotone on some interval $[\eta_0,\infty)$, since f''_τ and f'_τ are bounded. Assume that $|f''_\tau|^{n-1}f''_\tau$ is not monotone on any interval $[\eta_0,\infty)$. Then, there exists an increasing sequence (η_r) going to infinity with r, such that $\left(|f''_\tau|^{n-1}f''_\tau\right)'(\eta_r) = 0, |f''_\tau|^{n-1}f''_\tau(\eta_{2r})$ is a local maximum and $|f''_\tau|^{n-1}f''_\tau(\eta_{2r+1})$ is a local minimum. Setting $\eta = \eta_r$ in equation (1.9) yields

$$sf''_\tau(\eta_r) = -\frac{m(1-f'_\tau(\eta_r)^2) + M(1-f'_\tau(\eta_r)}{f_\tau(\eta_r)}. \tag{2.6}$$

Because f'_τ is bounded and $f(\eta)$ tends to infinity with η, we get from (2.6) $f''_\tau(\eta_r) \to 0$ as $n \to \infty$, and (then) $f''_\tau(\eta) \to 0$ as $\eta \to \infty$. ■

In the next result we shall prove that $f'_\tau(\eta)$ goes to 1 as η approaches infinity and this shows that problem (1.9)–(1.11) has an infinite number of solutions.

LEMMA 2.2. *Let f_τ be the (global) solution of* (1.9), (1.10), (2.1) *obtained in Lemma 2.1. Then*

$$\lim_{\eta\to\infty} f'_\tau(\eta) = 1.$$

PROOF. First we show that f'_τ has a finite limit at infinity. From the proof of Lemma 2.1 the function E hase a finite limit at infinity, E_∞, say, and this limit takes place in the interval $\left[\dfrac{4m+3M}{6}, 0\right]$. Since f''_τ goes to 0, we deduce that $-\frac{m}{3}{f'_\tau}^3 - \frac{M}{2}{f'_\tau}^2 + (M+m)f'_\tau$ tends to E_∞ as $\eta \to \infty$. Let L_1 and L_2 be two nonnegative real numbers given by

$$L_1 = \liminf_{\eta\to\infty} f'_\tau(\eta) \text{ and } L_2 = \limsup_{\eta\to\infty} f'_\tau(\eta)$$

and satisfy

$$E_\infty = -\frac{m}{3}L_i^3 - \frac{M}{2}L_i^2 + (M+m)L_i, \quad i = 1,2.$$

Suppose that $L_1 \neq L_2$ and fix L so that $L_1 < L < L_2$. Let $(\eta_r)_{n\in\mathbb{N}}$ be a sequence tending to infinity with n such that $\lim_{n\to\infty} f'_\tau(\eta_r) = L$. Using the function E we infer

$$E_\infty = -\frac{m}{3}L^3 - \frac{M}{2}L^2 + (M+m)L,$$

for all $L_1 < L < L_2$, which is impossible. Then $L_1 = L_2$. Hence, $f'_\tau(\eta)$ has a finite limit at infinity. Let us note this limit by L, which is nonnegative. Assume that $L = 0$. Then $E_\infty = 0$. Since E is a decreasing function, we deduce

$$E \equiv 0,$$

and get a contradiction. Hence $L > 0$. Next, we use identity (2.2) to deduce, as η approaches infinity,

$$|f''_\tau|^{n-1}f''_\tau(\eta) = -(M+m)\eta + ML\eta - sL^2\eta + \frac{1+3nm}{n+1}L^2\eta +$$

$$|f''_\tau|^{n-1}f''_\tau(\eta) = \left[mL^2 + ML - (M+m)\right]\eta + o(1),$$

and this is only satisfied if $mL^2 + ML - (M+m) = 0$, which implies $L = 1$, since L is positive. This ends the proof of the lemma and the proof of Theorem 2.1. ■

Lemma 2.2 shows also that $E_\infty = \frac{4m+3M}{6} < 0$. We finish this paper by a non–existence result in the case $m(2m-1) \leq -1, n > \frac{1}{2}$ and $b \geq \Gamma$.

THEOREM 2.2. *Problem* (1.9)-(1.10) *has no nonnegative solution for $M > 0$, $m < -M, m(2n-1) < -1$ and $b \geq \Gamma$.*

Proof. Let f be a nonnegative solution to (1.9)-(1.10)). As above, the function E satisfies $E' = -\frac{1+m(2n-1)}{n+1} f f''^2$, which is nonnegative. Clearly, $E(0) \leq \lim_{t\to\infty} E(t)$, hence

$$-\frac{m}{3}b^3 - \frac{M}{2}b^2 + (M+m)b \leq \frac{4m+3M}{6} < 0,$$

and this is not possible. ■

3. Numerical results

Now we presents the numerical results for differents values of n m and M:

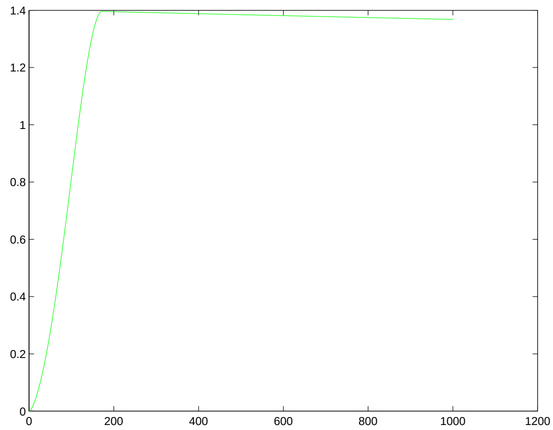

Figure 1: n=1.5, M=1, and m=-2.5

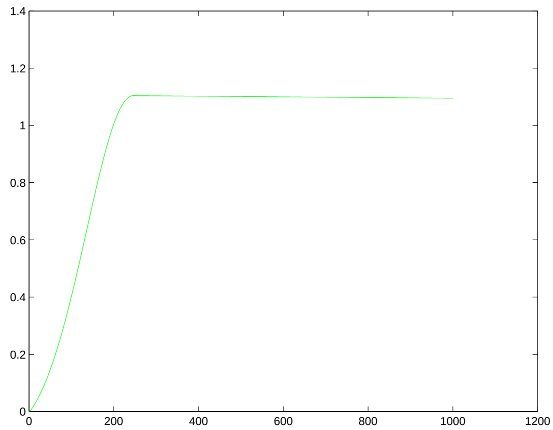

Figure 2: n=1.5, M=1.2, and m=-1.5

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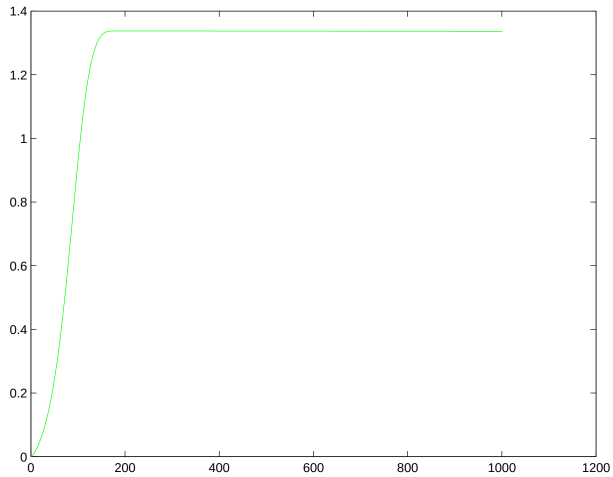

Figure 3: n=0.5, M=1, and m=-2.5

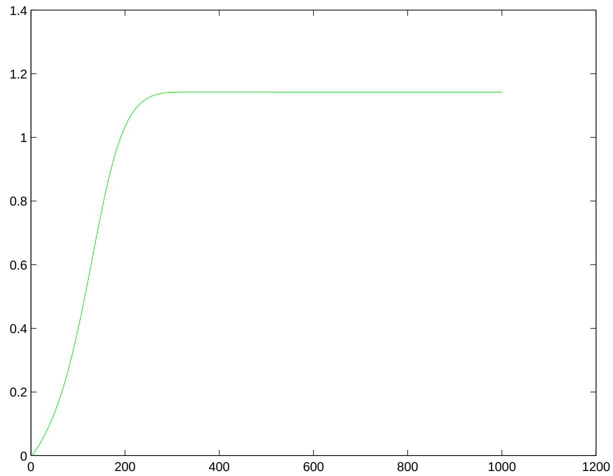

Figure 4: n=0.5, M=1.2, and m=-1.5

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