Magnetostatic Field calculations associated with thick Solenoids in the Presence of Iron using a Power Series expansion and the Complete Elliptic Integrals.

VASOS PAVLIKA

Harrow School of Computer Science
University of Westminster, School of Computer Science, Harrow, Middlesex, UK

Abstract:- The effect of iron on the uniformity of the field produced by an axisymmetric thick solenoid is considered. Using a power series expansion of the vector potential in the radial and axial coordinates the potential is found and from this the magnetic induction $\underline{B}$ is derived. The solution to the vector potential and field components is also achieved using the complete Elliptic Integrals of Legendre of the first and second kind respectively with numerical results using both methods of solution computed.

Key Words:- Time independent field, Power Series expansion and Elliptic Integrals of the first and second kind.

1. Introduction.

In this paper magnetostatic field calculations associated with an axisymmetric conductor of rectangular cross section situated equidistant from two semi-infinite regions of iron of finite permeability are computed. The magnetostatic field associated with iron-free axisymmetric systems has been considered by Boom and Livingstone [1], Garrett [2] and many others. Caldwell [3], Caldwell and Zisserman [4] and [5] have carried out work which takes account of the effects of the presence of iron on such systems. The main advantages of introducing iron are:

i. Higher fields are provided for the same current, producing substantial power savings over conventional conductors.

ii. The field uniformity is improved even for superconducting solenoids by placing the iron in a suitable position. The geometry considered is shown in figure 1, a toroidal conductor V' of rectangular cross section having inner radius A, outer radius B and length L-2ε, is located equidistant between two semi-infinite regions of iron of finite permeability a distance L apart, the axis of the conductor being perpendicular to the iron boundaries. The region V between the conductor and the iron is assumed insulating. Cylindrical coordinates *(r,ϕ,z)* are used where r and z are normalized in terms of L.

2. Problem Formulation

Prior to Caldwell [3] the presence of iron in axisymmetric systems had been largely ignored see

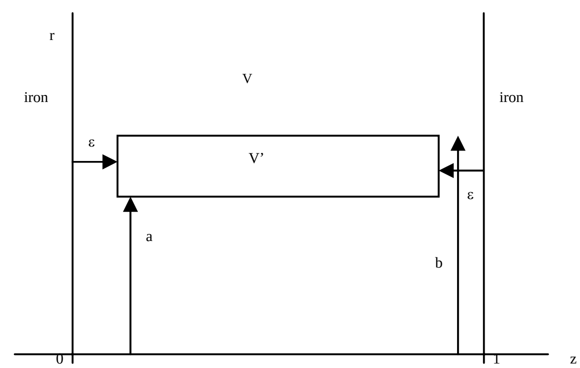

Figure1. A toroidal conductor V' of rectangular cross section located midway between two semi infinite regions of iron of finite permeability. The region V is assumed to be insulating.

Loney [6] and Garrett [2] et al. In cylindrical coordinates Maxwell's equations give:

$$\underline{\nabla} \wedge \underline{B} = \begin{cases} 0 \ \ in \ V \\ -Ce_{\phi} \ \ in \ V' \end{cases}$$

where $\boldsymbol{e_{\phi}}$ is a unit vector in the direction of increasing ϕ and C is a constant with

$\underline{\nabla}.\underline{B} = 0$ in V and V' (1)

Equation (1) suggests the introduction of a potential $\underline{A}$ such that $\underline{B} = \underline{\nabla} \wedge \underline{A}$, axial symmetry implies $B_r = -\frac{\partial A_{\phi}}{\partial z}$; $B_{\phi} = 0; B_z = \frac{1}{r}\frac{\partial (rA_{\phi})}{\partial r}$

By Maxwell's equation:

$$\underline{\nabla} \wedge \underline{B} = \underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{A}) = \begin{cases} 0 \;\; in \;\; V \\ Ce_\phi \;\; in \;\; V' \end{cases}$$

thus

$$\frac{1}{r}\begin{vmatrix} e_r & re_\phi & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ -\frac{\partial A_\phi}{\partial z} & 0 & \frac{1}{r}\frac{\partial (rA\phi)}{\partial r} \end{vmatrix} = \begin{cases} 0 \;\; in \;\; V \\ -Ce_\phi \;\; in \;\; V' \end{cases}$$

$$\Rightarrow \nabla_1^2 A_\phi = \begin{cases} 0 \;\; in\, V \\ Ce_\phi \;\; in\, V' \end{cases}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$

with boundary conditions for A_ϕ

$A_\phi = 0$ on r = 0; $A_\phi \to 0$ as $r \to \infty$

$\frac{\partial A_\phi}{\partial z} = 0$ on *z=0* and *z=1*. Axial symmetry gives

$\frac{\partial^2}{\partial \varphi^2} \equiv 0$. Using the integral representation of the vector potential this gives

$\underline{A}(\underline{r}) = \int_{V'} \frac{j(\underline{r})}{|\underline{r} - \underline{r}'|} dv'$, hence for finite μ,

$$A_\phi(r,z) = \frac{\mu_0 j}{4\pi}\sum_{n=-\infty}^{\infty} K^{|n|}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\frac{x\cos\vartheta\, dx\, d\vartheta\, dz'}{\{(z-z'-n)^2 + r^2 + x^2 - 2xr\cos\vartheta\}^{1/2}} \tag{2}$$

where $K = \frac{\mu - 1}{\mu + 1}$, known as the image factor. Noting that $A_\phi(r,z)$ is an odd function in r and an even function in z then A_ϕ can be expanded as a power series about the z axis giving:

$$A_\phi(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty} r^{2m+1} I_m(z) \tag{3}$$

where equation (2) gives

$$I_0(z) = \frac{1}{4}[[w\log_e|x+\alpha|]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

with w=z'-z-n and $\alpha^2 = x^2 + w^2$. Substituting expression (3) into equation (2) gives

$$\sum_{n=-\infty}^{\infty} K^{|n|}\left\{\sum_{m=1}^{\infty} 4m(m+1) r^{2m+1} I_m(z) + \sum_{m=1}^{\infty} r^{2m+1}\frac{\partial^2 I_m(z)}{\partial z^2}\right\} = 0$$

equating coefficients of $I_m(z)$ $\Rightarrow$

$$m(m+1) I_m(z) + \frac{\partial^2 I_{m-1}(z)}{\partial z^2} = 0, \;\; m = 0,1,2,...$$

so that $I_m(z) = \frac{(-1)^m I_0^{2m}(z)}{2^{2m} m!(m+1)!}$

and $A_\phi(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty}\frac{(-1)^m I_0^{2m}(z)}{2^{2m} m!(m+1)!} r^{2m+1}$

To relate this to the work of Garrett [2] let

$$a_1(x,w) = w\log_e|x+\alpha| \Rightarrow I_0 = \frac{j}{4}[[a_1(x,w)]_a^b]_{z=\varepsilon}^{1-\varepsilon} = \frac{A_1(z)}{2}$$

where $a_1(x,w) = \frac{j}{2}[[w\log|x+\alpha|]_a^b]_{z=\varepsilon}^{1-\varepsilon}$ (4)

and $A_m = \frac{j}{2}[[a_m(x,w)]_a^b]_{z=\varepsilon}^{1-\varepsilon}$ (5)

so that

$$A_\phi(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty}\frac{(-1)^m (2m)! A_{2m+1} r^{2m+1}}{2^{2m+1} m!(m+1)!} = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty}\frac{(-1)^m (2m-1)! A_{2m+1} r^{2m+1}}{(2m+2)!}$$

where (2m-1)!= 1.3.5…(2m-1),
and (2m+2)!= 2.4.6…(2m+2),

with $A_{m+1} = \frac{1}{m}\frac{\partial^m a_1}{\partial z^m}$ (6)

so for the field components

$B_z(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty}\frac{(-1)^m (2m-1)! A_{2m+1}(z) r^{2m}}{(2m)!}$ and

$$B_r(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(2m+1)! A_{2m+2}(z) r^{2m+1}}{(2m+2)!}$$

Hence $A_\phi(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\left(\frac{r}{2}A_1 - \frac{r^3}{8}A_3 + \frac{r^5}{16}A_5 + ...\right)$

$B_z(r,z) = \mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\left(A_1 - \frac{r^2}{2}A_3 + \frac{3r^4}{8}A_5 + ...\right)$ and

$$B_r(r,z) = -\mu_0 \sum_{n=-\infty}^{\infty} K^{|n|}\left(\frac{r}{2}A_2 - \frac{3r^3}{8}A_4 + \frac{5r^5}{16}A_6 + ...\right)$$

The first five terms will be quoted, the remainder can be obtained from the recurrence relations equations (4), (5) and (6). So that

$$A_2 = \frac{j}{2}[[\frac{x}{(w^2+x^2)^{1/2}} - \log_e(x + (w^2+x^2)^{1/2})]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

$$A_3 = \frac{j}{2}[[\frac{-x}{(w^2+x^2)^{1/2}} + \frac{xw}{(w^2+x^2)^{3/2}}]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

$$A_4 = \frac{j}{12}[[\frac{x}{(w^2+x^2)^{3/2}} - \frac{3xw}{(w^2+x^2)^{5/2}} + \frac{xw}{(w^2+x^2)^{3/2}}]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

$$A_5 = \frac{j}{48}[[\frac{3xw}{(w^2+x^2)^{5/2}} + \frac{6xw}{(w^2+x^2)^{3/2}} - \frac{15xw^3}{(w^2+x^2)^{7/2}} - \frac{x}{(w^2+x^2)^{3/2}} + \frac{3xw^2}{(w^2+x^2)^{5/2}}]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

and

$$A_6 = \frac{j}{240}[[\frac{-9x}{(w^2+x^2)^{5/2}} - \frac{9xw}{(w^2+x^2)^{3/2}} + \frac{15xw^2}{(w^2+x^2)^{7/2}} - \frac{xw^3}{(w^2+x^2)^{7/2}} - 105xw^4]_a^b]_{z=\varepsilon}^{1-\varepsilon}$$

3. The solution to the Magnetic Vector Potential using the Complete Elliptic Integrals.

In order to compare and validate the results of the previous section an independent solution for the magnetic vector potential and hence field components $B_r(r,z)$ and $B_z(r,z)$ must be derived. Using equation (2) if the integration with respect to ϑ is done first, the complete Elliptic Integrals of Legendre of the first and second kind respectively are obtained. Defining

$$I_\vartheta = \int_0^{2\pi} \frac{x\cos\vartheta}{(\beta^2 - \gamma\cos\vartheta)^{1/2}} d\vartheta$$

where $\beta^2 = w^2+x^2+r^2$ and $\gamma=2xr$ which can be written as:

$$I_\vartheta = \frac{x}{\beta}\int_0^{2\pi} \frac{\cos\vartheta}{(1-k\cos\vartheta)^{1/2}} d\vartheta$$

where $k = \frac{\gamma}{\beta^2}$, so that

$$I_\vartheta = -\frac{\beta}{2r}\int_0^{2\pi} \frac{1-k\cos\vartheta - 1}{(1-k\cos\vartheta)^{1/2}} d\vartheta$$

$$\Rightarrow I_\vartheta = -\frac{\beta}{2r}\int_0^{2\pi}(1-k\cos\vartheta)^{1/2}d\vartheta + \frac{\beta}{2r}\int_0^{2\pi}\frac{d\vartheta}{(1-k\cos\vartheta)^{1/2}}$$

with slight manipulation this can be written as

$$I_\vartheta = \frac{\beta}{r(1+k)^{1/2}}\left\{-(1+k)\int_{-\pi/2}^{\pi/2}(1-\delta^2\sin^2 u)du + \int_{-\pi/2}^{\pi/2}\frac{du}{(1-\delta^2\sin^2 u)^{1/2}}\right\}$$

where $\delta^2 = \frac{2k}{1+k}$ and $\frac{\vartheta}{2} = \frac{\pi}{2} - u$

so that $I_\vartheta = -\frac{2\beta^2}{r(\beta^2+\gamma)^{1/2}}\left((1+k)E(\delta) - K(\delta)\right)$

where K(δ) and E(δ) are the complete Elliptic integrals of Legendre of the first and second kind respectively. Provided 0<δ<1 these integrals may be expressed as a series which is uniformly convergent and thus may be integrated term by term. So considering this inequality with:

$k = \frac{2xr}{w^2+x^2+r^2}$ and $\delta^2 = \frac{2k}{1+k} = \frac{4xr}{w^2+(x+r)^2}$

hence for convergence $0 < \left(\frac{4xr}{w^2+(x+r)^2}\right)^{1/2} < 1$

i.e. 4xr > 0 which is true $\forall x, r > 0$. Similarly the second inequality gives $4xr < w^2 + (x+r)^2$ or $-(x-r)^2 < w^2$, which is again true $\forall x, r, w \neq 0$. So that the series is uniformly convergent. Hence using

$$K(\delta) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\delta + \left(\frac{1.3}{2.4}\right)^2\delta^2 + \left(\frac{1.3.5}{2.4.6}\right)^2\delta^3 + O(\delta^4)\right)$$

and

$$E(\delta) = \frac{\pi}{2}\left(1 - \left(\frac{1}{2}\right)^2\delta - \left(\frac{1.3}{2.4}\right)^2\frac{\delta^2}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{\delta^3}{5} - O(\delta^4)\right)$$

gives

$$A_\phi(r,z) = -\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}\int_a^b\int_{z-\varepsilon-n}^{z-1+\varepsilon-n}\frac{\beta^2}{r(\beta^2+\gamma)^{1/2}}\{k - (k+2)\left(\frac{1}{2}\right)^2\delta^2 - (k+4)\left(\frac{1.3}{2.4}\right)^2\frac{\delta^4}{3} - (k+6)\left(\frac{1.3.5}{2.4.6}\right)^2\frac{\delta^6}{5} - ....\}dxdz'$$

4. Considering the Higher Order, terms of δ^n.

Considering the order δ^0 term which will be denoted by I_0, say where

$$I_0 = \int_a^b\int_{z-\varepsilon-n}^{z-1+\varepsilon-n}\frac{\beta^2 k}{r(\beta^2+\gamma)^{1/2}}dxdz'$$

$$\Rightarrow I_0 = -2\int\int\frac{u-r}{(w^2+u^2)^{1/2}}dudw$$

where $x+r=u$ and $w=z-z'$, so that

$$A_\phi(r,z) = \frac{\mu_0 j}{4\pi}\sum_{n=-\infty}^{\infty}K^{|n|}[[(u^2-2ru)\log_e(w+(w^2+u^2)^{1/2}) + w(w^2+u^2)^{1/2} - 2rw\log_e(u+(w^2+u^2)^{1/2})]_{a+r}^{b+r}]_{z'=z-\varepsilon-n}^{z-1+\varepsilon-n} + O(\delta^2)$$

Evaluating the higher order terms as shown in Pavlika [8], it can be shown that:

$$A_\phi(r,z)=\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[-rw\log_e(u+(w^2+u^2)^{1/2})$$

$$+\frac{r^2w}{(w^2+u^2)^{1/2}}-\frac{r^3(27u^4+70w^4+102u^2w^2)}{6uw(w^2+u^2)^{3/2}}$$

$$+\frac{r^4w(59u^2+49w^2)}{3u^2(w^2+u^2)^{3/2}}-\frac{r^5(4u^6+6u^4w^2+21u^2w^4+14w^6)}{u^3w^3(w^2+u^2)^{3/2}}$$

$$+\frac{10r^6w(3u^2+2w^2)}{3u^4(w^2+u^2)^{3/2}}-$$

$$\frac{5r^7(8u^8+12u^6w^2+3u^4w^4+12u^2w^6+8w^8)}{2(w^2+u^2)^{3/2}}]_{a+r}^{b+r}]_{z'=z-\varepsilon-n}^{z-1+\varepsilon-n}$$

$$+O(\delta^8) \qquad (6)$$

or $A_\phi(r,z)=\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[r\alpha_{3,1}(u,w)+r^2\alpha_{3,2}(u,w)$

$$+r^3\alpha_{3,3}(u,w)+r^4\alpha_{3,4}(u,w)+$$

$$r^5\alpha_{3,5}(u,w)+r^6\alpha_{3,6}(u,w)+r^7\alpha_{3,7}(u,w)]_{a+r}^{b+r}]_{z=z'-\varepsilon-n}^{z-1+\varepsilon-n}$$

$$+O(\delta^8)$$

where the $\alpha_{i,j}$, $i=3, j=2,3,\ldots7$ are defined by expression (6)

5. Calculating the Radial and Axial Field Components.

Since $\underline{B}=\underline{\nabla}\wedge\underline{A}$, using cylindrical coordinates this gives

$$B_z(r,z)=\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[2\alpha_{3,1}(u,w)+3r\alpha_{3,2}(u,w)$$

$$+4r^2\alpha_{3,3}(u,w)+5r^3\alpha_{3,4}(u,w)+$$

$$6r^4\alpha_{3,5}(u,w)+7r^5\alpha_{3,6}(u,w)+8r^6\alpha_{3,7}(u,w)]_{a+r}^{b+r}]_{z=z'-\varepsilon-n}^{z-1+\varepsilon-n}$$

$$+O(\delta^8) \qquad (7)$$

and differentiating with respect to z gives

$$B_r(r,z)=\frac{\mu_0 j}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[-r\{\log_e(u+(w^2+u^2)^{1/2})$$

$$+\frac{w^2}{(w^2+u^2)^{1/2}(u+(w^2+u^2)^{1/2})}\}+\frac{r^2u^2}{(w^2+u^2)^{3/2}}+$$

$$\frac{r^3u(9u^4+2u^2w^2-2w^4)}{2w^2(w^2+u^2)^{5/2}}+\frac{r^4(59u^2+29w^2)}{3(w^2+u^2)^{5/2}}+$$

$$\frac{3r^5u(4u^4+10u^2w^2+w^4)}{w^4(w^2+u^2)^{5/2}}+\frac{10r^6}{(w^2+u^2)^{5/2}}-$$

$$\frac{25r^7w^5(15u^4+20u^2w^2+8w^4)}{2(w^2+u^2)^{5/2}}]_{a+r}^{b+r}]_{z'=z-\varepsilon-n}^{z-1+\varepsilon-n}+O(\delta^8)$$

(8)

Results for $A_\phi(r,z)$, $B_r(r,z)$ and $B_z(r,z)$ using expressions (6), (7) and (8) with a=0.9, b=1.1, ε = 0.05 and μ_0j = 100 were found to be in good agreement with the solution using the Power series expansion as shown in tables 1, 2, 3, 4 and 5.

6. Conclusions

The two methods of solution were found to be in good agreement. The summations were performed from -200 to 200 with a change only in the fourth decimal place occurring when the number of terms in the summation was doubled. The effect of the permeability of the iron is shown in figures 2, 3, 4 and 5. The two methods described can be easily computerised and provide a quick and flexible method for calculating and thus demonstrating the effects of the permeability of iron μ, on the field components. It is clear that the accuracy of the methods and the region of applicability can be extended by taking more terms in both the Power Series and in the series obtained using the Complete Elliptic Integrals of the 1st and 2nd kind respectively. The effects of the iron in boosting and improving the field homogeneity are clearly evident.

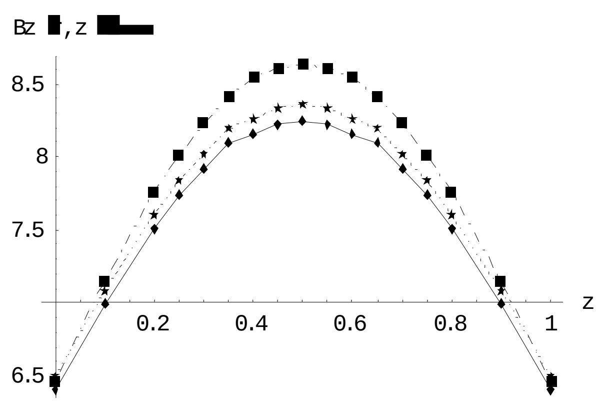

Figure 2. The variation of $B_z(r,z)$ with r and z for two semi-infinite regions of iron of unit permeability. ■:*r=0.3*, *:*r=0.2*, ●:*r=0.1*

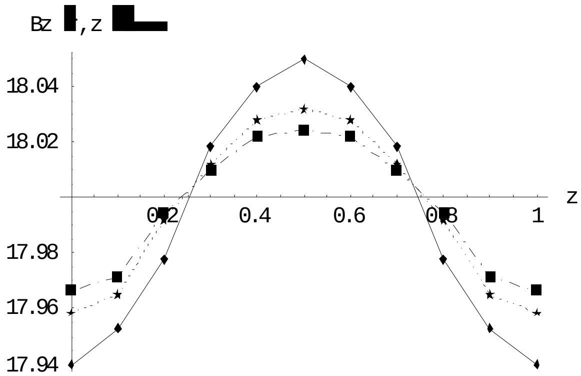

Figure 3. The variation of $B_z(r,z)$ with r and z for two semi-infinite regions of iron of infinite permeability. ■:r=0.1,*:r=0.2, ●:r=0.3

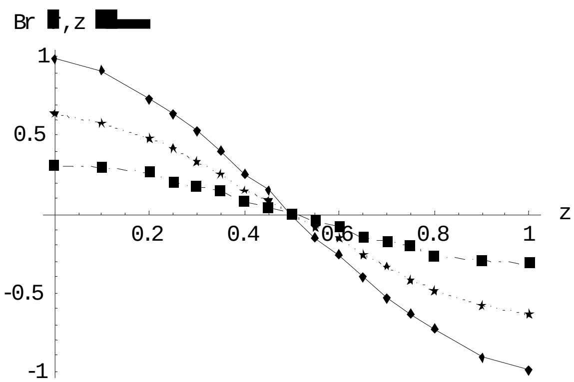

Figure 4. The variation of $B_r(r,z)$ with r and z for two semi-infinite regions of iron of unit permeability. ■:r=0.1, *:r=0.2, ●:r=0.3

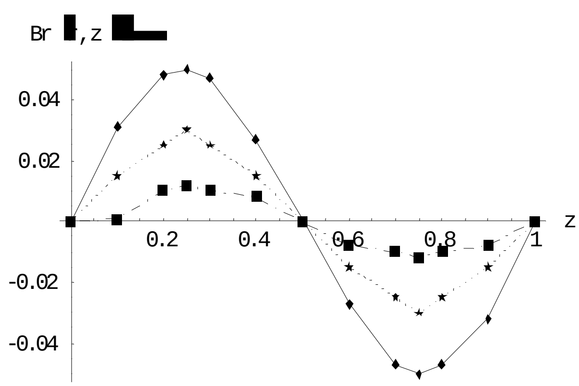

Figure 5. The variation of $B_r(r,z)$ with r and z for two semi-infinite regions of iron of infinite permeability. ■:r=0.1, *:r=0.2, ●:r=0.3

7. References

[1] Boom, R.W., and Livingstone. R.S., Proc. IRE, 274 (1962).

[2] Garrett, M.W., Axially symmetric systems for generating and measuring magnetic fields. J. Appl. Phys., **22**, 1091 (1951).

[3] Caldwell. J., Magnetostatic field calculations associated with superconducting coils in the presence of magnetic material, IEEE, Transactions on Magnetics, Vol. MAG-18, 2, 397 (1982).

[4] Caldwell, J and Zisserman A., Magnetostatic field calculations in the presence of iron using a Green's Function approach. J.Appl. Phys.D 54, 2, (1983a).

[5] Caldwell, J and Zisserman A., A Fourier Series approach to magnetostatic field calculations involving magnetic material accepted for publication in J.Appl. Phys (1983b).

[6] Loney, S.T., The Design of Compound Solenoids to Produce Highly Homogeneous Magnetic Fields. J.Inst. Maths Applics (1966) 2, 111-125.

[7] Pavlika, V., Vector Field Methods and the Hydrodynamic Design of Annular Ducts, Ph.D thesis, University of North London, Chapter II, 1995.

[8] Pavlika, V., Vector Field Methods and the Hydrodynamic Design of Annular Ducts, Ph.D thesis, University of North London, Chapter III, 1995.

8. Tables

r	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0	0.1	0	0	0	0
0.1	0.1	0.8957	0.8807	0.7590	0.3495
0.2	0.1	1.7911	1.7613	1.5177	0.7021
0.3	0.1	2.6852	2.6414	2.2797	1.0607
0.4	0.1	3.5812	3.5232	3.0396	1.4285
0.5	0.1	4.4730	4.4001	3.8051	1.8094
0.1	0.2	0.8976	0.8835	0.7655	0.3743
0.1	0.3	0.8985	0.8836	0.7710	0.3952
0.1	0.4	0.8992	0.8859	0.7735	0.4070
0.1	0.5	0.8992	0.8860	0.7747	0.4123

Table 1. Values of $A_\phi(r,z)$ using the Power Series Expansion accurate O(r^4).

r	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0.1	0.1	5.584E-3	0.0127	0.0718	0.2815
0.2	0.1	1.131E-2	0.0272	0.1472	0.5776
0.3	0.1	2.350E-2	0.0451	0.2297	0.9026
0.4	0.1	3.826E-2	0.0680	0.3226	1.2710
0.5	0.1	5.896E-2	0.0976	0.4297	1.6972
0.1	0.2	8.727E-3	0.0141	0.0607	0.2316
0.1	0.3	8.493E-3	0.0122	0.0443	0.1647
0.1	0.4	5.153E-3	0.0070	0.0234	0.0855

0.1	0.5	0	0	0	0

Table 2. Values of $B_r(r,z)$ using the Power Series Expansion accurate O(r^5).

r	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0	0.1	17.9169	17.6163	15.1601	6.9821
0.1	0.1	17.0149	17.6150	15.1601	7.0022
0.2	0.1	17.9090	17.6111	15.1601	7.0627
0.3	0.1	17.8990	17.6046	15.1601	7.1634
0.4	0.1	17.8851	17.5964	15.1601	7.3045
0.5	0.1	17.8672	17.5838	15.1601	7.4860
0.1	0.2	17.9731	17.6545	15.2902	7.5232
0.1	0.3	17.9722	17.6770	15.4011	7.9258
0.1	0.4	17.9860	17.6995	15.4602	8.1802
0.1	0.5	17.9866	17.7014	15.4923	8.2672

Table 3. Values of $B_z(r,z)$ using the Power Series Expansion accurate O(r^4).

r	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0	0.1	0	0	0	0
0.1	0.1	0.89171	0.881237	0.7575	0.3480
0.2	0.1	1.79492	1.762866	1.5140	0.6901
0.3	0.1	2.69390	2.645276	2.2679	1.0200
0.4	0.1	3.59465	3.528857	3.0178	1.3318
0.5	0.1	4.49779	4.414001	3.7624	1.6195
0.1	0.2	0.89781	0.882507	0.7641	0.3732
0.1	0.3	0.89595	0.883736	0.7692	0.3925
0.1	0.4	0.89919	0.884628	0.7725	0.4048
0.1	0.5	0.89942	0.884954	0.7737	0.4090

Table 4. Values of $A_\phi(r,z)$ using the Elliptic Integrals of the 1st and 2nd kind, accurate O(δ^8).

r	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0.1	0.1	5.831E-3	0.0162	0.1041	0.0361
0.2	0.1	1.314E-2	0.0342	0.2119	0.0775
0.3	0.1	2.343E-2	0.0555	0.3673	0.1425
0.4	0.1	3.818E-2	0.0819	0.4520	0.1598
0.5	0.1	5.886E-2	0.1150	0.5913	2.0971
0.1	0.2	8.425E-3	0.0165	0.0851	0.2936
0.1	0.3	8.082E-3	0.0135	0.0606	0.2071
0.1	0.4	4.897E-3	0.0070	0.0315	0.0106
0.1	0.5	0	0	0	0

Table 5. Values of $B_r(r,z)$ using the Elliptic Integrals of the 1st and 2nd kind, accurate O(δ^8).