Road traffic Modelling and Simulating with fluid-dynamic approach

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Abstract: - In this work we present a mathematical model for fluid-dynamic flows on road networks, based on conservation laws according with Piccoli et al. approach. The road network is represented by means of a links and nodes structure. A description of Riemann Solver at junction is given and a complete implementation of this model is then presented, together with some results from computational experiences on different real case studies of road networks.

Key-Words: - Scalar Conservation Law, traffic flow, fluid-dynamic model.

1. Introduction

With this paper we want to focus on some road traffic problems. In particular we consider a road network with some junctions and, from a macroscopic point o view, we want to analyze a model describing this problem by means of some significant functions. We formulate it on the basis of conservation laws, proposed by Lighthill and Whitham ([14], [15]) and Richards [16].

Thus, on each single road, the evolution of this nonlinear model is governed by the scalar hyperbolic conservation law:

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad (1)$$

where $\rho = \rho(t,x) \in [0, \rho_{\max}]$, $(t,x) \in \mathbb{R}^2$, is the density of cars, $\rho_{\max}$ is the maximal density of cars, $f(\rho) = \rho v$ is the flux and v the average velocity. We further assume that v is a smooth decreasing function of the density ρ and f is concave.

Such a conservation law describes a fluid-dynamic approach useful to perform macroscopic phenomena as shock waves formation and propagation. Recently, fluid-dynamical approaches were extended to flows on urban networks: some based on the LWR model (1) and some others based on the Aw-Rascle second order model [2].

For the urban setting, the simple LWR model, ([14], [15], [16]) is sufficient to describe most of the important traffic behaviour features and it is the only one for which a fairly complete theory and numerics are available.

Let's to describe a road network as a finite collection of roads meeting at some junctions that play a key role, since the system at a junction is under-determined even after imposing the conservation of cars. In order to obtain a unique solution of the Riemann problem at junctions (problem with constant initial data on each road), we need to assume some rules, so we can construct solutions via wave-front tracking technique, by means of defining some right of way parameters.

The paper is organized as follows. We introduce the model for traffic flow on a road network in Section 2. The next Section 3 deals with prototype description and simulation cases developing. Finally, we discuss, in the last Section about results and future works.

2. Fluid-dynamic model for traffic simulation

Let's consider a road network as a finite number of roads, modelled by intervals $I_i = [a_i, b_i] \subset \mathbb{R}$, $i = 1, \ldots, N$, $a_i < b_i$, with one of the endpoints that can be infinite. The roads are connected by some junctions, and each junction J has a finite number of incoming and outgoing roads. On each road the problem agrees with equation (1).

We make the following assumptions on the flux function: $f : [0, \rho_{\max}] \to \mathbb{R}$ is a smooth, strictly concave function, $f(0) = f(\rho_{\max}) = 0$, $|f'(x)| \leq C < +\infty$. Hence, there exists a unique $\sigma \in]0, \rho_{\max}[$ such that $f'(\sigma) = 0$.

One example of velocity function ensuring (H) is:

$$v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right),$$

where $v_{\max}$ is the maximal velocity of cars, which travel along the road. Then the flux is given by

$$f(\rho) = v_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right).$$

For a single conservation law (1), a Riemann Problem (RP) is a Cauchy problem for an initial data piecewise constant with only one discontinuity. The solutions are either formed by continuous waves (rarefactions) or by travelling discontinuities (shocks). The condition at the junctions (*Rankine-Hugoniot* relation) holds:

$$\sum_{i=1}^{n} f(\rho_i(t,b_i^+)) = \sum_{j=n+1}^{n+m} f\left(\rho_j\left(t,a_j^-\right)\right),$$

where $\rho_i, i = 1,...,n$, are the incoming densities and $\rho_i, i = 1,...,n$ the outgoing ones. It represents a different way of writing the conservation of cars: it expresses the equality of incoming and outgoing fluxes.

Riemann Problems at junctions are under-determined even after prescribing the conservation of cars. Existence and Uniqueness of solution are guaranteed by three following rules:

(A) There are some fixed coefficients representing the drivers' preferences. These coefficients denote the traffic's distribution from incoming to outgoing roads. For this reason, it's useful to define a traffic *distribution matrix*:

$$A = \{\alpha_{ji}\}_{j=n+1,...,n+m,\ i=1,...,n} \in \mathbb{R}^{m\times n},$$

such that

$$0 < \alpha_{ji} < 1, \ \sum_{j=n+1}^{n+m} \alpha_{ji} = 1, i = 1,...,n; \ j = n+1,...,n+m.$$

(B) Respecting (A), the drivers choose roads such that the flux can be maximized, that is we suppose that no car can stop without cross the junction.

(C) Assuming that $m < n$ ($m = 1$ and $n = 2$), let C be the amount of cars that can enter the outgoing road. We fix a right of way parameter $p \in \left]0,1\right[$. Then pC cars come from the first incoming road and $(1-p)C$ cars come from the second one.

2.1 Riemann Solver

In this section, we recall the construction of the Riemann solver at junctions, which satisfy rules (A), (B) and (C). Particularly, we treat two case studies: junctions of type 2×1 (two incoming roads and one outgoing road) and junctions of type 1×2 (one incoming road and two outgoing roads).

Proposition. Let $(\rho_{1,0}, \rho_{2,0}, \ldots, \rho_{n+m,0})$ be the initial densities of a RP at junction and $\gamma_\varphi^{\max}, \ \varphi = 1,\ldots,n$ and $\gamma_\psi^{\max}, \ \psi = n+1,\ldots,n+m$ be the maximum fluxes that can be obtained on incoming roads and outgoing ones, respectively. Then:

$$\gamma_\varphi^{\max} = \begin{cases} f(\rho_{\varphi,0}), & \text{if } \rho_{\varphi,0} \in [0,\sigma], \\ f(\sigma), & \text{if } \rho_{\varphi,0} \in \left]\sigma,1\right], \end{cases} \qquad \varphi = 1,..,n \qquad (2)$$

$$\gamma_\psi^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{\psi,0} \in [0,\sigma], \\ f(\rho_{\psi,0}), & \text{if } \rho_{\psi,0} \in \left]\sigma,1\right], \end{cases} \qquad \psi = n+1,..,n+m. \qquad (3)$$

Let us consider a junction of type 2×1 (a and b are the incoming roads and c is the only outgoing road. Considering rule (C), the solution to the Riemann problem with initial data $(\rho_{a,0}, \rho_{b,0}, \rho_{c,0})$ is constructed in the following way. Since we want to maximize the through traffic (rule (B)), we set:

$$\widehat{\gamma}_c = \min\{\gamma_a^{\max} + \gamma_b^{\max}, \gamma_c^{\max}\},$$

where $\gamma_i^{\max}$, $i = a, b$, is defined as in (2) and $\gamma_c^{\max}$ as in (3). In fact, $\widehat{\gamma}_c$ is the maximal through flux, which can respect the Rankine-Hugoniot condition at the junction, i.e. the conservation of cars through the junction.

Notice that in this case the matrix A (or rule (A)) is simply given by the column vector (1,1), thus it gives no additional restriction. This is due to the fact that there is a single outgoing road, so cars must flow to that outgoing road necessarily.

Consider now the space (γ_a, γ_b) and the line:

$$\gamma_b = \frac{1-q}{q}\gamma_a, \qquad (4)$$

defined according to the rule (C). Let P be the point of intersection of the line (4) with the line $\gamma_a + \gamma_b = \widehat{\gamma}_c$. The final fluxes must belong to the region

$$\Omega = \left\{(\gamma_a, \gamma_b) : 0 \le \gamma_i \le \gamma_i^{\max}, \ 0 \le \gamma_a + \gamma_b \le \widehat{\gamma}_c, \ i = a,b\right\}.$$

There are two different cases:

- P belongs to Ω,
- P does not belong to Ω.

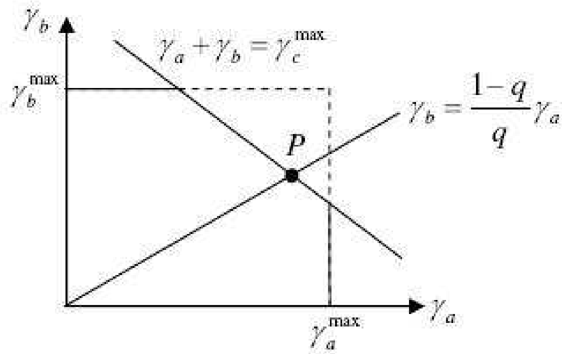

Fig 1. First case.

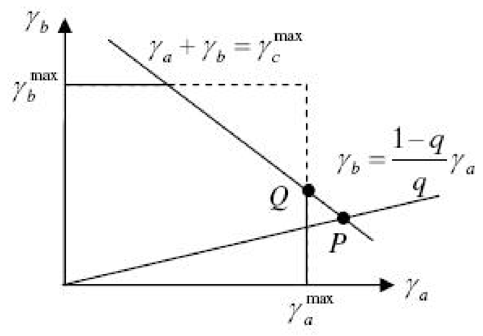

Fig 2. Second case.

The two cases are represented in Figures 1 and 2. In the first case, we set $(\widehat{\gamma}_a, \widehat{\gamma}_b) = P$, while in the second case we set $(\widehat{\gamma}_a, \widehat{\gamma}_b) = Q$, where Q is the point of $\Omega \cap \{(\gamma_a, \gamma_b) : \gamma_a + \gamma_b = \widehat{\gamma}_c\}$ closest to the line (4). Once we have determined $\widehat{\gamma}_a$ and $\widehat{\gamma}_b$ (and $\widehat{\gamma}_c$), we can find in a unique way $\widehat{\rho}_i$, $i \in \{a,b,c\}$. This is again due to restrictions on waves velocities.

Let us now consider the junction 1×2, with one incoming (*a*) and two outgoing (*b*, *c*) rods. In detail, is the only incoming road while and are the outgoing roads.

Here, no additional rule is needed thus only rules (A) and (B) are used. The distribution matrix A, of rule (A), takes the form

$$A = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix},$$

where $\alpha \in \,]0,1[$ and $(1-\alpha)$ denotes the percentage of cars which, from road *a*, goes to road *b* and *c*, respectively. Thanks to rule (B), the solution to a RP is:

$$\widehat{\gamma} = (\widehat{\gamma}_a, \widehat{\gamma}_b, \widehat{\gamma}_c) = (\widehat{\gamma}_a, \alpha\widehat{\gamma}_a, (1-\alpha)\widehat{\gamma}_a),$$

where

$$\widehat{\gamma}_a = \min\left\{\gamma_a^{\max}, \frac{\gamma_b^{\max}}{\alpha}, \frac{\gamma_c^{\max}}{1-\alpha}\right\}.$$

Once we have obtained $\widehat{\gamma}_a$, $\widehat{\gamma}_b$ and $\widehat{\gamma}_c$, it is possible to find in a unique way $\widehat{\rho}_i$, $i \in \{a,b,c\}$.

3 Implementing the FLUIDISM prototype

According to the model described in the previous sections, it was realized a prototype for the simulation of vehicular traffic: FLUIDSIM.

The prototype describes the behaviour of cars densities along the network as function of time.

In order to realize the simulation, it is necessary to insert a series of data input, related to the network topology, that has to be analyzed, and the starting initial configuration.

The topological informations (name of the road, length, impacts on some junctions) can be detected by GIS (Geographical Information System) database (file .dbf), that describes the network, so as to obtain all the informations related to roads (arcs) and junctions (nodes), that constitute the network itself [1].

Then, data related to the initial configuration for the simulation will have to be given. Particularly, the initial values of densities along the roads, the duration of simulation, and the numerical scheme to use, will be required. Moreover, as you can have different values of densities on same roads, it is necessary to use as input parameter a discretization, that allows to segment roads in subintervals of length $\left\lfloor \frac{l_i}{deltax} \right\rfloor$, where l_i represents the length of the *i-th* road, *i = 1, …, N*, where *N* indicates the numbers of considered roads and *deltax* the parameter given as input for the discretization.

The tool produces as output a series of *dbf* files, as function of simulation final time. In every file, for each road and for each its segment, the recorded vehicular density value is reported. Starting from an initial configuration of empty network, you will obtain that the first dbf file will have, for each road, values of densities on every segment equal to zero.

In order to have a consistent vision of these products, it is interesting to report the values of densities, that are in every file *.dbf* obtained on the studied network loading, in every GIS application, the shape file (*.shp*), that described the examined network. In this way, it is possible to have a direct and immediate graphical vision of the density along the network in a given instant of time.

Furthermore, reproducing in sequence the obtained various maps, it is possible to have a video, that describes the real behaviour of densities along the whole studied network, as function of the given input parameters.

The prototype was implemented in language ANSI C ++ and compiled with *gcc* in *linux* environment.

In what follows, its architecture is reported.

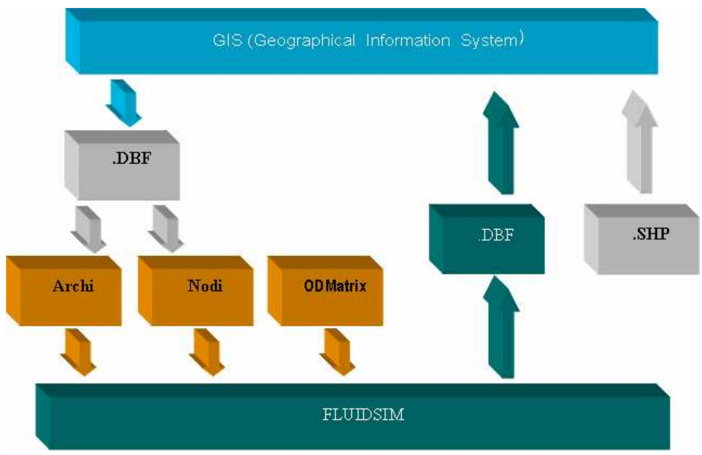

Fig 3. Architecture of the FLUIDSIM prototype.

3.1 Details of the prototype code

As function of the simulation final time, for every iteration the functions for nodes and arches network elaboration are required in sequence, and also the routine for the generation of output data. Particularly, in what follows, we present the *pseudocode* of the main function of the prototype:

```
For every iteration
 For every node n ∈ N
          Elaborate_node ( n )
 For every arc e ∈ E
          Elaborate_arc( e  )
Give_output( t )
```

In what follows, a short description of the required functions in the main of the tool.

Elaborate_node: the function, solving a linear programming problem, determines the optimal value of the flux and of the density to assign to the endpoints of an assigned incident link. At the beginning, this function used an extern routine “lp_solve.exe” in order to solve the linear programming problem. As such routine made the tool execution very heavy, it was then substituted by an inner function, that simulates the simplex method for the resolution of the problem. The performances of the prototype, in terms of execution times using an extern function or an inner one to solve the PL problem, are reported in section 4.

Elaborate_arch: as we told before, such function, for each road, can determine the density value on each segment, that constitutes itself. When the number of roads increases, according to the precision of the discretization parameter *deltax* and to the numerical scheme used [6], the execution of this routine can lead to some bottlenecks for the prototype performances.
Give_output: for each instant of time t, the corresponding dbf file is produced as output and, as already explained, it will contain, for each road, the value of density for each segment that constitutes itself.

Definitely, the prototype is able to verify how a given traffic flux can act on the whole urban network. Moreover, modifying the initial configuration of the given data, it is possible to analyze the whole network.
In this way, it is possible to give improvement to the vehicular flux in neuralgic zones of the urban network, zones that are very sensitive to heavy traffic situations.

4 Running Time Analysis

All the test results presented in this section were performed on a example network composed by 24 roads and 12 junctions, as presented in Figure 4.Thetest was performed on a single cpuPentium 41.7GHzPC with2GbRAM.All times are reported in seconds, and do not include the time needed for the exportation and the processing of the output data for successive visualization.

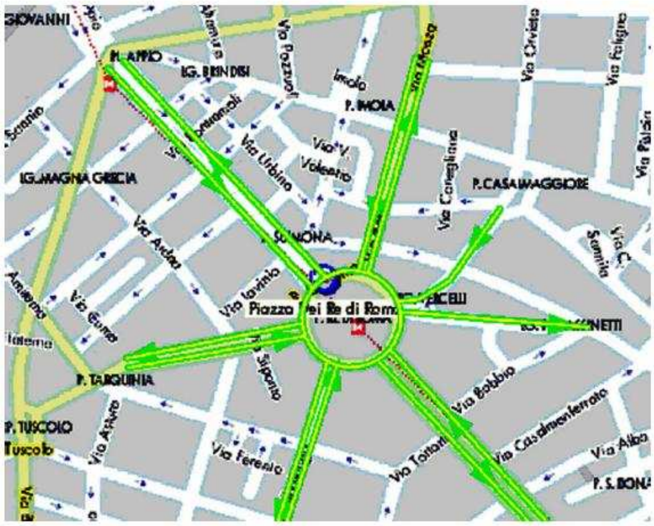

Fig. 4. The extracted network used for the code test.

In Figure 5 the computational time is reported at varying the simulated period. For 30 minutes simulations, the Godunov [12] and the 1-*st* order kinetic schemes require less that 60 seconds of computation, while the 2-*nd* order scheme needs about 10 seconds for each a minute of simulation. In Figure 6 a similar analysis is presented with respect to the size of the space step s. Results confirm the strong computational effort required by the 2-*nd* order approximation scheme.

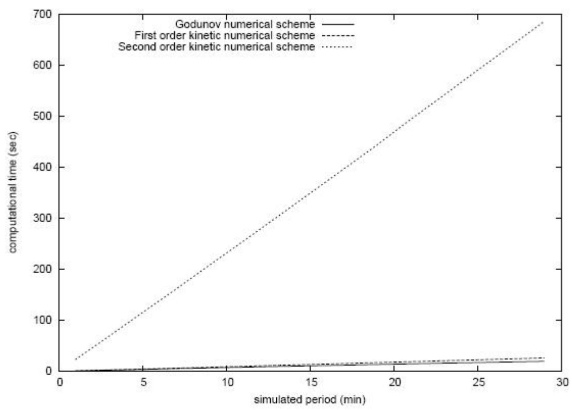

Fig. 5. Comparing numerical approximation schemes: computational time (seconds) vs simulated period (minutes).

In Figures 7,8 the computational time profile is presented for the Godunov approximation scheme, comparing the performance obtained through the embedded simplex algorithm routine with respect to the use of the external LP solver. For this numerical scheme the efficiency gain is relevant, while the use of the 2-nd order approximation scheme(see results in Figure 9) strongly reduces the impact of invoking an external solver, since in that case the computational bottleneck is in the arc processing routine.

4.1 Real networks case studies

Simulations were performed on real urban networks were made to test the suitability of this prototype in order to reproduce real traffic situations, and to test the scalability of the code on large size networks. In particular the code was tested on the whole traffic network extracted from the city of Salerno, Italy(see Figures 10,11) composed

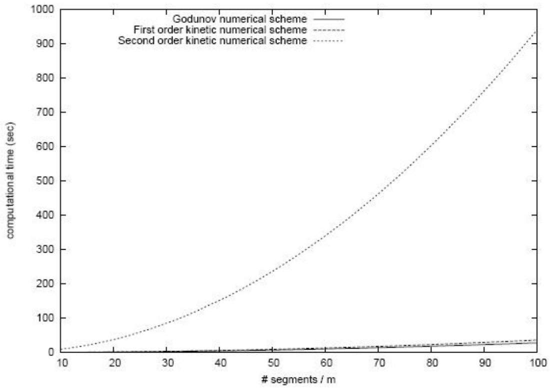

Fig. 6. Comparing numerical approximation schemes: computational time (seconds) vs (number of segments)/meter

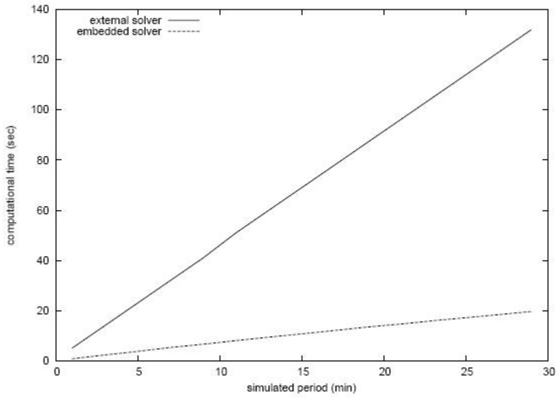

Fig. 7. Computational gain when embedding the PL solver algorithm in the code(Godunov numerical approximation scheme): computational time (seconds) vs simulated period(minutes) by almost 1200 roads. A focus was then performed on the junction of Salerno Fratte, as reported in Figures 10, 11.

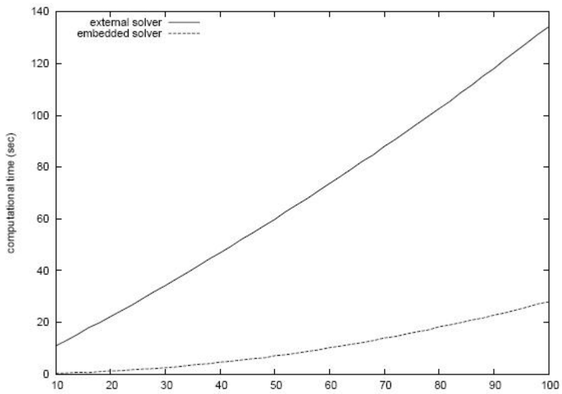

Fig. 8. Computational gain when embedding the LP solver algorithm in the code (Godunov numerical approximation scheme): computational time(seconds) vs (number of segments)/meter

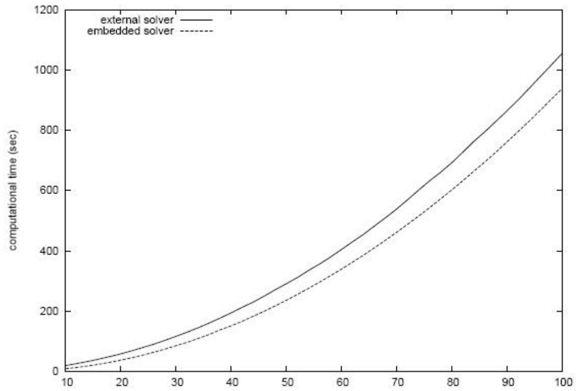

Fig. 9. Computational gain when embedding the LP solver algorithm in the code (2-*nd* order Kinetic numerical approximation scheme): computational time(seconds) vs (number of segments)/meter

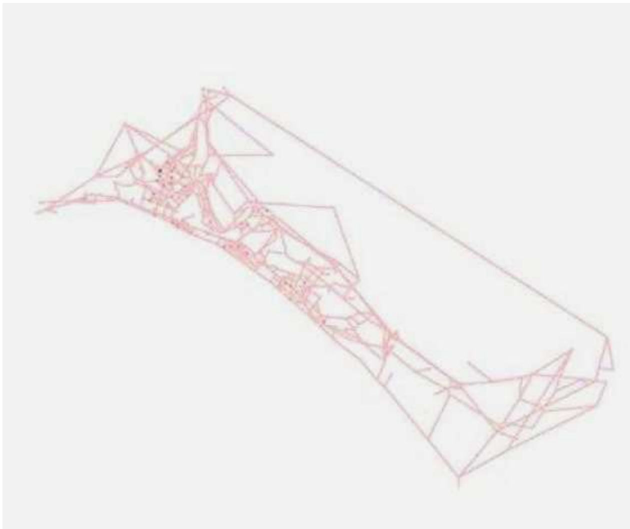

Fig. 10. The full network of Salerno (before the simulation).

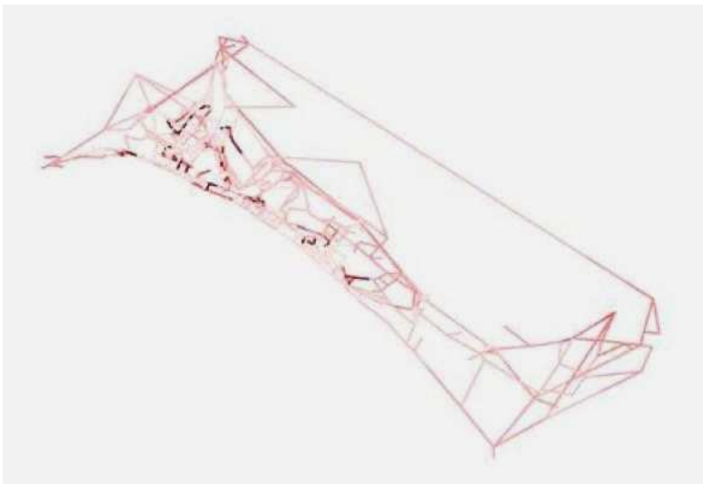

Fig. 11. The full network of Salerno (after the simulation).

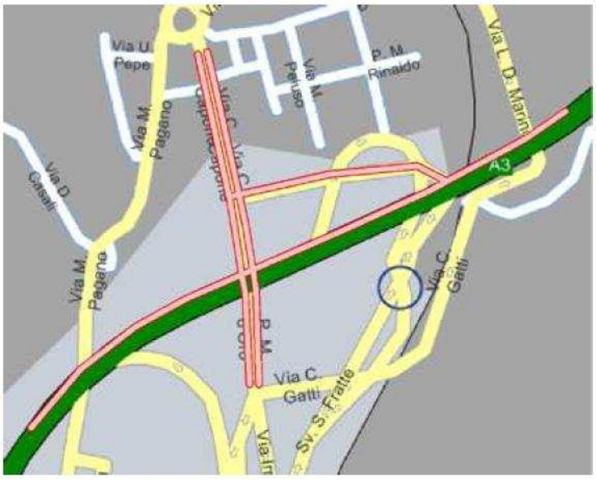

Fig. 12. A focus on a relevant junction: Salerno Fratte (before the simulation).

Fig. 13. A focus on a relevant junction: Salerno Fratte (after the simulation).

5 Conclusion

In this paper we deal with traffic problems on road network according with Piccoli et al. [8] fluid-dynamic approach that analyzes traffic by means of conservation law on each road of networks. Then we show some simulations carried out by a simulation prototype. Finally we want to end this work opening a new future aim about optimization of traffic behaviour along road networks. In fact we intend to compute some optimal value for distribution and right of way parameters in order to optimize networks performance.

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