Generation and processing of complex signals

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Abstract: We use the method for building fractal functions, from periodic scaled components, for the generation of optical and electronics fractal signals. For the processing, we consider the conditions for the application of the Shannon-Whittaker theorem for such cases. We use an expression of the sampling theorem for periodic (band-limited) functions.

Key-Words: Signal processing, digital and analog fractal signals, electronic signals, interferometers, optical signals, Shannon-Whittaker theorem, sampling theorem.

1 Introduction

The signals with certain degree of complexity have diverse applications in sciences and technological development [1-4]. It is for this reason that many authors are focused not only in the identification of such types of signals in natural or socioeconomics systems, but also in the generation and processing of them [5-8].

In previous works [9-11], it has been demonstrated that some prefractal structures can be obtained starting from periodic distributions (with a scaling factor between them). This fact is important for applications in the processing of optical and electrical signals, where some type of geometry can be required in the final signal. Also, an extension of the mathematical basis has been included in a recent development [12]. In the present work we are interested in showing a method for the generation of fractal signals using the procedure previously mentioned. We also include a variant of the Shannon-Whittaker theorem to exemplify the measurement procedure and processing of such signals. We consider two types of systems: 1) electronics signals, just as it is the case of a carrier signal and the periodic components included as products, 2) array of interferometers in cascade that allows to obtain a fractal structure at the output, from the superposition of fringes.

2 Fractal signals with periodic components

There are three basic transformations for building fractal objects: change of scale, translation and rotation. In several works we have used these transformations for the construction of fractal structures [9-11]. In these cases, we used periodic domains which are defined through distribution of disjoined sets included in 1D or 2D Euclidean space. The mathematical expression to obtain such fractal structures is:

$$C(x,y) = \prod_{k=1}^{N} \left\{ P\left[s^{k}; x, y \right] \right\}^{h_k} , \qquad (1)$$

where *P[]* is a set of scaled periodic (band-limited) functions, s is the scaling function and h_k is an exponent which permits filtering any periodic component individually.

Now, we include a method for building fractals sets, which was developed with more details in Ref. [12].

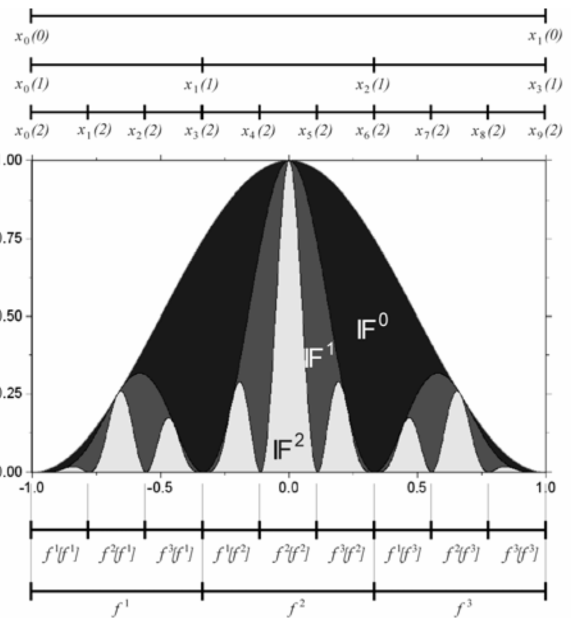


Figure 1 – Sequence of sets $\mathbb{F}^k$, obtained as the product of scaled periodic components, $f^i(f^j)$ are the successive contractions and $x_m()$ are the fixed points for the boundary of region equal to zero.

The foundation to represent regular fractal sets through a product superposition can be obtained by using a sequence of sets, which are the union of disjoined intervals (or contractions). If we have a sequence of sets, defined in the (X,d) space, whose boundary is obtained through the transformation:

$$f^k(x) = T\left[s^k;x\right]f^{k-1}(x) \quad \text{with} \quad f^0(x) = T\left[s^0;x\right] \quad \text{and} \quad k = 0,1,\ldots,N, \tag{2}$$

where k refers to each periodic component, $T\left[s^k;x\right]$ define contractions and each $f^k(x)$. These contractions are shown in Fig. 1, together with the fixed points obtained in each iteration. This fact allows to define a non-linear Iterated Function System (or IFS) [13].

So, using the theory of IFS function, it can be seen that the sets contained into the boundary defined in Eq. (1), have the following property for the sequence of sets F^k:

$$\mathrm{F}^k = \left\{(x,y) : |x| \le L,\, y \le f^k(x)\right\}, \quad \mathrm{F}^0 \supset \mathrm{F}^1 \supset \ldots \supset \mathrm{F}^N \tag{3}$$

where $x \in [-L,L]$ and $0 \le y \le 1$.

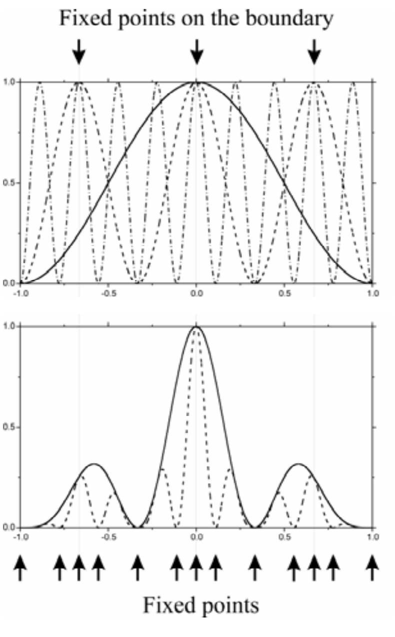


Figure 2 – Positions of fixed points in the product, related with the periodic components (up). Fixed points in the product (boundary equal to 0 or scaled components equals to 1).

If we consider the periodic signal functions as the boundary of sets (see Fig. 1), we can obtain the resulting signal (boundary of the attractor sets of Fig. 1). The process, to determine the fixed points for each iteration, is schematized in Fig. 2. We can see that a fixed point is given at the value 1 for the periodic function, from a certain periodic component and the successive scaled component from it.

2.1 Samplig of complex signals

The Shannon-Whittaker theorem (or sampling theorem) [14-16] relates the measured points of a certain signal and the possibility of reconstruction of such signal completely. Here, we use a consequence, expressed for the case of periodic band-limited functions [17]. Then, our interest is the inclusion of this formulation for the case of prefractal signals, obtained through a product superposition of periodic functions. The sampled function can be represented as a simple function, which is defined as:

A function $\boldsymbol{f : X \to R}$ is called simple if [12]:

$$f(x) = \sum_{i=1}^{K} R_i\, \chi\left[I_i;x\right], \text{ with } \chi\left[I_i;x\right] = \begin{cases} 1 \text{ if } x \in I_i \\ 0 \text{ if } x \notin I_i \end{cases}, \tag{4}$$

being $I_i \in B$ (a Borel set), $R_i \in \Re$ for $i=1,2,\ldots,K$; $\chi[I_i;x]$ is the characteristic function.

This way, we want to show a consequence of the sampling theorem for the reconstruction of signals with complex geometry. In Ref. [17] was obtained the expression for the sampling theorem, which is given by:

$$g(x) = \frac{\sin(2\pi\Lambda_x x)}{2K} \sum_{n=-L}^{M-1} (-1)^n F\left[\frac{n}{2\Lambda_x}\right] \tag{5}$$

$$\left[(-1)^{K+1}\tan\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K}\right) + \cot\left(2\pi\Lambda_x \frac{x - \frac{n}{2\Lambda_x}}{2K}\right)\right]$$

being K the number of samples and Λ_x is the size of the interval in the variable x.

Using Eq. (1), we can represent functions such as the one plotted in Fig. 1, employing a certain sampling frequency (for a single period). So, the sampling theorem is applied for the product superposition of periodic functions, which initially can be independently obtained. Then, using the product of functions of Eq. (1), the Shannon-Whittaker theorem can be expressed as:

$$g(x)=\frac{\sin(2\pi\Lambda_x x)}{2K}\prod_{k=1}^{N}\left\{\sum_{n=-L}^{M-1}(-1)^n\left[R_n\chi\left[I_n;x\right]\right]^{(k)}\right. \qquad (6)$$

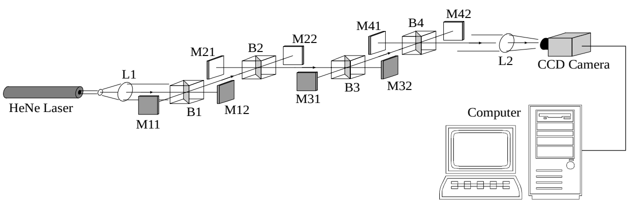
$$\left.\left[(-1)^{K+1}\tan\left(2\pi\Lambda_x\frac{x-\dfrac{n}{2\Lambda_x}}{2K}\right)+\cot\left(2\pi\Lambda_x\frac{x-\dfrac{n}{2\Lambda_x}}{2K}\right)\right]\right\}^{h_k}$$

where now $\chi[I_n;x]$ (= *0,1*) is the characteristic function for each periodic components. The interval I_n is related with the width of the corresponding sampling interval and the supra-index k indicates each periodic component. Furthermore, R_n is the value of the function at the point n.

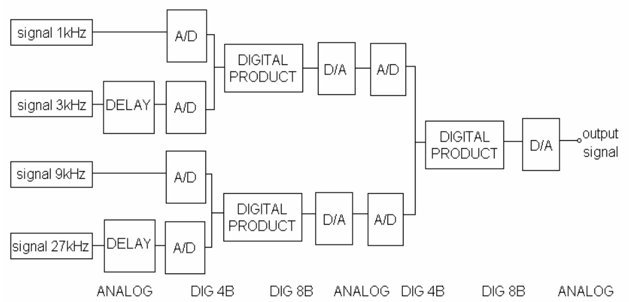
This means that if there is a system with several inputs (one for each periodic component), a signal described by Eq. (1) at the output; then, the sampling for each component and for the output signal are related through Eq. (5). Also, we can see that the sampling interval will be given by the corresponding interval of the component with smaller period.

3 Results obtained and discussion

Until now we have shown the method to obtain structures of fractal using periodic band-limited functions.



(a)



(b)

Figure 3 – (a) Optical setup. (b) Block diagram of the electronic system.

The Shannon-Whittaker sampling theorem assures us that we have a good representation of a function which possesses an experimental base, since the function is represented by discrete points obtained, for example, when a CCD-camera is used for optical signals.

When we implement practically the previous results, for the registration of complex signals, the infinite points of the fractal objects are never obtained. In the measurement of a certain signal only discrete points are obtained. Then, we can relate these points with the representation as a simple function. For example, the cosine functions (*cos^2*) of the Fig. 1 are built with finite number of points, with a scaling factor between each periodic component.

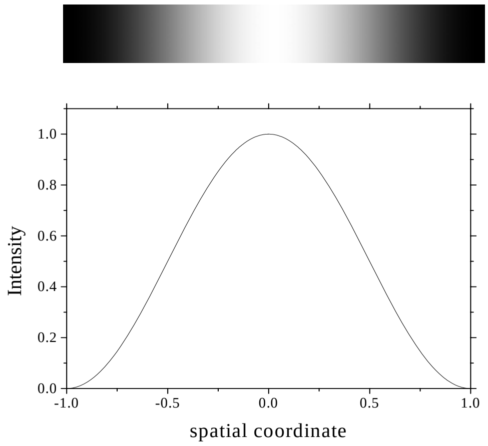
In Fig. 3a the experimental setup, with interferometers in cascade, are shown. The elements indicated with M are mirrors, with L are lenses and with B are beamsplitters. The fringes (and the corresponding profiles) are obtained from a CCD camera, and can be visualized in Fig. 4. The block diagram in Fig. 3b shows the way in which the electronic signals are superposed through the following process: analog/digital converter, product and normalization (D/A-A/D). The electronic signals, registered with an osciloscope are shown in Fig. 5.

In both cases periodic components (of the type $\cos^2$) were generated, and it is seen that fractal structures are obtained at the output, whose mathematical foundation for the generation and processing was exposed in section 2.

4 Conclusions

In this work, we used the well-known results for the construction of prefractal functions through periodic components. As an example, we used cosine functions, which can be obtained from the measurement of optical or electronic signals. Since the function has a periodic envelope, we use a version of the sampling theorem which permits us to represent it (and their scaled periodic components) from finite number of points. The sampling interval that must be used is the one corresponding to the component with the smallest period.

The result presented here represents a simple method for the generation and processing of fractal signals that can be used in diverse fields of technology, such as encryption, barcodes, images, general metrology, etc.



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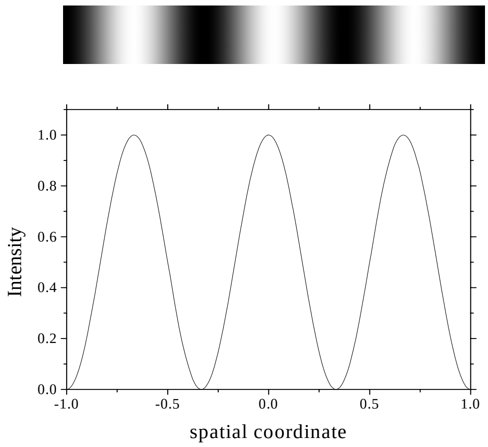

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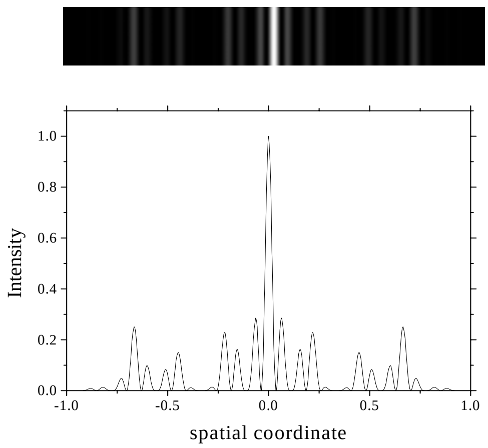
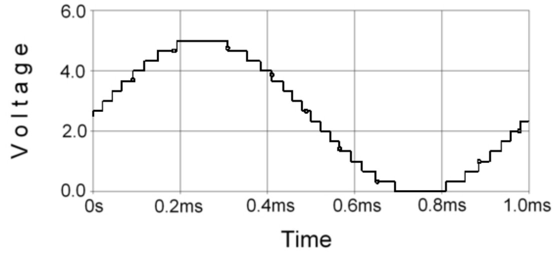
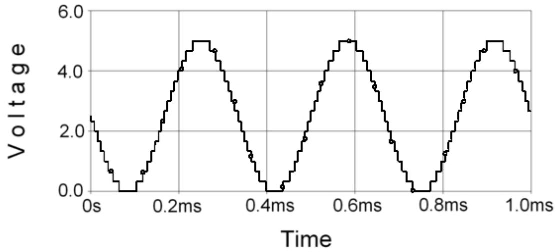


(c)

Figure 4 – Intensity vs. spatial coordinate for the interferometers in cascade of Fig. 3: (a) first periodic component, (b) second periodic component, (c) fractal structure with order 3.



(a)



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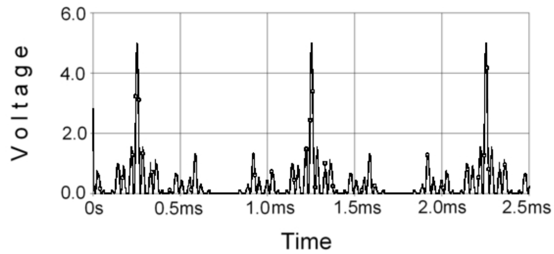

(d)

Figure 5 – Voltage vs. Time for the electronic signals for the block diagram of Fig. 3: (a) first periodic component, (b) second periodic component, (c) fractal component with order 3.

Acknowledgments
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