Computational Results for Integral Modeling in some Problems of Electrical Engineering

LECHOSŁAW HĄCIA
Institute of Mathematics
KAROL BEDNAREK , ANDRZEJ TOMCZEWSKi
Institute of Electrical Engineering and Electronics
Faculty of Electrical Engineering, Poznań University of Technology
Piotrowo 3a, 60-965 Poznań,
POLAND
http://www.math.put.poznan.pl/~lhacia

Abstract: - In this paper the method of integral equations is proposed for some problems of electrical engineering (current density, radiative heat transfer, heat conduction). Presented models lead to a system of Fredholm integral equations, integro-differential equations or Volterra-Fredholm integral equations, respectively. We propose various numerical methods (discretization method and projection methods) providing to a system algebraic equations. Computational results for integral modeling are given.

Key-Words: Integral equations method, current density, Fredholm integral equation, radiosity problems, Volterra-Fredholm integral equation, numerical methods.

1. Introduction

In natural sciences and engineering a mathematical model is generally defined by differential equations which can be solved using mostly: finite-difference method (FDM), finite-element method (FEM), boundary-element method (BEM). The choice of calculation method is affected by many factors. First of all, it is determined by the possibility of an accurate definition of the problem and regard to the boundary-initial conditions. It also depends on the system of algebraic equations and parameters of computer hardware used for carrying out the task. Unfortunately, the above-mentioned methods are of little avail in determining the voltage gradient distribution in electro-insulation systems. The integral equation method (IEM) seems to be the most appropriate for this purpose, as electric potential distribution is described with the help of integral equations [4]. Such formulation of the problem, including boundary conditions (potentials for the conductors or their total charges) leads to a system of integral equations the numerical solution of which makes it possible to determine the charge density distribution of conducting parts and potential distribution in the surrounding space . In general, IEM is directly conducive to a mathematical model described by various classes of integral equations or, indirectly, to a boundary-initial problem of some differential equations. The method is applied to the theory of heat conduction and to the theory of diffusion. It is also used in electromagnetism, for example in determining selected electrodynamical parameters in three-phase systems of shielded heavy-current busways . The choice of the method in such a case is justified, as the distribution of current density in phase conductors is obtained by solving a system of integral equations. The current density distribution of a single live conductor described in [5-8] was found with the use of geometry of the system. The kernel of the equation was formulated with the account of skin effect of phase conductors, their approaching, and inducing of eddie currents in the shield. By such an approach, solutions were obtained that could be used to analyze electromagnetic phenomena occurring in current busways and the neighboring space, confining the considerations to a particular part of the field under investigation, i.e. to the surface area of one of the live conductors.

The aim of this paper is to present the advantages of the method of integral equations (IEM) and the possibilities of its application to various branches of engineering, particularly to the problems arising in power engineering [4]. It is an analytical-numerical method and requires great efforts from highly skilled specialists (mathematicians, computer scientists, engineers). IEM seems to be a natural method, especially in the field of electrodynamics. This is a case of electromagnetic field being described with integral equations, the kernels of which are searched for by integral transformations in the domain of space variables, while in the time domain the expected system response has a form of integral

formulas. It confines the expected solution valid for the whole domain to a predefined part of the space subject to analysis. This enables a considerable reduction of the size of the system of equations. In such a case, the minimization of the size of the systems of equations and the reduction of computation time, while maintaining the accuracy, becomes highly important.
Integral equations, or rather their systems, are often matched with mathematical models describing the current density distribution at the cross-section of a working conductor or in the cartridge of an induction heater. The knowledge of the current density distribution may be a ebasis for determining some electrodynamic values such as magnetic induction or distribution of electrodynamic forces acting at selected points of the conductors. Moreover, some problems of the radiative energy transferare reducible to a system of Fredholm integral equations.
We restrict to the following mathematical models in electrical engineering: radiative heat transfer, current density problems and Fourier's problems. Presented models lead to a system of Fredholm integral equations, integro-differential equations and Volterra-Fredholm integral equations, respectively. We propose various computational methods (discretization method and projection methods) providing to a system algebraic equations.

2. Integral modeling in the current density theory

In [5,7,8] a current density in the conductor with rectangular cross-section provides to Fredholm integral equations of the second kind, which were solved by discretization method. Presented numerical method is reduced to a system of linear algebraic equations, can be calculated by various methods (iterative method, Gauss method, Croutte'a method and other methods). The above current density problem was treated as stationary. In [4,5] was considered nonstationary problem leading to a system of integro-differential equations, which by discretization in time and space provides also to linear algebraic equations. In this paper we generalize these results. Moreover, we give other propositions to solve Fredholm integral equations and integro-differential equations [2].
We will generalize results of papers [4-7,10]. The nonmagnetic cylinders of arbitrary cross-section S_k (k=1,2,...,m, where m - number of conductors) are situated parallel to z-axis of the rectangular coordinate systems. The currents $i_1(t)$, $i_2(t)$,..,$i_m(t)$ flow through the conductors. The magnetic vector potential $\boldsymbol{A}$ generated by these currents is parallel to the z-axis and independent of z. The conducting cylinders are considered to be infinitely long. Thus the problem is two dimensional.

$$\mathbf{J} = \mathbf{1}_z J(x,y,t) \qquad (1)$$

$$\mathbf{A} = \mathbf{1}_z A(x,y,t) \qquad (2)$$

Presented problem provides to a system of integro-differential equations (see [5]) and is a generalization of papers [4-7,9] for few conductors in time.

Example 1

As an example we consider the two conductors of a=b=10 mm, d=2,5 mm, and of the conductivity γ=56 MS/m. The conductors carrying the currents $i_1(t)$=-$i_2(t)$=100*t A. the number of rectangular subsections ΔS for each conductors was equal to 625 (N_x=N_y=25).

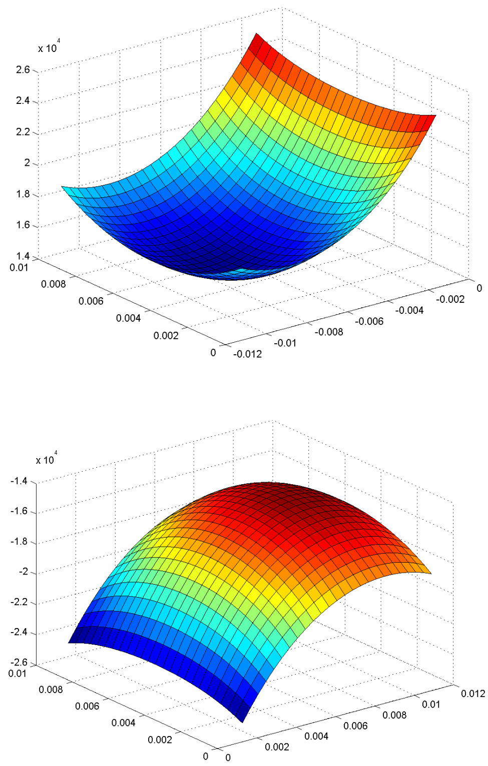

Fig. 1. Distribution of the current density within conductors for t=18 ms

Remark 1. Current density problems lead to a system of Fredholm integral equations [6,7] or respecting time to integro-differential equations [5], which using the discretization can be solved by a system of algebraic equations.

3. Fredholm integral equations in the exchange of the radiosity

The paper [1] presents theoretical foundations for the modelling of phenomena related to visualisation performed by means of computer graphics software and for the modelling of radiative heat transfer. Since the equations describing both of these processes are very similar, there is a possibility of applying certain computer graphics programmes to resolve problems related to radiative heat transfer. It is explores all necessary supplements making it possible to perform such calculations. Thermokinetics, describing radiative heat transfer; lighting engineering, investigating problems of determination of surface illumination and computer-generated graphics, resolving issues connected with visualisation (it is the creation of seemingly three-dimensional representations of virtual reality on a two-dimensional screen, based on mathematical descriptions) – they all examine, to a greater or smaller extent, the same phenomena of emission, transmission and absorption of optical radiation energy. The similarity of phenomena occurring in all of these cases additionally offers the possibility to use similar research tools to investigate them. Only the simplest tasks involving radiative heat transfer or lighting engineering can be solved in [1,9] using analytical methods. Practically, all more demanding problems in these fields are currently solved using numerical methods or by means of modelling and simulation: (see [1,9]). It thus seems interesting to adapt such sophisticated computer graphics software to solve very complex problems involved in radiative heat transfer.

Contemporary advanced computer graphics software and interior visualisation applications are based on the visualisation equation given below:

$$L\left(x^0,x^1\right)=g\left(x^0,x^1\right)\cdot\left[L_e\left(x^0,x^1\right)+\int_\Omega \rho\left(x^0,x^2,x^1\right)L\left(x^2,x^1\right)dx^2\right] \quad (3)$$

where $L(x^0,x^1)$ is luminance of point x^0: the total of luminance of radiation emitted $L_e(x^0, x^1)$ and reflected (integral value) in the direction of point x^1; $g(x^0, x^1)$ - factor dependent on the geometry of the system, defining the "visibility" of point x^1 from x^0; $\rho(x^0,x^0,x^1)$ – specular reflectance of radiation for point x^0, with radiation propagating from the direction of point x^2 and reflected in the direction of x^1. Integration is performed along the whole hemisphere Ω surrounding x^0. This is illustrated by figure 1.

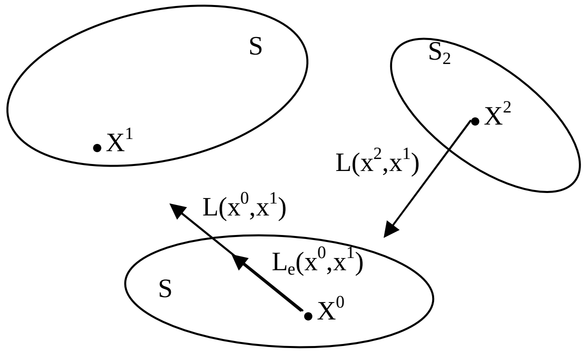

Fig. 2. Illustration of equation (3).

The eqn (3) was written in the terminology used in lighting engineering and computer graphics, where the concept of luminance L_v [lm/m^2/sr] is used, referring to visible radiation. Thermokinetics, however, uses the concept of radiance L [W/m^2/sr] referring to all optical radiation (including thermal radiation).

The solution of eqn. (3) for every point of surfaces $S_0...S_n$ under consideration consists of determination of luminance of each of these points. This is the basic information, necessary for further construction of visual images of surfaces examined. Unfortunately, the eqn. (3) cannot be solved analytically. Only simulation methods can be applied. A commonly used method is backward ray tracing.

The equation describing heat balance of point x^0 in Siegel and Howell [8] has a form that is similar to (3), as given below :

$$p_{eff}(x^0,T,\theta,\phi) = p_e(x^0,T,\theta,\phi) + \int_\Omega \rho(x^0,\theta,\phi,\theta_{in},\phi_{in})p_{in}(x^2,T_j,\theta_{in},\phi_{in})\cos\theta d\omega \quad (4)$$

where, in radiative heat transfer terminology: p_{eff} stands for surface density of effective radiant intensity (radiance) of point x^0 in the direction of x^1, defined by angles (θ, ϕ); T represents temperature and the index 'in' concerns incident radiation. The eqn. (4), when only diffuse radiation is considered, is simplified to a system of linear equations, solved (when the number of points is limited) using exact methods (e.g. matrix methods) or approximate methods. When taking into account both diffuse and specular reflection, the eqn (4) is solved applying simulation methods, usually radiosity method. However, this method calls for considerable computer resources (memory capacity and the number of calculations), which is a significant limitation in the case of radiative heat transfer systems which are geometrically more complex.

Remark 2. Presented problem lead to a system of integral equations of the Fredholm type, which by the discretization method can be reduced to a system of algebraic equations[1,2,9]. Moreover, Fredholm integral equations are particular case of integral equations arise in the Fourier's theory [3,4,6].

4. Integral equations in the Fourier's theory

We consider the integral equations of the mixed type

$$u(x,t) = f(x,t) + \int_0^t \int_M k(x,t,y,s)u(y,s)dyds \qquad (5)$$

or shortly

$$u = f + Ku \qquad (6)$$

which generalize Volterra and Fredholm integral equations. The presented equations play a very important role in electromagnetics. These equations arise in the heat conduction theory. Some initial-boundary problems for a number differential partial equations in physics are reducible to the above integral equation. Consider this equation in space-time, where g is a given function in the domain $D = M \times [0,T]$ (M is a compact subset of *m*-dimensional Euclidean space) and u is anunknown function in *D*. The kernel *k* is defined in the domain $\Omega = \{(x,t,y,s) : x, y \in M, 0 \le s \le t \le T\}$.

3.1. Galerkin method

Classical Galerkin method for integral equation (5) leads to approximate solution of the form

$$u_n(x,t) = \sum_{j=1}^{n} c_j \chi_j(x,t),$$

where $\{\chi_j\}$ is the orthogonal basis in the space $L^2(D)$. Because it is difficult to define such a system we propose the following formula

$$u_n(x,t) = \sum_{i,k=1}^{n} c_{ik}\phi_i(x)\psi_k(t), \qquad (7)$$

where $\{\varphi_i\},\{\psi_k\}$ are orthogonal bases in spaces $L^2(D)$ and $L^2[0,T]$, respectively.

Coefficients c_{ik} *(i,k=1,2,...,n)* are determined by the orthogonality condition in $L^2(D)$ of the form

$$(\varepsilon_n, \phi_i\psi_k) = 0 \ \ (i,k = 1,2,\ldots,n),$$

where

$$\varepsilon_n = u_n - f - Ku_n,$$

is a deviation function.

In practice, we restrict our considerations to the orthonormal bases. Then we get the following system of linear algebraic equations

$$c_{ik} = f_{ik} + \sum_{j,m=1}^{n} c_{jm} k_{jm,ik} \quad (i,k = 1,2,\ldots,n) \qquad (8)$$

where

$$f_{ik} = \int_0^T \int_M f(x,t)\varphi_i(x)\psi_k(t)dxdt,$$

$$k_{jm,ik} = \int_0^T \int_M \left[\int_0^t \int_M k(x,t,y,s)\varphi_j(y)\psi_m(s)dyds \right] \varphi_i(x)\psi_k(t)dxdt \cdot$$

Theorem 1

Let $\{\varphi_i\},\{\psi_k\}$ to be orthonormal complete systems in the spaces $L^2(M)$ and $L^2[0,T]$, respectively. If $f \in L^2(D)$ and $k \in L^2(\Omega)$, then system (8) is uniquely solvable and the sequence defined by formula (7) converges to unique solution of equation (5) in the space $L^2(D)$.

The proof is similar as in the case of the Fredholm integral equation and it is based on the Fourier series theory.

The presented method we illustrate in the following integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_{-1}^1 k(x,t,y,s)u(y,s)dyds, \ \ t \in [0,1] \qquad (9)$$

using in (3.3) as a basis the orthonormalized Legendre polynomials.

Example 2

Consider equation (9) with $k(x,y,t,s) = x^2t^2e^s$ and $f(x,t) = x^2\left(e^{-t} - \frac{2}{3}t^3\right)$.

x \ t	0.0	0.2	0.4	0.6	0.8	1.0
-1.0	.9619e-	.2089e-	-1,0	.1273e-	.3487e-	.2317e-
-0.6	.9625e-	.2093e-	-0,6	.1270e-	.3480e-	.2315e-
-0.2	.9685e-	.2146e-	-0,2	.1266e-	.3432e-	.2300e-
0.2	.9685e-	.2146e-	0,2	.1266e-	.3432e-	.2300e-
0.6	.9625e-	.2093e-	0,6	.1270e-	.3480e-	.2315e-
1.0	.9619e-	.2089e-	1,0	.1273e-	.3487e-	.2317e-

Table 1. The relative errors for n=6

n	4	5	6	7
E	.337e-3	.163e-4	.665e-6	.328e-7

Table 2. Dependence of average relative errors of a number of basic functions.

Example 3

In equation (9) we take $k(x,y,t,s)=e^{-y}x^2$, $f(x,t)=e^x t^2-\frac{2}{3}x^2t^3$

n	4	5	6	7
E	.506e-2	.549e-3	.615e-5	.344e-8

Table 3**. A**verage relative errors

3.2. The modern Galerkin method

In this section we propose a projection method for equation (5) leading to solve a system of Volterra linear integral equations. Approximate solution of (5) we seak in the form

$$u_n(x,t)=\sum_{j=1}^{n}a_j(t)\varphi_j(x) \qquad (10)$$

for $(x,t)\in D$, $D=M\times[0,T]$, where:

- $\{\varphi_j\}$ is an orthonormal and complete basis in w $L^2(M)$;
- $\{a_j\}$ is a solution to system of the following Volterra integral equations

$$a_j(t)=f_j(t)+\sum_{k=1}^{n}\int_0^t k_{jk}(t,s)a_k(s)ds\,, \qquad (11)$$

$$k,\,j=1,2,\dots,n,$$

where f_j and k_{jk} are Euler-Fourier coefficients respect to orthonormal system $\{\varphi_j\}$ for functions *f* and *k*, respectively.

Theorem 2

If $f\in L^2(D)$ and $k\in L^2(\Omega)$, where $\Omega=\{(x,t,y,s):0\le s\le t\le T;\,x,y\in M\}$, then function (10) is a unique solution in the space $L^2(D)$ of the equation

$$u_n(x,t)=f_n(x,t)+\int_0^t\int_M k_n(x,t,y,s)u_n(y,s)dyds\,, \qquad (12)$$

with

$$f_n(x,t)=\sum_{k=1}^{n}f_k(t)\varphi_k(x),$$

$$k_n(x,t,y,s)=\sum_{j=1}^{n}\sum_{k=1}^{n}k_{jk}(t,s)\varphi_j(x)\varphi_k(y)\,. \qquad (13)$$

Proof. Putting (12) and (13) to (11) and using linear independence of system $\{\varphi_k\}$ we get

$$u_n(x,t)=\sum_{k=1}^{n}u_k(t)\varphi_k(x)\,, \qquad (14)$$

where

$$u_k(t)=f_k(t)+\sum_{j=1}^{n}\int_0^t\int_M k_{jk}(t,s)\varphi_j(y)u_n(y,s)dyds \qquad (15)$$

$$k=1,2,\dots,n\,.$$

By orthonormality of $\{\varphi_k\}$) we obtain the Volterra a system of integral equations

$$u_k(t)=f_k(t)+\sum_{j=1}^{n}\int_0^t k_{jk}(t,s)u_j(s)ds\,,\ k=1,2,\dots,n\,. \qquad (16)$$

From assumptions and Volterra theory it follows, this system has unique solution $\{u_k\}$ in space $L^2[0,T]$ such that $u_k(t)=a_k(t)$ for every $k=1,2,\dots,n$..

Equations (11) we can rewrite in the operator form

$$u_n=f_n+K_nu_n\,, \qquad (17)$$

where K_n is Volterra-Fredholm integral operator of form (6) determined by kernel k_n (13).

Theorem 3

If $f\in L^2(D)$ and $k\in L^2(\Omega)$, then sequence $\{u_n\}$ defined by formula (10) converges in the space $L^2(D)$ to unique solution of equation (5) and the estimate error

$$\|u_n-u\|_{L^2(D)}\le\frac{c}{1-c\delta_n}\left[\|f-f_n\|_{L^2(D)}+\|u\|_{L^2(D)}\delta_n\right]$$

holds with

$$c=\left\|(I-K)^{-1}\right\| \quad\text{and}\quad \delta_n=\|k_n-k\|_{L^2(\Omega)}\,.$$

Proof. Subtracting (17) and (6) we get

$$u_n-u=f_n-f+(K_n-K)u_n+K(u_n-u).$$

Hence

$$\|u_n-u\|_{L^2(D)}\le\left\|(I-K)^{-1}\right\|\left[\|f_n-f\|_{L^2(D)}+\|k_n-k\|_{L^2(\Omega)}\|u_n\|_{L^2(D)}\right],$$

From [2] we obtain the following estimate

$$c=\left\|(I-K)^{-1}\right\|\le\sum_{j=0}^{\infty}\frac{\|k\|^j_{L^2(\Omega)}}{\sqrt{j!}}\,,$$

where

$$\|k\|_{L^2(\Omega)}=\left\{\int_0^T\int_M\left[\int_0^t\int_M k^2(x,t,y,s)dyds\right]dxdt\right\}^{\frac{1}{2}}.$$

Using the theory of Fourier series and properties Lebesgue'a integrals we have

$$F_n^2(t) = \| f_n(\cdot,t) - f(\cdot,t)\|^2_{L^2(M)} \to 0 \text{ for every } t \in [0,T]$$

and

$$\|F_n\|_{L^2[0,T]} = \left[\int_0^T \| f_n(\cdot,t) - f(\cdot,t)\|^2_{L^2(M)} dt\right]^{\frac{1}{2}} = \| f_n - f\|_{L^2(D)} \xrightarrow{n\to\infty} 0 .$$

Similarly

$$\|k_n(\cdot,t,\cdot,s) - k(\cdot,t,\cdot,s)\|^2_{L^2(M)\times L^2(M)} \to 0,$$
$$\text{for every } 0 \le s \le t \le T$$

and

$$\delta_n = \|k_n - k\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0 .$$

Then we get

$$\|u_n - u\|_{L^2(D)} \xrightarrow{n\to\infty} 0 .$$

To obtain an error estimate let us notice

$$\|u_n\|_{L^2(D)} \le \|u_n - u\|_{L^2(D)} + \|u\|_{L^2(D)},$$

where

$u = (I-K)^{-1} f$ is a unique solution of equation (5).

By the above considerations we have

$$\|u_n - u\|_{L^2(D)} \le c\left[\| f_n - f\|_{L^2(D)} + \delta_n \|u_n\|_{L^2(D)}\right] \le$$
$$\le c\| f_n - f\|_{L^2(D)} + c\delta_n \|u_n - u\|_{L^2(D)} + c\delta_n \|u\|_{L^2(D)} .$$

From here we get estimate.

Presented theory is illustrated by Legendre'a polynomials $\{P_j\}$ in the orthonormalized form

$$\phi_j(x) = \frac{P_j(x)}{\|P_j\|_{L^2(-1,1)}} .$$

($\{\phi_j\}$ is complete system in $L^2(-1,1)$). Functions $a_j\ (j = 1,2,...,n)$ being a solution to system of Volterra integral equations (11) are calculated by ng the Newton-Cotes quadrature.

Example 5

Aproximate solution of the integral equation

$$u(x,t) = e^{xt} - 2x^2 t + \int_0^t \int_{-1}^1 x^2 e^{-sy} u(y,s)\,dyds$$

is proposed by the presented method for *n=4 h=0,1*

x/ t	0,2	0,4	0,6	0,8	1,0
-1,0	0,2833e-4	0,0035	0,00214	0,0820	0,02435
-0,6	0,1020e-4	0,00012	0,00017	0,0255	0,00707
-0,2	0,1131e-5	0,5804e-4	0,00029	0,00096	0,00240
0,2	0,5585e-5	0,5567e-4	0,00028	0,00088	0,00216
0,6	0,3000e-4	0,8005e-4	0,00036	0,00104	0,00231
1,0	0,6409e-5	0,00017	0,00073	0,00197	0,00411

4. Table of relative errors

Remark 3. By comparison examples 2 and 3 we get better results for the modern Galerkin method. The best results we obtain for *n=3* or *n=4*.

4. Conclusion

In this paper we restrict to the following mathematical models in electrical engineering: radiative heat transfer, current density problems and Fourier's problems. Presented models lead to a system of Fredholm integral equations, integro-differential equations and Volterra-Fredholm integral equations, respectively. We propose numerical methods (discretization method and projection methods) providing to a system algebraic equations.

References:

[1] Jaluria Y., Torrance K.E.: *Computational heat transfer*. Hemisphere Pub. Co.: Washington, 1986.

[2] Hącia L., Domke K., *Fredholm integral equations In the radiative heat transfer problems*, Academic Journals – Electrical Engineering 52, 2006, 115-123.

[3] Hącia L., Computational methods for Volterra-Fredholm integral equations, *Computational Methods in Science and Technology*, Vol 8, No 2, 2002, 13-26,.

[4] Hącia, L. Some integral mathematical models in electrotechnics, *Computer Applications in Electrical Engineering-Monograph* 1(2004),106-123, WPP, Poznań

[5] Hącia L., Patecki A., *Analysis of the current density in the conductor by integro- differential equations*, VIII Conference on Applications of Computers in Electrotechnics, Poznań/Kiekrz 2003, 159-162.

[6] Hącia L., Nawrowski R., *Numerical methods for integral equations in the heat conduction theory of electrotechnics*, Computer Applications in Electrical Engeenering, Monograph 2002, 60-78.

[7] Nawrowski R., Pabian J., *Obliczanie rozkładu gęstości prądu w przewodach metodą równań całkowych Fredholma*, VII Konferencja Naukowo-Techniczna – Zastosowanie Komputerów w Elektrotechnice", Poznań/Kiekrz 2002, 183-186.

[8] Nawrowski R., Pabian J., Tomczewski A., *The use of object-oriented software for computation of current density distrubution,* Academic Journals – Electrical Engineering 52, 2006, 131-144.

[9] Siegel. H.,Howell J.R.: *Thermal radiation heat Transfer*, Mc-Graw Hill Book Co. New York, 1972.

[10] Szymański G., *Zastosowanie metod całkowych w elektrodynamice technicznej*, WPP, Rozprawy nr 144, Poznań 1983.