On propagation of discontinuity waves in thermo-piezoelectric bodies

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Abstract: With regard to a body composed of a linear thermo-piezoelectric medium, referred to a natural configuration, we consider processes for it constituted by small displacements, thermal deviations and small electric fields superposed to the natural state. We show that any discontinuity surface of order $r \geq 1$ for the above processes is characteristic for the linear thermo-piezoelectric partial differential equations. We show that discontinuity surfaces of order 0 generally are not characteristic; hence the conditions are written which characterize the discontinuity surfaces of order 0 that are characteristic.

Key–Words: Thermodynamics, Piezoelectricity, Discontinuity waves, Characteristic surfaces, Strong waves, Weak waves.

1 Introduction

We consider a solid body $\mathcal{B}$ which is composed of a linear thermo-piezoelectric medium, that is, a non-magnetizable linearly elastic dielectric medium that is heat conducting and not electric conducting.

We assume that $\mathcal{B}$ has a natural configuration, say a placement $\kappa[\mathcal{B}]$ that $\mathcal{B}$ can occupy with zero stress, uniform temperature θ_0 and uniform electric field. Such natural configuration will be used as reference.

We consider processes of $\mathcal{B}$ constituted by small displacements, thermal deviations and small electric fields

$$(\mathbf{u},\, T,\, \mathbf{E}\,)$$

superposed to $\kappa[\mathcal{B}]$; we adopt the linearized theory for thermo-piezoelectricity which is developed in [1], [2]; such a general framing contains many particular theories; for example, the theory in [3] is a particular case of it.

A smooth singular surface (or discontinuity surface) of order r in the triple of fields $(\mathbf{u},\, T,\, \mathbf{E}\,)$ is referred to as a weak (thermo-piezoelectric) wave if $r \geq 2$ and a strong (thermo-piezoelectric) wave if $r = 0$ or 1.

Here we show that (i) any singular surface of order $r \geq 1$ is characteristic (for the linear thermo-piezoelectric partial differential equations); moreover, (ii) singular surfaces of order 0 generally fail to be characteristic.

Hence strong waves of order $r = 1$ and all weak waves of any given order $r \geq 2$ have the same propagation conditions.

Such results generalize to piezoelectric heat-conducting bodies the results of [3] that hold for not heat-conducting piezoelectric bodies.

2 Linear thermo-piezoelectricity

2.1 Constitutive equations

We assume that the body $\mathcal{B}$ occupies the region $\mathbf{B} = \kappa[\mathcal{B}]$, which is the closure of a regular, open and connected subset of the three-dimensional Euclidean space. A unique system of coordinates $(x_1,\, x_2,\, x_3)$ for both the reference configuration and the ambient space will be used, so that the notations of [1], [2] can be adopted by unifying the symbolism used there for the material and spatial descriptions.

Hence, the following terminology is adopted here.

- $\mathbf{t}$ mechanical Cauchy stress tensor
- $\mathbf{E}$ electric vector
- ϕ electrostatic potential
- T incremental absolute temperature
- $\mathbf{D}$ electric displacement vector.

The linear constitutive equations are specified in terms of the constitutive quantities listed below.

σ_{klij} = elastic moduli

e_{ikl} = piezoelectric moduli
β_{kl} = thermal stress moduli
κ^E_{kl} = dielectric susceptibility
$\tilde{\omega}_k$ = pyroelectric polarizability
ϵ_{kl} = permittivity moduli
κ_{kl} = Fourier coefficients
γ = heat capacity
η_o = entropy at the natural state
T_o = absolute temperature at the natural state
ρ_o = mass-density at the natural state

We assume the following constitutive equations respectively for the Cauchy stress, electric displacement vector, heat flux vector and specific entropy:

$$t_{kl} = \sigma_{klij}\, u_{i,j} - e_{ikl}\, E_i - \beta_{kl}\, T \quad (1)$$
$$D_k = e_{kij}\, u_{i,j} + \epsilon_{ki}\, E_i + \tilde{\omega}_k\, T \quad (2)$$
$$\rho_o\, \theta\, \dot{\eta} - q_{k,k} = \rho_o\, h \quad (3)$$
$$q_k = \kappa_{kl}\, T_{,l} + \kappa^E_{kl}\, E_l \quad (4)$$
$$\eta = \eta_0 + \frac{\gamma}{T_0}\, T + \frac{1}{\rho_o}\,(\beta_{ij}\, u_{i,j} + \tilde{\omega}_i\, E_i) \quad (5)$$

where $E_i = -\phi_{,i}$ and the following symmetries hold:

$$\sigma_{klij} = \sigma_{ijkl} = \sigma_{lkij} = \sigma_{klji} \quad (6)$$
$$e_{kij} = e_{kji}\,, \qquad \beta_{ij} = \beta_{ji} \quad (7)$$
$$\kappa_{kl} = \kappa_{lk}\,, \qquad \kappa^E_{kl} = \kappa^E_{lk} \quad (8)$$

2.2 Balance laws

The field equations corresponding to the (i) balance law of linear momentum, (ii) Maxwell's equation, and (iii) balance law of conservation of energy, write as

$$t_{kl,k} + \rho_o(f_l - \ddot{u}_l) = 0 \quad (9)$$
$$D_{k,k} = q_e \quad (10)$$
$$\rho_o\, \theta\, \dot{\eta} - q_{k,k} = \rho_o\, h \quad (11)$$

where

- f_l is the body force density
- q_e is the free (or prescribed) body charge density
- h is the heat source per unit mass.

2.3 Field equations

The linearized field equations, which are obtained by replacing the constitutive equations in the balance laws and neglecting the higher order terms, in the homogeneous case write as

$$\sigma_{klij} u_{i,jk} + e_{ijl}\, \phi_{,ij} - \beta_{kl}\, T_{,k} = \rho_o(\ddot{u}_l - f_l) \quad (12)$$
$$e_{kji}\, u_{j,ik} - \epsilon_{kj}\, \phi_{,jk} + \tilde{\omega}_k\, T_{,k} = q_e \quad (13)$$
$$-\kappa_{kj}\, T_{,jk} + \kappa^E_{jk}\, \phi_{,jk} + + T_0\, \beta_{kj}\, \dot{u}_{k,j} + \rho_o\, \gamma\, \dot{T} - T_0\, \tilde{\omega}_k\, \dot{\phi}_{,k} = \rho_o\, h \quad (14)$$

Instead in the inhomogeneous case the linearized field equations write as

$$\sigma_{klij}\, u_{i,jk} + \sigma_{klij,k}\, u_{i,j} + e_{ijl}\, \phi_{,ij} + e_{ijl,j}\, \phi_{,i} + -\beta_{kl}\, T_{,k} - \beta_{kl,k}\, T = \rho_o(\ddot{u}_l - f_l)\,, \quad (15)$$
$$e_{kji}\, u_{j,ik} - \epsilon_{kj}\, \phi_{,jk} + \tilde{\omega}_k\, T_{,k} = q_e\,, \quad (16)$$
$$-\kappa_{kj} T_{,jk} + \kappa^E_{jk} \phi_{,jk} + + T_0\, \beta_{kj} \dot{u}_{k,j} + \rho_o \gamma \dot{T} - T_0 \tilde{\omega}_k \dot{\phi}_{,k} = \rho_o h\,. \quad (17)$$

We note that in both cases the field equations can be put in the form

$$\sigma_{klij}\, u_{i,jk} + e_{ijl}\, \phi_{,ij} - \rho_o\, \ddot{u}_l = \Sigma_l - \rho_o f_l \quad (18)$$
$$e_{kji} u_{j,ik} - \epsilon_{kj} \phi_{,jk} = \Sigma_4 + q_e \quad (19)$$
$$-\kappa_{kj} T_{,jk} + \kappa^E_{jk} \phi_{,jk} + T_0 \beta_{kj} \dot{u}_{k,j} - -T_0\, \tilde{\omega}_k \dot{\phi}_{,k} = \Sigma_5 + \rho_o h \quad (20)$$

where Σ_1 through Σ_5 represent sums of external sources with terms involving only first derivatives of the dependent variables and of the material functions.

3 Characteristic hypersurfaces of the linear thermo-piezoelectric equations

Consider a linear differential operator, in Schwartz notation,

$$L(\mathbf{y}, D)u = \sum_{|\alpha| \le m} A_\alpha(\mathbf{y}) D^\alpha u\,, \quad (21)$$

where

$$\mathbf{y} = (x_1, x_2, x_3, t) \in I\!R^4,\ u : I\!R^4 \to I\!R,\ u = u(\mathbf{y}),$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \partial t^{\alpha_4}},$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4),\ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$

the α_i being non-negative integers.

The same formula describes the general mth-order system of N differential equations in N unknowns if we interpret u as column vector with N components and the A_α as $N \times N$ square matrices.

A *characteristic manifold* of the linear differential equation (21) is a surface in $I\!R^4$ which is exceptional for the assignment of data in the appropriate Cauchy initial value problem.

More in detail, the (generalized) *Cauchy Problem* consists of finding a solution u of

$$L(\mathbf{y},\, D)u = \sum_{|\alpha|\leq m} A_\alpha(\mathbf{y}) D^\alpha u = 0 \qquad (22)$$

having prescribed *Cauchy data* on a hypersurface $S \subset I\!R^4$ given by $\Psi(\mathbf{y}) = 0$, where one assumes that Ψ has m continuous derivatives and the surface is regular in the sense that

$$D\Psi = (\Psi_{x_1},\, \Psi_{x_2},\, \Psi_{x_3},\, \Psi_t) \neq 0\,.$$

The *Cauchy data* on S for an mth-order equation consist of the derivatives of u of order less than or equal $m-1$. They cannot be given arbitrarily but have to satisfy the compatibility conditions valid on S for all functions regular near S (instead normal derivatives of order less than m can be given independently from each other).

We call S *noncharacteristic* if we can get all $D^\alpha u$ for $|\alpha| = m$ on S from the linear algebraic system of equations consisting of the compatibility conditions for the data and the partial differential equation (or system of equations) (22) taken on S.

We call S *characteristic* if at each point $\mathbf{y} \in S$ the surface S is not noncharacteristic.

The principal part $L^{(pr)}$ of L is defined as the operator consisting of the highest order terms of L:

$$L^{(pr)} \;=\; \sum_{|\alpha|=m} A_\alpha D^\alpha\,. \qquad (23)$$

It can be expressed in matrix form by putting

$$\Lambda(\xi) \;=\; \sum_{|\alpha|=m} A_\alpha\, \xi^\alpha\,, \quad \xi \in I\!R^4\,. \qquad (24)$$

If (21) represents an mth-order system of N differential equations in N unknowns, hence u is a column vector with N components and the A_α are $N \times N$ square matrices, then a surface S of equation $\Psi = \Psi(x_1,\, x_2,\, x_3,\, t)$ is characteristic for (21) if

$$det\Big[\Lambda(\nabla\Psi)\Big] \;=\; 0\,. \qquad (25)$$

If the surface S has equation

$$\Psi(x_1,\, x_2,\, x_3,\, t) = 0\,, \qquad (26)$$

then putting

$$n_i = |\nabla\Psi|^{-1}\frac{\partial\Psi}{\partial x_i}\,, \quad V = -|\nabla\Psi|^{-1}\frac{\partial\Psi}{\partial t}\,, \qquad (27)$$

equation (25) becomes

$$det\Big[\Lambda(n_1,\, n_2,\, n_3,\, -V)\Big] \;=\; 0\,. \qquad (28)$$

This is called *characteristic equation* of the system of partial differential equation (22).

3.1 Characteristic equation of the thermo-piezoelectric field equations

Now let we identify the system of equations (22) with the linear thermo-piezoelectric equations (briefly, l.t.p.e.) (12)-(14), or (15)-(17) in the inhomogeneous case, thus we have $m = 2$, $N = 5$, and the characteristic equation (28) becomes the vanishing of the determinant of the coefficients of the system of five equations

$$(\sigma_{klij}n_j n_k - \rho_o V^2\delta_{li})\lambda_i + e_{ilj}n_i n_j\varphi = 0 \qquad (29)$$

$$e_{kji}n_i n_k\,\lambda_j - \epsilon_{kj}n_j n_k\varphi = 0 \qquad (30)$$

$$-\kappa_{kj}n_j n_k\,\tau + T_0 V\beta_{ij}n_j\lambda_i + +(\kappa^E_{jk}n_j n_k - T_0 n_k\,\tilde\omega_k V)\varphi = 0 \qquad (31)$$

in the five scalar unknowns $\tau,\, \lambda_i,\, \varphi$.

Now, putting

$$A_{li} = \sigma_{klij}n_j n_k - \rho_o V^2\delta_{li},\; B_l = e_{ilj}n_i n_j$$
$$D = n_k\epsilon_{kj}n_j,\; E = -n_k\kappa_{kj}n_j\,, \qquad (32)$$
$$F_l = T_0 V\beta_{lj}n_j,\; G = \kappa^E_{jk}n_j n_k - T_0 n_k\tilde\omega_k V\,,$$

the 5×5 system (29)-(31) in the variables $(\,\tau,\, \lambda_i,\, \varphi\,)$ writes as

$$A_{li}\,\lambda_i + B_l\,\varphi = 0 \qquad (33)$$

$$B_i\,\lambda_i - D\,\varphi = 0 \qquad (34)$$

$$E\tau + F_i\,\lambda_i + G\,\varphi = 0\,. \qquad (35)$$

By the substitution

$$\varphi \;=\; D^{-1}B_i\,\lambda_i \qquad (36)$$

the system (33) becomes

$$(A_{li} + B_l D^{-1}B_i)\,\lambda_i = 0 \qquad (37)$$

$$D^{-1}B_i\,\lambda_i - \varphi = 0 \qquad (38)$$

$$E\tau + (F_i + GD^{-1}B_i)\lambda_i = 0 \qquad (39)$$

whose matrix $\mathcal{M}$ is

$$\begin{array}{ccccc} \tau & \lambda_1 & \lambda_2 & \lambda_3 & \varphi \end{array}$$
$$\begin{pmatrix} 0 & H_{11} & H_{12} & H_{13} & 0 \\ 0 & H_{21} & H_{22} & H_{23} & 0 \\ 0 & H_{31} & H_{32} & H_{33} & 0 \\ 0 & L_1 & L_2 & L_3 & -1 \\ E & M_1 & M_2 & M_3 & 0 \end{pmatrix}$$

where

- $H_{ij} \;=\; A_{ij} + D^{-1}B_i B_j \quad (i,\, j \;=\; 1,\, 2,\, 3)$
- $L_i \;=\; D^{-1}B_i \quad (i, \;=\; 1,\, 2,\, 3)$
- $M_i \;=\; F_i + GD^{-1}B_i \quad (i, \;=\; 1,\, 2,\, 3)$.

Hence, the characteristic equation for the l.t.p.e.s is

$$det\mathcal{M} = E \times det[H_{ij}] = 0 \qquad (40)$$

Since $E \neq 0$, such characteristic equation coincides with the characteristic equation for the partial differential equations of a not heat conducting piezoelectric medium - cf. [3].

In the latter case the characteristic equation reduces to the vanishing of the determinant of the coefficients of the system of four equations

$$(\sigma_{klij}n_jn_k - \rho_o V^2 \delta_{li})\lambda_i + e_{ilj}n_in_j\,\varphi = 0 \qquad (41)$$

$$e_{kji}n_in_k\,\lambda_j - \epsilon_{kj}n_jn_k\,\varphi = 0 \qquad (42)$$

in the four scalar unknowns $\lambda_i,\, \varphi$ – cf. [3]. That is,

$$det[H_{ij}] = 0 \qquad (43)$$

4 Compatibility conditions for jumps of partial derivatives

In this section we follow the treatment of the subject given in [5]. Let E^3 denote the three-dimensional Euclidean ambient space, $I = [t_o,\, t_1]$ a time interval and $\mathcal{E} = I \times E^3$. We consider a smooth hypersurface $\mathcal{S}$ in $\mathcal{E}$ which admits a suitably regular representation

$$x_i = \psi_i(t,\, \xi_1,\, \xi_2), \quad i = 1,\, 2,\, 3\,, \qquad (44)$$

with the parameter pair belonging to an open subset of $I\!R^2$. For any value of t equation (44) defines a surface $\mathcal{S}_t$ in E^3, referred to the curvilinear coordinates $\xi_1,\, \xi_2$. The totality of surfaces $\mathcal{S}_t$ for $t \in I$ is a moving surface in E^3. Thus $\mathcal{S}$ can be interpreted as both the hypersurface of $\mathcal{E}$ of equations (44) and the associated moving surface in E^3.

The comma notation $f_{,\alpha}$ is used to denote covariant derivative in the ξ coordinate system. For all $t \in I$, at each point of $\mathcal{S}_t$, there is a unit normal $\mathbf{n}$ whose x components are denoted by n_i.

The ξ components of the metric tensor on $\mathcal{S}_t$ are denoted by

$$g_{\alpha\beta} = \psi_{i,\alpha}\psi_{i,\beta}\,. \qquad (45)$$

The *speed* $\mathbf{V}$ of the surface $\mathcal{S}$ at time t has x components

$$V_i = \frac{\partial \psi_i}{\partial t} \qquad (46)$$

and the *speed* of $\mathcal{S}$ in direction of $\mathbf{n}$ is

$$V = V_i\, n_i\,. \qquad (47)$$

Now let $f : \mathcal{N} \to I\!R$ be a real scalar-valued function and let $\mathcal{N} = I \times N$ with N open subset of E^3 having, for all $t \in I$, non-empty intersection with $\mathcal{S}_t$. Since the results below refer only to the part of $\mathcal{S}$ contained in $\mathcal{N}$, we replace $\mathcal{S}\cap\mathcal{N}$ by $\mathcal{S}$ and $\mathcal{S}_t\cap\mathcal{N}$ by $\mathcal{S}_t$. Let $\partial f/\partial n$ denote the derivative of f in the direction of $\mathbf{n}$ on $\mathcal{S}_t$, where n is distance measured from $\mathcal{S}_t$. Hence $\partial/\partial n \equiv n_i\,\partial/\partial x_i$.

If the hypersurface $\mathcal{S}$, with representation (44), is singular in $\mathcal{E}$ of order 0 for the real scalar-valued function $f = f(x_1,\, x_2,\, x_3,\, t)$, then the *compatibility conditions* below hold for discontinuities in the first partial derivatives of f across $\mathcal{S}$

$$\Big[\frac{\partial f}{\partial x_a}\Big] = g^{\sigma\tau}\Big[f\Big]_{,\sigma}\psi_{a,\tau} + \Big[\frac{\partial f}{\partial n}\Big]n_a\,, \qquad (48)$$

$$\Big[\frac{\partial f}{\partial t}\Big] = \frac{\delta[f]}{\delta t} - V\Big[\frac{\partial f}{\partial n}\Big]\,, \qquad (49)$$

where

$$\frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t} + V\,\frac{\partial}{\partial n}$$

denotes the $\delta-$time derivative of Thomas.

If $\mathcal{S}$ is a singular hypersurface in $\mathcal{E}$ of order 1 for the continuous function $f = f(x_1,\, x_2,\, x_3,\, t)$, then the following compatibility conditions (Hadamard [6] , pp.103-104)

$$\Big[\frac{\partial f}{\partial x_a}\Big] = \Big[\frac{\partial f}{\partial n}\Big]n_a\,,\ \Big[\frac{\partial f}{\partial t}\Big] = -V\Big[\frac{\partial f}{\partial n}\Big]\,. \qquad (50)$$

hold for the discontinuities in the first partial derivatives of f across $\mathcal{S}$.

If $\mathcal{S}$ is a singular hypersurface in $\mathcal{E}$ of order $r \geq 2$ for the function $f = f(x_1,\, x_2,\, x_3,\, t)$, then the compatibility conditions (Hadamard [6] , pp.103-104)

$$\Big[\frac{\partial^r f}{\partial x_i\partial x_j\ldots\partial x_l\partial t^{r-s}}\Big] = (-V)^{r-s}\Big[\frac{\partial^r f}{\partial n_r}\Big]n_i\,n_j\ldots n_l\,, \qquad (51)$$

hold on $\mathcal{S}$, where $0 \leq s \leq r$,

$$\frac{\partial^r f}{\partial n^r} = \frac{\partial^r f}{\partial x_p\ldots\partial x_q}n_p\ldots n_q \quad (r \text{ indexes})\,, \qquad (52)$$

and V is the local speed of propagation with respect to the medium, apply to the derivatives of f.

5 Weak waves

We assume that

(a) the material functions

$$(\,\rho_o,\, \sigma_{klij},\, e_{ijl},\, \beta_{kl},\, \epsilon_{kj},\, \tilde{\omega}_k,\, \kappa_{kj},\, \kappa^E_{jk},\, \gamma\,)$$

are of class C^r and the external fields $\mathbf{f}$, q_e and h are of class C^{r-2}, where r is any given integer ≥ 2.

The *l.p.d.e.*s (12)-(14) and (15)-(17) are of second order; thus the adjective *weak* is applied to singular hypersurfaces $\mathcal{S} \subset \mathcal{E} := I \times I\!R^3$ of the dependent variables $(u_i,\, \phi,\, T\,)$ of order $r \geq 2$.

Proposition 1 *Assume* (a). *Then weak thermo-piezoelectric singular hypersurfaces are characteristic for the l.p.d.e.s (15)-(17).*

PROOF. Let $\mathcal{S}$ be a weak wave; then, across $\mathcal{S}$ the jumps of the rth partial derivatives of $(u_i,\, \phi,\, T\,)$ are defined and the jumps of the partial derivatives of order lower than r identically vanish.

For $r > 2$ the *l.p.d.e.*s (15)-(17) hold on $\mathcal{B}' := I \times \mathcal{B}$ and for $r = 2$ they hold on $\mathcal{B}' \setminus \mathcal{S}$.

As a consequence, for all $r \geq 2$ the three equations below, which are obtained by applying to (15)-(17) the differential operator

$$\frac{\partial^{r-2}}{\partial x_a \ldots x_c} \quad (r-2 \quad \text{summed indexes})\,, \qquad (53)$$

hold on $\mathcal{B}' \setminus \mathcal{S}$. That is, we have

$$\sigma_{klij}\frac{\partial^r u_i}{\partial x_a \ldots x_c \partial x_j \partial x_k} + e_{ijl}\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_i \partial x_j} - \rho_o \frac{\partial^r u_l}{\partial x_a \ldots x_c \partial t^2} = \frac{\partial^{r-2}(\Sigma_l - \rho_o f_l)}{\partial x_a \ldots x_c} \qquad (54)$$

$$e_{kij}\,\frac{\partial^r u_j}{\partial x_a \ldots x_c \partial x_i \partial x_k} - \epsilon_{kj}\,\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_j \partial x_k} = \frac{\partial^{r-2}(\Sigma_4 + q_e)}{\partial x_a \ldots x_c} \qquad (55)$$

$$-\kappa_{kj}\frac{\partial^r T}{\partial x_a \ldots x_c \partial x_j \partial x_k} + \kappa^E_{jk}\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_j \partial x_k} + T_0 \beta_{kj}\frac{\partial^r u_k}{\partial x_a \ldots x_c \partial x_j \partial t} - T_0 \tilde{\omega}_k \frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_k \partial t} = \frac{\partial^{r-2}(\Sigma_5 + \rho_o h)}{\partial x_a \ldots x_c} \qquad (56)$$

Now, by (c) it follows that the right-hand sides in equations (54), (55) and (56) are terms involving derivatives of order lower than r. Thus their jumps across $\mathcal{S}$ identically vanish. As a consequence, forming the jumps across $\mathcal{S}$ of the *l.p.d.e.*s (15)-(17) yields

$$\sigma_{klij}\Big[\frac{\partial^r u_i}{\partial x_a \ldots x_c \partial x_j \partial x_k}\Big] + e_{ijl}\Big[\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_i \partial x_j}\Big] = \rho_o\Big[\frac{\partial^r u_l}{\partial x_a \ldots x_c \partial t^2}\Big] \qquad (57)$$

$$e_{kij}\,\Big[\frac{\partial^r u_j}{\partial x_a \ldots x_c \partial x_i \partial x_k}\Big] = \epsilon_{kj}\,\Big[\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_j \partial x_k}\Big] \qquad (58)$$

$$-\kappa_{kj}\,\Big[\frac{\partial^r T}{\partial x_a \ldots x_c \partial x_j \partial x_k}\Big] + \kappa^E_{jk}\,\Big[\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_j \partial x_k}\Big] + T_0 \beta_{kj}\,\Big[\frac{\partial^r u_k}{\partial x_a \ldots x_c \partial x_j \partial t}\Big] = T_0 \tilde{\omega}_k \Big[\frac{\partial^r \phi}{\partial x_a \ldots x_c \partial x_k \partial t}\Big]\,. \qquad (59)$$

Now, the compatibility conditions for the jumps (51) apply to each of the functions $(u_i,\, \phi,\, T\,)$ where, in the spatial picture, V must be interpreted as the local speed of propagation w.r.t. the medium. Substituting them in equations (57)-(59) and then multiplying each term by $n_a \ldots n_c$ and summing on the repeated indexes $a, \ldots\, c$ we have the equations for the jumps

$$(\sigma_{klij} n_j n_k - \rho_o V^2 \delta_{li})\lambda_i + e_{ilj} n_i n_j \varphi = 0 \qquad (60)$$

$$e_{kji} n_i n_k \lambda_j - \epsilon_{kj} n_j n_k\, \varphi = 0 \qquad (61)$$

$$-\kappa_{kj} n_j n_k \tau + T_0 V \beta_{ij} n_j \lambda_i + (\kappa^E_{jk} n_j n_k - T_0 n_k \tilde{\omega}_k V)\varphi = 0\,, \qquad (62)$$

where

$$\lambda_i = \Big[\frac{\partial^r u_i}{\partial n_r}\Big]\,,\ \varphi = \Big[\frac{\partial^r \phi}{\partial n_r}\Big]\,,\ \tau = \Big[\frac{\partial^r T}{\partial n_r}\Big]\,. \qquad (63)$$

Note that equations (60)-(62) just coincide with equations (29)-(31).

Q.E.D.

6 Strong waves

Let $\mathcal{S}$ be a singular hypersurface of the dependent variables u_i, T and ϕ, of order 1.

Let $\mathbf{n}$ denote a unit oriented normal vector on $\mathcal{S}$. For points on $\mathcal{S}_t$ the equation of jump corresponding to the balance equation of linear momentum is

$$[\mathbf{t}]\,\mathbf{n} \,=\, -\rho_o V[\frac{\partial \mathbf{u}}{\partial t}]\,, \qquad (64)$$

the equation of jump corresponding to the first Maxwell's equation is

$$[\mathbf{D}]\,\cdot\,\mathbf{n} \,=\, 0\,, \qquad (65)$$

the equation of jump corresponding to the balance equation of energy is

$$-\rho_0\,[\eta]V + T_o^{-1}[\mathbf{q}]\,\cdot\,\mathbf{n} \,=\, 0\,. \qquad (66)$$

6.1 Strong waves of order 1

Now we use the kinematical compatibility conditions and the constitutive equations to prove the

Proposition 2 *Strong waves of order* 1 *for* $(\mathbf{u},\, \phi,\, T)$ *are characteristic for the l.p.d.e.s (15)-(17).*

Proof. In fact, we show that if $\mathcal{S}_t$ is singular of order 1 for $(\mathbf{u},\, \phi,\, T)$, then at each point $(x_1,\, x_2,\, x_3) \in \mathcal{S}_t$ the jumps

$$\lambda_i = \Big[\frac{\partial u_i}{\partial n}\Big]\,, \quad \varphi = \Big[\frac{\partial \phi}{\partial n}\Big]\,, \quad \tau = \Big[\frac{\partial T}{\partial n}\Big] \qquad (67)$$

and the speed of propagation V satisfy the characteristic equation of (18)-(20).

Indeed, from the compatibility condition $(50)_2$ for u_l and the jump law (64) we obtain

$$\Big[t_{al}\Big]\, n_a \;=\; \rho_o V^2 \Big[\frac{\partial u_l}{\partial n}\Big]\,; \tag{68}$$

by replacing the stress response law (1) we have

$$\Big(\sigma_{alij}\Big[u_{i,\,j}\Big] + e_{ial}\Big[\Phi_{,\,i}\Big] - \beta_{al}\,\Big[T\Big]\Big)\, n_a = \rho_o V^2 \lambda_l\,; \tag{69}$$

and using the compatibility conditions (50) for u_i and $\phi_{,\,i}$, since T is continuous, we have

$$\sigma_{alib}\,\lambda_i\, n_b n_a \;+\; e_{ial}\,\varphi\, n_b n_a \;=\; \rho_o V^2 \lambda_l\,. \tag{70}$$

which just is equation (29).

Now we apply the procedure above to Maxwell's equation; by replacing the constitutive law (2) in the jump law (65) we obtain

$$-\epsilon_{ai}\Big[\phi_{,\,i}\Big] n_a + e_{aij}\Big[u_{i,\,j}\Big] n_a + \tilde{\omega}_k \Big[T\Big] n_a = 0\,; \tag{71}$$

now using the compatibility conditions (50) for u_i and $\Phi_{,\,i}$, since T is continuous, the latter equality becomes

$$-\epsilon_{ai}\,\varphi\, n_i n_a \;+\; e_{aij}\,\lambda_i\, n_j n_a \;=\; 0\,; \tag{72}$$

which just is (30).

Lastly we apply the procedure above to the law of consevation of energy; by replacing the constitutive law (4) in the jump law (66) we obtain

$$-\rho_0\Big(\Big[\eta_0 + \frac{\gamma}{T_0}T\Big] + \frac{1}{\rho_o}\Big(\beta_{ij}\,\Big[u_{i,\,j}\Big] - \tilde{\omega}_i\Big[\phi_{,\,i}\Big]\Big)\Big)V + T_o^{-1}\Big(\kappa_{al}\,\Big[T_{,\,l}\Big] - \kappa_{al}^{E}\Big[\phi_{,\,l}\Big]\Big) n_a = 0\,; \tag{73}$$

by the continuity of T this equality becomes

$$\Big(-\beta_{ij}\,\Big[u_{i,\,j}\Big] + \tilde{\omega}_i\,\Big[\phi_{,\,i}\Big]\Big)V + T_o^{-1}\Big(\kappa_{al}\Big[T_{,\,l}\Big] - \kappa_{al}^{E}\Big[\phi_{,\,l}\Big]\Big) n_a = 0; \tag{74}$$

now using the compatibility conditions (50) for u_i, $\phi_{,\,i}$ and $T_{,\,l}$, we have

$$\Big(-\beta_{ij}\,\lambda_i\, n_j \;+\; \tilde{\omega}_i\,\varphi\, n_i\Big)V + T_o^{-1}\Big(\kappa_{al}\,\tau\, n_l \;-\; \kappa_{al}^{E}\,\varphi\, n_l\Big)\, n_a \;=\; 0\,, \tag{75}$$

which just coincides with (31).

Q.E.D.

6.2 Strong waves of order 0

Now we use the kinematical compatibility conditions and the constitutive equations to prove the

Proposition 3 *Let S be a strong wave of order* 0 *for* $(\mathbf{u},\,\phi,\,T)$ *such that* $[\mathbf{u}] = \mathbf{0}$.

Then S is characteristic if and only if

$$\Big(e_{ial}\, g^{\sigma\tau}\Big[\phi\Big]_{,\,\sigma}\psi_{i,\,\tau} \;-\; \beta_{al}\,\Big[T\Big]\Big) n_a \;=\; 0 \tag{76}$$

$$\Big(-\epsilon_{ai}\, g^{\sigma\tau}\Big[\phi\Big]_{,\,\sigma}\psi_{i,\,\tau} \;+\; \tilde{\omega}_a\,\Big[T\Big]\Big)\, n_a \;=\; 0 \tag{77}$$

$$\Big(-\gamma\,\Big[T\Big] \;+\; T_o\,\tilde{\omega}_i\, g^{\sigma\tau}\Big[\phi\Big]_{,\,\sigma}\psi_{i,\,\tau}\Big)\, V + \Big(\kappa_{al}\Big[T\Big]_{,\,\sigma} \;-\; \kappa_{al}^{E}\Big[\phi\Big]_{,\,\sigma}\Big)\, g^{\sigma\tau}\,\psi_{i,\,\tau}\, n_a \;=\; 0\,. \tag{78}$$

<u>Proof</u>. Let $\mathcal{S}$ be singular of order 0 for $(\mathbf{u},\,\phi,\,T)$, with

$$[\mathbf{u}] = \mathbf{0}\,, \qquad [\phi] \neq 0\,, \qquad [T] \neq 0\,;$$

then, at each point $(x_1,\,x_2,\,x_3) \in \mathcal{S}_t$, from the jump law (64) and the compatibility condition (49), since $[\mathbf{u}] = \mathbf{0}$, we have

$$\Big[t_{al}\Big]\, n_a \;=\; \rho_o V^2 \Big[\frac{\partial u_l}{\partial n}\Big]\,; \tag{79}$$

by replacing the stress response law (1) we obtain

$$\Big(\sigma_{alij}\,\Big[u_{i,\,j}\Big] + e_{ial}\,\Big[\phi_{,\,i}\Big] - \beta_{al}\,\Big[T\Big]\Big)\, n_a \;=\; \rho_o V^2 \lambda_l\,; \tag{80}$$

and using the compatibility conditions (48) for u_i and $\phi_{,\,i}$, we have

$$\sigma_{alib}\,\lambda_i\, n_b n_a \;+\; e_{ial}\,\Big(g^{\sigma\tau}\Big[\phi\Big]_{,\,\sigma}\psi_{i,\,\tau} + \varphi n_i\Big) n_a - \beta_{al}\,\Big[T\Big]\, n_a = \rho_o V^2 \lambda_l\,; \tag{81}$$

this equation differs from equation (29) by the presence in the left side of the term

$$\Big(e_{ial}\, g^{\sigma\tau}\Big[\phi\Big]_{,\,\sigma}\psi_{i,\,\tau}\beta_{al}\,\Big[T\Big]\Big) n_a\,. \tag{82}$$

Now we apply the procedure above to Maxwell's equation; by replacing the constitutive law (2) in the jump law (65) we obtain

$$-\epsilon_{ai}\,\Big[\phi_{,\,i}\Big]\; n_a \;+\; e_{aij}\,\Big[u_{i,\,j}\Big]\, n_a \;+\; \tilde{\omega}_a\,\Big[T\Big]\, n_a \;=\; 0\,; \tag{83}$$

now using the compatibility conditions (48) for u_i and $\phi_{,i}$, the latter equality becomes

$$-\epsilon_{ai}\left(g^{\sigma\tau}\Big[\phi\Big]_{,\sigma}\psi_{i,\tau} + \varphi\, n_i\right) n_a + e_{aij}\,\lambda_i\, n_j n_a + \tilde{\omega}_a\Big[T\Big]\, n_a = 0\,; \qquad (84)$$

which differs from equation (30) by the presence in the left side of the term

$$\Big(-\epsilon_{ai}\, g^{\sigma\tau}\Big[\phi\Big]_{,\sigma}\psi_{i,\tau} + \tilde{\omega}_a\Big[T\Big]\Big)\, n_a \qquad (85)$$

Lastly we apply the procedure above to the law of consevation of energy; by replacing the constitutive law (4) in the jump law (66) we obtain

$$-\rho_0\Big(\Big[\eta_0 + \frac{\gamma}{T_0}T\Big] + \frac{1}{\rho_o}\Big(\beta_{ij}\Big[u_{i,j}\Big] - \tilde{\omega}_i\Big[\phi_{,i}\Big]\Big)\Big)V + T_o^{-1}\Big(\kappa_{al}\Big[T_{,l}\Big] - \kappa^E_{al}\Big[\phi_{,l}\Big]\Big)\, n_a = 0\,; \qquad (86)$$

that is,

$$\Big(-\frac{\gamma}{T_o}\Big[T\Big] - \beta_{ij}\Big[u_{i,j}\Big] + \tilde{\omega}_i\Big[\phi_{,i}\Big]\Big)V + T_o^{-1}\Big(\kappa_{al}\Big[T_{,l}\Big] - \kappa^E_{al}\Big[\phi_{,l}\Big]\Big)\, n_a = 0\,; \qquad (87)$$

now using the compatibility conditions (48) for u_i, $\phi_{,i}$ and T, this equality becomes

$$\Big(-\gamma\Big[T\Big] - T_o\beta_{ij}\lambda_i n_j + T_o\tilde{\omega}_i(g^{\sigma\tau}\Big[\phi\Big]_{,\sigma}\psi_{i,\tau} + \varphi n_i)\Big)V + \Big(\kappa_{al}(g^{\sigma\tau}\Big[T\Big]_{,\sigma}\psi_{l,\tau} + \tau n_l) - \kappa^E_{al}(g^{\sigma\tau}\Big[\phi\Big]_{,\sigma}\psi_{l,\tau} + \varphi n_l)\Big)n_a = 0\,, \qquad (88)$$

which differs from equation (31) by the presence in the left side of the term

$$\Big(-\gamma\Big[T\Big] + T_o\,\tilde{\omega}_i\, g^{\sigma\tau}\Big[\phi\Big]_{,\sigma}\psi_{i,\tau}\Big)V + \Big(\kappa_{al}\Big[T\Big]_{,\sigma} - \kappa^E_{al}\Big[\phi\Big]_{,\sigma}\Big)g^{\sigma\tau}\,\psi_{l,\tau}\, n_a\,. \qquad (89)$$

Hence equations (29)-(31) hold if and only if equations (76)-(78) hold.

Q.E.D.

As a consequence of the last proposition we have that, generally, strong waves of order 0 are not characteristic.

For $p = 1,\, 2,\, 3,\ \sigma = 1,\, 2$ put

$$a_p{}^\sigma = e_{ipl}\, g^{\sigma\tau}\psi_{i,\tau}\,, \qquad a_p{}^3 = -\beta_{ap}\, n^a \qquad (90)$$

$$b_\sigma = -\epsilon_{ai}\, g^{\sigma\tau}\psi_{i,\tau} n^a \qquad (91)$$

$$c_\sigma = (\,T_o\,\tilde{\omega}_i\, V - \kappa^E_{ai} n^a) g^{\sigma\tau}\psi_{i,\tau} \qquad (92)$$

$$c_{\sigma+3} = k_{al}\, g^{\sigma\tau}\psi_{i,\tau}\, n^a\,. \qquad (93)$$

Hence by ordering the variables as $\Big([\phi]_{,\sigma},\,[T],\,[T]_{,\sigma}\Big)$ the matrix of the system of equations (76)-(78) is

$$\mathcal{M} = \begin{bmatrix} a_1{}^1 & a_1{}^2 & a_1{}^3 & 0 & 0 \\ a_2{}^1 & a_2{}^2 & a_2{}^3 & 0 & 0 \\ a_3{}^1 & a_3{}^2 & a_3{}^3 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 \\ c_1 & c_2 & -\gamma & c_4 & c_5 \end{bmatrix} \qquad (94)$$

Note that we have $det\mathcal{M} = 0$ for any possible choice of the material parameters and of the propagation direction. Thus equations (70)-(88) are compatible with the existence of characteristic strong waves of order 0.

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