A Modified Theory of Turbulent Flow over a Flat Plate

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Abstract:- A scale-invariant statistical theory of turbulence is described. The modified and invariant form of the equation of motion is then solved at the scale of eddy-dynamics, cluster-dynamics, and molecular-dynamics to reveal the internal structure of turbulent boundary layer over a flat plate. The predicted velocity profile is found to be in good agreement with the large body of experimental data reported in the literature. The results suggest that the classical logarithmic law of the wall should be modified. Also, based on an invariant definition of kinematic viscosity, a scale invariant definition of *Reynolds* number $\mathrm{Re}_\beta = L_{x\beta} w_{x\beta} / \lambda_{x\beta-1} v_{x\beta-1}$ is presented.

Key-Words: - Theory of turbulence, turbulent boundary layer theory, the logarithmic law of the wall.

1 Introduction

The truly universal character of turbulent phenomena from the very small scales of stochastic quantum fields [1-16] to the intermediate and exceedingly large scales of classical hydrodynamics and cosmology [17-26] resulted in recent introduction of a scale-invariant model of statistical mechanics and its application to the field of thermodynamics [27]. Following the classical methods, the implications of the model to the study of transport phenomena and the invariant forms of conservation equations in reactive fields have also been addressed [28, 29].

As two examples of the exact solutions of the modified form of the equation of motion, the classical problems of two-dimensional and axi-symmetric laminar [30] and turbulent [31] jets were recently investigated. According to the theory, the solutions for turbulent jets at LED scale were identical to those for laminar jets at the smaller scale of LCD (Fig.1). A close agreement was found between the predicted turbulent velocity profiles and the experimental data without any adjustable parameters.

In another recent study [32], following *Blasius* [33], the modified form of the equation of motion was solved for the classical problem of laminar flow over a flat plate. The predicted velocity profile was found to be in close agreement with the early experimental data of *Nikuradse* [34] and in excellent agreement with the more recent experimental data of *Dhawan* [35].

In the present study, the implications of the scale invariant model of statistical mechanics to the statistical theory of turbulence are further examined. Also, the problem of turbulent flow over a flat plate will be investigated and it will be shown that the analytical solution of the modified equation of motion closely agree with the large body of experimental data available in the literature. The results suggest that the logarithmic law of the wall should be modified.

2 Scale Invariant Forms of the Conservation Equations for Reactive Fields

Following the classical methods [36-38], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [27]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \tag{2}$$

The scale-invariant model of statistical mechanics for equilibrium fields of … -, eddy-, cluster-, molecular-, and atomic-dynamics at the scale β = .., e, c, m, a,...

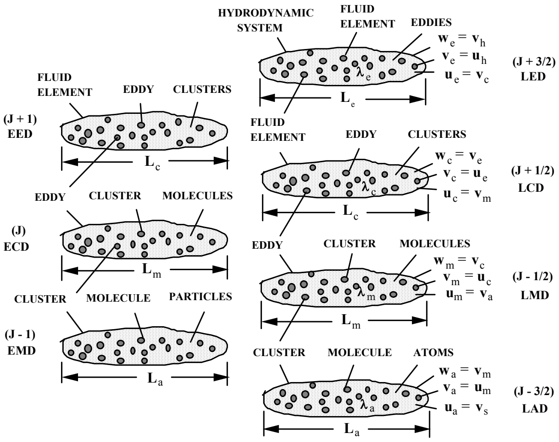

Fig.1 Hierarchy of statistical fields for equilibrium eddy-, cluster-, and molecular-dynamic scales and the associated laminar flow fields.

and the corresponding *non-equilibrium* laminar flow fields are schematically shown in Fig.1. Each statistical field, described by a distribution function $f_\beta(\mathbf{u}_\beta) = f_\beta(\mathbf{r}_\beta, \mathbf{u}_\beta, t_\beta)\, d\mathbf{r}_\beta d\mathbf{u}_\beta$, defines a "system" that is composed of an ensemble of "elements", each element is composed of an ensemble of small particles viewed as *point-mass* "atoms". The element (system) of the smaller scale (β) becomes the atom (element) of the larger scale (β+1). The characteristic lengths associated with the "atoms", the elements, and the system are ($l_\beta = \lambda_{\beta-1}$, λ_β, $L_\beta = \lambda_{\beta+1}$) where $\lambda_\beta = <l^2_\beta>^{1/2}$ is the cluster length that is also equal to the mean-free-path of the "atoms" [28].

The invariant definitions of the peculiar and the diffusion velocities have been introduced as [27]

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \qquad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta = \mathbf{V}'_{\beta+1} \tag{3}$$

For the equilibrium statistical fields shown on the left side of Fig.1, $f_\beta(\mathbf{u}_\beta)$ will be the *Maxwell-Boltzmann* distribution function.

Following the classical methods [36-38], the scale-invariant forms of mass, thermal energy and linear momentum conservation equations at scale β are written as [29]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.\left(\rho_\beta \mathbf{v}_\beta\right) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla.\left(\varepsilon_\beta \mathbf{v}_\beta\right) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla.\left(\mathbf{p}_\beta \mathbf{v}_\beta\right) = -\nabla.\mathbf{P}_\beta \tag{6}$$

that involve the *volumetric density* of thermal energy $\varepsilon_\beta = \rho_\beta h_\beta$ and linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$. Also, Ω_β is the chemical reaction rate and h_β is the absolute enthalpy [28].

The local velocity $\mathbf{v}_\beta$ in (4)-(6) is expressed in terms of the convective $\mathbf{w}_\beta = \langle \mathbf{v}_\beta \rangle$ and the diffusive $\mathbf{V}_\beta$ velocities [28]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{7a}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{7b}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \tag{7c}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$) are respectively the diffusive, the thermo-diffusive, and the linear hydro-diffusive velocities. For unity *Schmidt* and *Prandtl* numbers, one may express

$$\mathbf{V}_{\beta tg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta t} \quad , \quad \mathbf{V}_{\beta t} = -\alpha_\beta \nabla \ln(h_\beta) \tag{8}$$

$$\mathbf{V}_{\beta hg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta h} \quad , \quad \mathbf{V}_{\beta h} = -\nu_\beta \nabla \ln(\mathbf{v}_\beta) \tag{9}$$

that involve the thermal $\mathbf{V}_{\beta t}$, and linear hydrodynamic $\mathbf{V}_{\beta h}$ diffusion velocities. Since for an ideal gas $h_\beta = c_{p\beta} T_\beta$, when $c_{p\beta}$ is constant and $T = T_\beta$, (8) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_\beta = \rho_\beta h_\beta \mathbf{V}_{\beta t} = -\kappa_\beta \nabla T \tag{10}$$

where κ_β and $\alpha_\beta = \kappa_\beta / (\rho_\beta c_{p\beta})$ are the thermal conductivity and diffusivity. Similarly, (9) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [28]

$$\boldsymbol{\tau}_{ij\beta} = \rho_\beta \mathbf{v}_{j\beta} \mathbf{V}_{ij\beta h} = -\mu_\beta \partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_i \tag{11}$$

Substitutions from (7a)-(7c) into (4)-(6), neglecting the cross-diffusion terms and assuming constant transport coefficients with $Sc_\beta = Pr_\beta = 1$, result in [29]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta.\nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{12}$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta.\nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -h_\beta \Omega_\beta / (\rho_\beta c_{p\beta}) \tag{13}$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta.\nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\frac{\nabla p_\beta}{\rho_\beta} - \frac{\mathbf{v}_\beta \Omega_\beta}{\rho_\beta} \tag{14}$$

An important feature of the modified equation of motion (14) is that it is linear since it involves a convective velocity $\mathbf{w}_\beta$ that is different from the local fluid velocity $\mathbf{v}_\beta$. Because the convective velocity $\mathbf{w}_\beta$ is not *locally defined* it cannot occur in *differential form* within the conservation equations [28]. To determine $\mathbf{w}_\beta$, one needs to go to the next higher scale (β+1) where $\mathbf{w}_\beta = \mathbf{v}_{\beta+1}$ becomes a local velocity. However, at this new scale one usually encounters yet another convective velocity $\mathbf{w}_{\beta+1}$ which is not known, thus requiring consideration of the higher scale (β+2). This unending chain constitutes the *closure problem* of the statistical theory of turbulence schematically shown in Fig.1.

3 Connections between the Modified Form of the Equation of Motion and the Navier-Stokes Equation

The original form of the *Navier-Stokes* equation with constant coefficients is given as [36, 37]

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}.\nabla\mathbf{v} = -\nabla P + \mu\nabla^2\mathbf{v} + \frac{1}{3}\mu\nabla(\nabla.\mathbf{v}) \qquad (15)$$

The stress P in (15) is not the thermodynamic pressure p but rather it is defined in terms of the total stress tensor $T_{ij} = -p\delta_{ij} + \tau_{ij}$ and is known as the *mechanical pressure* [39]

$$P_m = -(1/3)T_{ii} = p - (1/3)\tau_{ii} \qquad (16)$$

Since the normal viscous stress is given by (11) as $(1/3)\tau_{ii} = (1/3)\rho v_i V_{ii} = -(1/3)\mu\nabla.\mathbf{v}$, the gradient of (16) gives

$$\nabla P = \nabla P_m = \nabla\frac{1}{3}\mu(\nabla.\mathbf{v}) = \nabla p + \frac{1}{3}\mu\nabla(\nabla.\mathbf{v}) \qquad (17)$$

Substituting from (17) into (15), the *Navier-Stokes* equation assumes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} - \nu\nabla^2\mathbf{v} = -\nabla p/\rho \qquad (18)$$

that is almost identical to (14) with $\Omega_\beta = 0$ except that in (14) the convective velocity is different from the local velocity $\mathbf{v}_\beta$. However, because (18) includes a diffusion term and $\mathbf{w}_\beta$ and $\mathbf{v}_\beta$ are related by (7c) as $\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_\beta$, it is clear that (18) should in fact be written as (14).

4 Perspectives on a Scale Invariant Statistical Theory of Turbulence

According to Fig.1, and in harmony with *Richardson's* well known rhyme about big and little eddies [23], the "atom" of the statistical field of equilibrium eddy-dynamics EED *(J+1)* is the turbulent eddy, that is considered to be composed of a large number of molecular-clusters and hence may be called a super-cluster. It is emphasized that the *hierarchical model of turbulence* being presented (Fig.1) is different from the classical cascade models according to which larger eddies are considered to be composed of successively smaller eddies. According to the present model, the element of the statistical field of EED at the scale *(j+1)* is considered to be composed of an *entirely new statistical field* of ECD at the smaller scale *(j)* as shown in Fig.1.

4.1 The Nature of Brownian Motions

According to Fig.1, the statistical fields of equilibrium eddy-dynamics EED and molecular-dynamics EMD are separated from each other by an intermediate-scale statistical field called equilibrium cluster-dynamics ECD. The evidence for the existence of the statistical field of equilibrium cluster-dynamics is the phenomena of Brownian motions [23, 40-46]. Modern theory of Brownian motion starts with the *Langevin* equation [23]

$$\frac{d\mathbf{u}_p}{dt} = -\beta\mathbf{u}_p + \mathbf{A}(t) \qquad (19)$$

where $\mathbf{u}_p$ is the particle velocity. The drastic nature of the assumptions inherent in the division of forces in (19) was emphasized by *Chandrasekhar* [23].

Because of the much larger mass and size of particles as compared with individual molecules, the Brownian motion of particles may not be attributed to their collision with single molecules as first noted by *Gouy* [40]. In fact, such a description based on the kinetic theory failed, since several experiments have shown that the kinetic energy of the particle and the molecule may differ by a factor of 100, 000 [40, 46]. According to the classical descriptions, multiple collisions of large numbers of molecules with a single suspended particle are believed to be responsible for the Brownian motions. However, since the background fluid is in *thermodynamic equilibrium*, either *simultaneous* or *successive* collisions of many molecules preferentially from one side of the particle and not the other would be unacceptable on account of their *Maxwellian* velocity distribution.

To account for Brownian motions, one must assume collision between collections of molecules and the particle. Because Brownian motion is an equilibrium phenomenon such collections of molecules must themselves possess Brownian motions. However, this would mean the existence of the equilibrium statistical field of cluster-dynamics as shown in Fig.1.

Let us consider suspended particles in equilibrium with molecular clusters that are undergoing Brownian motions themselves as schematically shown in Fig.2.

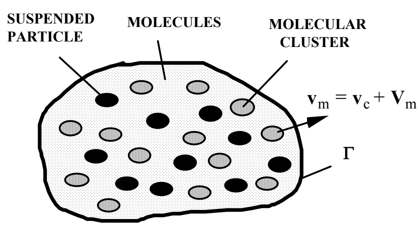

Fig. 2 Brownian motions of suspended particles u_p due to collisions with molecular-clusters u_c that themselves undergo Brownian motions.

The coefficient of diffusion of the Brownian particle [43] written here in invariant form is

$$D_\beta = \frac{kT_\beta}{6\pi\mu_\beta d_\beta} \tag{20}$$

The above relation could be expressed as

$$D_\beta = \frac{m_\beta u_\beta^2/3}{6\pi\rho_\beta \nu_\beta d_\beta} \tag{21}$$

The density and the kinematic viscosity are then expressed as

$$\rho_\beta = m_\beta n_\beta \quad , \quad \nu_\beta = \ell_\beta u_\beta/3 \tag{22}$$

where the mean-free-path ℓ_β is

$$\ell_\beta = 1/(\sqrt{2}\pi n_\beta \sigma_\beta^2) \tag{23}$$

By substitutions from (22)-(23) into (21) one obtains

$$D_\beta = u_\beta \sigma_\beta^2/(3\sqrt{2}d_\beta) \tag{24}$$

The result (24) reduces to the classical result of *Maxwell* $D_\beta = u_\beta d_\beta/3$ based on the kinetic theory of an ideal gas if the equality $\sigma_\beta^2 = \sqrt{2}d_\beta^2$ holds.

The result (24) suggests that Brownian particles of various sizes behave exactly the same as molecular clusters of various sizes. The absence of viscous dissipations effects in Brownian motions is due to this equilibrium between the particles and clusters. The only difference between them is that molecules composing the particles as rigid bodies are always the same, while those composing molecular clusters will be always changing since clusters are only stochastically stationary. Based on the ultra-simplified models of kinetic theory of ideal gas one expects the equality of diffusivity of momentum, mass, and heat $\nu = D = \alpha$ [47] thus leading to

$$\nu_\beta = \frac{1}{3}\lambda_{\beta-1}v_{\beta-1} = \frac{\lambda_{\beta-1}}{\sqrt{3}}\frac{v_{\beta-1}}{\sqrt{3}} = \lambda_{x\beta-1}v_{x\beta-1} \tag{25}$$

that is the scale-invariant definition of the kinematic viscosity [28]

5 Theory of Laminar Boundary Layer over a Flat Plate

The invariant model of statistical mechanics, Fig.1, described in the previous section suggests that the equation of motion for turbulent flows should be identical to that for laminar ones with the only difference being that for the former the "atoms" of the field are eddies rather than clusters. Such a correspondence was indeed recently established by the solution of the modified equation of motion (14) for the classical problems of axi-symmetric and two-dimensional turbulent jets [31]. It was found that the predicted velocity profiles identical to those for laminar flow are in excellent agreement with the experimental data without any adjustable parameters.

In another recent study [32] the solution of (14) for the problem of laminar flow over a flat plate was presented and the results were found to be in close agreement with the classical numerical solution of *Blasius* [33] as well as the experimental data [34, 35]. The objective of the present study is to extend the solution of the modified equation of motion (14) to the important problem of turbulent flow over a flat plate. However, to facilitate the presentation of the theory for turbulent flow, it is best to first review the solution of (14) for laminar flow over the flat plate. This is helpful since no reference to any particular scale such as LED, LCD, needs to be made until the following section where the problem of turbulent boundary layer is addressed.

The laminar boundary layer is schematically shown in Fig.3 and the uniform convective velocity

$$w'_{x\beta} = w'_{o\beta} \qquad w'_{y\beta} = 0 \tag{26}$$

is considered to be known.

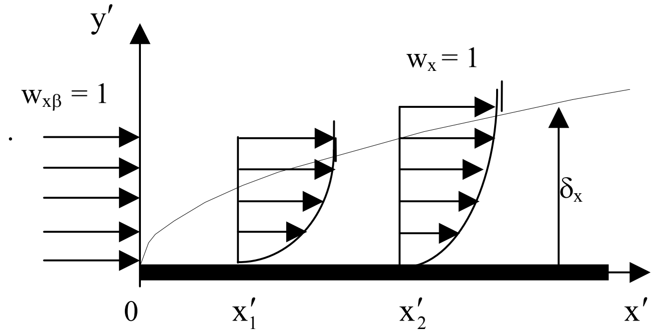

Fig.3 Laminar boundary layer over a flat plate.

Hence, the "atomic", element, and system velocities are $(\mathbf{u}'_\beta, \mathbf{v}'_\beta, \mathbf{w}'_\beta)$ and the corresponding length scales are $(l_\beta, \lambda_\beta, L_\beta)$. The conventional boundary layer assumptions $\partial^2/\partial x'^2 \ll \partial^2/\partial y'^2$ and $\nabla p_\beta \simeq 0$ are introduced along with the dimensionless velocities

$$(\mathbf{v}_{x\beta}, \mathbf{v}_{y\beta}, \mathbf{w}_{x\beta}) = (\mathbf{v}'_{x\beta}, \mathbf{v}'_{y\beta}, \mathbf{w}'_{x\beta})/\mathbf{w}'_{o\beta} \qquad (27)$$

and coordinates

$$x_\beta = x'/\delta_\beta \quad , \quad y_\beta = y'/\delta_\beta \quad , \quad \delta_\beta = \nu_\beta / w'_{o\beta} \quad (28)$$

where δ_β is the length for diffusion of momentum due to eddy viscosity given by (25) as $\nu_\beta = \lambda_{\beta-1} v_{\beta-1}/3$ [5]. Because usually $\delta_\beta = \nu_\beta / w'_{o\beta} \ll 1$, the boundary layer coordinates (x_β, y_β) in (28) are stretched coordinates. The steady forms of (4) and (14) in the absence of chemical reactions $\Omega_\beta = 0$ and negligible transverse convection (26) reduce to

$$\frac{\partial v_{x\beta}}{\partial x_\beta} + \frac{\partial v_{y\beta}}{\partial y_\beta} = 0 \qquad (29)$$

$$w_{x\beta}\frac{\partial v_{x\beta}}{\partial x_\beta} = \frac{\partial^2 v_{x\beta}}{\partial y_\beta^2} \qquad (30)$$

that are subject to the boundary conditions

$$y_\beta = 0 \qquad v_{x\beta} = v_{y\beta} = 0 \qquad (31)$$

$$y_\beta \to \infty \qquad v_{x\beta} = w_{x\beta} = 1 \qquad (32)$$

According to Fig.3, the local axial velocity $v_{x\beta}$ within the boundary layer must vanish at the plate and match the outer convective velocity field $w_{x\beta} = 1$ at the edge of the turbulent boundary layer, i.e. in the limit $y_\beta \to \infty$ (Fig.3). Therefore, the convective velocity that is the mean of the local velocity $w_{x\beta}$ = $\langle v_{x\beta} \rangle$ within the boundary layer will have the constant value of $w_{x\beta}$ = ½ at all axial locations. By introducing the value $w_{x\beta}$ = ½ and the similarity variable

$$\xi_\beta = y_\beta / (2\sqrt{2x_\beta}) \qquad (33)$$

into (30) one obtains

$$\frac{d^2 v_{x\beta}}{d\xi_\beta^2} + 2\xi_\beta \frac{dv_{x\beta}}{d\xi_\beta} = 0 \qquad (34)$$

that is subject to the boundary conditions

$$\xi_\beta = 0 \qquad v_{x\beta} = 0 \qquad (35)$$

$$\xi_\beta \to \infty \qquad v_{x\beta} = 1 \qquad (36)$$

The solution of (34)-(36) is

$$v_{x\beta} = \mathrm{erf}(\xi_\beta) \qquad (37)$$

To facilitate the comparisons, the solution (37) is also expressed in terms of a new coordinate

$$v_{x\beta} = \mathrm{erf}[\eta_\beta / 2\sqrt{2}] \qquad \eta_\beta = y_\beta / \sqrt{x_\beta} \qquad (38)$$

where η_β is the similarity variable of the classical theory [33, 37]. The boundary layer thickness is obtained from (37) as the position $\xi^*_\beta \simeq 1.8$ where $v_{x\beta}$ = 0.99 that by (33) leads to

$$\delta_\beta \simeq 5.1\, x_\beta^{1/2} = 5.1\sqrt{\mathrm{Re}_{x\beta}} \qquad (39)$$

in close agreement with the classical numerical result of *Blasius* [33, 37]

$$\delta_\beta \simeq 5.0\sqrt{\mathrm{Re}_{x\beta}} \qquad (40)$$

The calculated velocity profile from (38) and the experimental data of *Dahwan* [35] are shown in Fig.4.

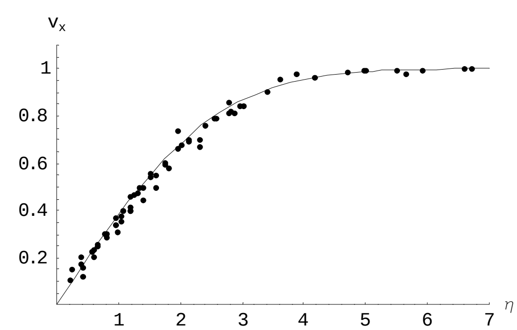

Fig.4 Comparison between the predicted axial velocity profile from (38) and the experimental data of *Dhawan* [35].

As shown in Fig.4, the predicted velocity profile from (34) is in excellent agreement with the experimental data of *Dhawan* [35], which are more recent as compared to 1942 data of *Nikuradse* [34]. The earlier experimental data of *Nikuradse* [34] are found to always locate on the lower boundary of the more recent data shown in Fig.4 as discussed earlier [32].

One can express the solution (34) in terms of the stream function

$$\Psi_\beta = 2\sqrt{2x_\beta}\int_0^{\xi_\beta} \mathrm{erf}(\xi_\beta)d\xi_\beta \qquad (41)$$

The transverse velocity that is obtained from (41) as

$$v_{y\beta} = \sqrt{\frac{2}{x_\beta}}\left[\xi_\beta \mathrm{erf}(\xi_\beta) - \int_0^{\xi_\beta} \mathrm{erf}(z_\beta)dz_\beta\right] \qquad (42)$$

Some of the streamlines calculated from (41) in terms of (x_β, y_β) coordinates are shown in Fig.5.

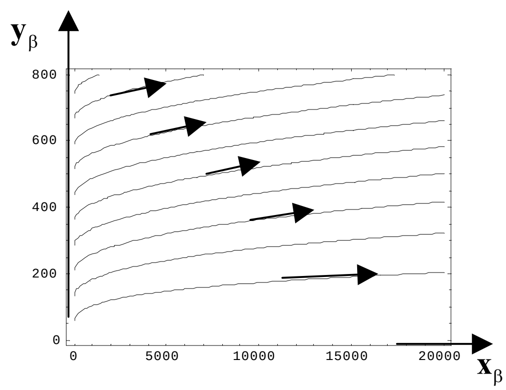

Fig.5 Calculated streamlines from (41) for both laminar and turbulent flow over a flat plate.

6 Theory of Turbulent Boundary Layer over a Flat Plate

The turbulent boundary layers over flat plates or in pipes are usually described by phenomenological methods leading to the well-known law of the wall [37, 50, 51]. The structure of turbulent boundary layer over a flat plate is schematically shown in Fig.6. The free stream turbulent flow away from the wall is separated from the wall by three distinguishable regions respectively called (1) the turbulent boundary layer at LED scale (2) the laminar boundary layer at LCD scale, and (3) the laminar sub-layer at LMD scale. The coordinates as well as the length scales associated with each of the three "boundary layers" are described next.

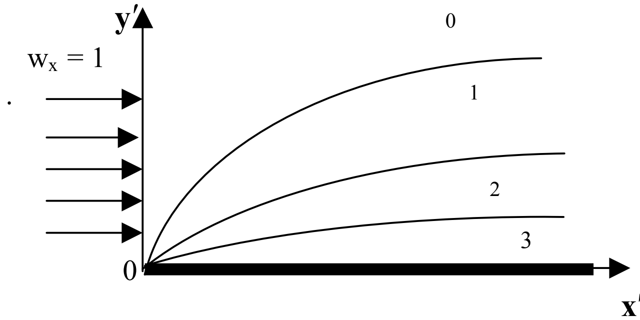

Fig.6 Turbulent flow over a flat plate, (0) turbulent free stream (1) turbulent boundary layer at LED scale (2) laminar boundary layer at LCD scale (3) laminar sub-layer at LMD scale.

Recently, a scale-invariant logarithmic definition of coordinates was introduced [48] where the coordinate at scale β is related to that at the lower adjacent scale β −1 as schematically shown below

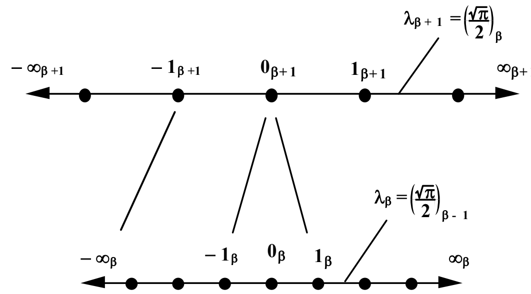

Fig.7 Hierarchy of normalized coordinates for cascades of embedded statistical fields.

According to Fig.7, the range $(-1_\beta, 1_\beta)$ of the outer coordinate x_β will correspond to the range $(-\infty_{\beta-1}, \infty_{\beta-1})$ of the inner coordinate $x_{\beta-1}$. Also, the zero of the outer scale $(-0_\beta, +0_\beta)$ decompactifies to the unity of the inner scale $(-1_{\beta-1}, 1_{\beta-1})$ as shown in Fig.7. An analogy between the hierarchy of embedded boundary layers shown in Fig.6 and the hierarchy of embedded finite interval $(-1_\beta, 1_\beta)$ on a line shown in Fig.7 may be noted. This analogy is further supported by the fact that the solution of the velocity field (37) involves *Gauss's* error function that also formed the basis for the "measure" employed for the normalization of coordinates [48]. In order to reveal the relative sizes of the various coordinates close to the wall, the region near the origin of Fig.6 is expanded in Fig.8.

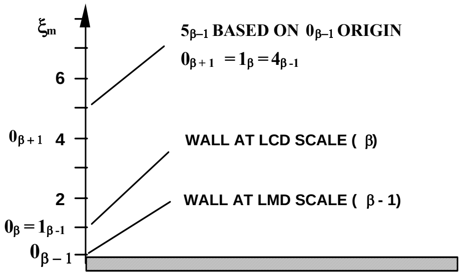

Fig.8 Coordinates near the wall and the associated origins at LED, LCD, and LMD scales.

Next, the relationship between the length scales of adjacent boundary layers (Fig.6) is addressed. In a recent investigation on opposed finite jets [49], it was found that the appropriate scale factor between adjacent generations of statistical fields such as LED and LCD is

$$\delta_{\beta+1} / \delta_\beta = \delta_e / \delta_c \simeq 4 \tag{43}$$

The result (43) originates from the fact that the boundary layer thickness in opposed finite jets is [49]

$$L_\beta = \delta_{\beta+1} \simeq 2\sqrt{\nu_\beta / \Gamma_\beta} \tag{44}$$

where the stretch rate is defined as

$$\Gamma_\beta = w'_\beta / L_\beta \tag{45}$$

Finally, the inner viscous length scale is defined as

$$\delta_\beta \simeq \nu_\beta / w'_\beta \tag{46}$$

By (44)-(46) one arrives at the scaling expressed in terms of *Reynolds* number

$$Re_\beta = w'_\beta L_\beta / \nu_\beta = \delta_{\beta+1} / \delta_\beta \simeq 4 \tag{47}$$

that is in accordance with (43). Since the coordinates (x_β, y_β) are measured in units of δ_β (28), by (47) the coordinates of adjacent scales will relate as

$$\xi_\beta / \xi_{\beta-1} = \delta_{\beta-1} / \delta_\beta \simeq 1/4 \tag{48}$$

Also, in view of the scale-invariant definition of kinematic viscosity (25), one arrives at an invariant definition of *Reynolds* number

$$Re_\beta = L_\beta w'_\beta / \nu_\beta = L_{x\beta} w'_{x\beta} / (\lambda_{x\beta-1} v'_{x\beta-1}) \tag{49}$$

With the above concepts of coordinates of different scales, the solution for velocity distribution in regions (1)-(3) of Fig.6 can be addressed. The solution of (34) in region (3) at LMD will correspond to laminar sub-layer. This thin layer will have a velocity profile given by (37) and involves the characteristic "atomic", element, and system velocities $(\mathbf{u}'_m, \mathbf{v}'_m, \mathbf{w}'_m)$ and lengths scales $(l_m = 10^{-9},\ \lambda_m = 10^{-7},\ L_m = 10^{-5})$ m. Because of the very small lengths, the structure of the viscous sub-layer is not observed at the resolution of usual fluid mechanic experiments and will not be further examined here.

For the laminar boundary layer in region (2) of Fig.6 at LCD, the characteristic "atomic", element, and system velocities and the associated lengths are $(\mathbf{u}'_c, \mathbf{v}'_c, \mathbf{w}'_c)$ and $(l_c = 10^{-7},\ \lambda_c = 10^{-5},\ L_c = 10^{-3})$ m. Hence, typical system length $L_c \approx 10^{-3}$ m is about the thickness of the boundary layer and the dissipative length is the cluster size $l_c \approx \lambda_m = 10^{-7}$ m that is also the mean free path of molecules. The *friction velocity* is defined as [37]

$$v'^2_\tau = \tau / \rho = \nu \left(\partial v' / \partial y' \right)_{y'=0} \tag{50}$$

along with the definitions

$$v^+ = v' / v'_\tau \qquad , \qquad y^+ = y' / (\nu / v'_\tau) \tag{51}$$

that lead to the classical result [37]

$$\left(dv^+ / dy^+ \right)_{y^+=0} = 1 \tag{52}$$

Also, by (28) one obtains from (50) the dimensionless friction velocity

$$v^2_\tau = v'^2_\tau / w'^2_o = \left(\partial v / \partial y \right)_{y=0} \tag{53}$$

The velocity field (37) when applied to (50)-(53) results in

$$v^2_\tau = 1 / \sqrt{2\pi x} \tag{54}$$

With (54) the similarity coordinate (33) and the classical coordinate (51) will be related as

$$\xi_\beta = y_\beta / (2\sqrt{2 x_\beta}) = (\sqrt{\pi} / 2) v_{\tau\beta} y^+_\beta \tag{55}$$

and hence

$$\tilde{\xi}_\beta = (2/\sqrt{\pi}) \xi_\beta = v_{\tau\beta} y^+_\beta \tag{56}$$

In terms of the new dimensionless velocity and coordinate in (51), the solution (37) becomes

$$v^+_\beta = v_\beta / v_{\tau\beta} = \frac{1}{v_{\tau\beta}} \frac{2}{\sqrt{\pi}} \operatorname{erfn}(v_{\tau\beta} y^+_\beta) \tag{57}$$

where the coordinate-normalized error function is defined as

$$\operatorname{erfn}(\tilde{\xi}) = \int_0^{\tilde{\xi}} \exp[-\pi \tilde{\xi}^2 / 4] d\tilde{\xi} = \operatorname{erf}(\xi) \tag{58}$$

From (52) and (35) one obtains $v^+ = v / v_\tau = y^+$ such that the friction velocity may be expressed as

$$v_{\tau\beta} = v_\beta / y^+_\beta \tag{59}$$

By (37), $v_c \simeq 0.995$ at the edge of boundary layer $\xi_{cb} \simeq 2$, and (33) results in $\delta_x \simeq 4\sqrt{2x}$ such that $\xi \simeq 2y / \delta_x$. At $y^+_{cb} \simeq 2$ where $v_c \simeq 1$, the friction velocity by (59) and (48) is approximately

$$v_{\tau c} \simeq 1/2 \quad , \quad v_{\tau m} \simeq 1/8 \tag{60}$$

In addition to the scaling (48), the coordinates of adjacent scales are normalized, expressed in measureless form, as shown in Fig.7 such that

$$y^+_\beta = (2/\sqrt{\pi}) y^+_{\beta-1} / 4 \tag{61}$$

With (60) and (61), the solution (57) assumes the form

$$v^+_c = 0_c + (1/v_{\tau c})\,(2/\sqrt{\pi})\ \mathrm{erfn}(y^+_c v_{\tau c})$$
$$= 8\ (2/\sqrt{\pi})\ \mathrm{erf}(y^+_m / 8) \tag{62}$$

The normalization constant $2/\sqrt{\pi}$ in (61) converts erfn in (62) to the classical error function.

Finally, the region (1) of Fig.6 is the turbulent boundary layer at LED scale with the "atomic", element, and system velocities $(\mathbf{u}'_e, \mathbf{v}'_e, \mathbf{w}'_e)$ and lengths $(l_e = 10^{-5},\ \lambda_e = 10^{-3},\ L_e = 10^{-1})$ m. The velocity profile in turbulent boundary layer at LED scale is again obtained from solution of the invariant equation of motion (14) that leads to (34) resulting in the solution similar to (62)

$$v^+_e = 0_e + 8(2/\sqrt{\pi})\ \mathrm{erf}(\ y^+_c / 8)$$
$$= 5 + 8(2/\sqrt{\pi})^2\ \mathrm{erf}(\ y^+_m / 32) \tag{63}$$

The factor $8(2/\sqrt{\pi})^2$ in (63) is because of the additional factor of $2/\sqrt{\pi}$ due to the coordinate renormalization (61) in moving from y^+_c to y^+_m coordinate. The number 5 in (63) results from the choice of the origin shown in Fig.8 and the scale factor (48). According to the logarithmic definition of coordinates [48] the zero of LED corresponds to the unity of LCD $(0_\beta = 0_e \Leftrightarrow 1_{\beta-1} = 1_c)$. Also, the scale relation (48) leads to the equivalence $(1_\beta \Leftrightarrow 4_{\beta-1})$ in terms of LCD coordinate y^+_m. Finally, because of the relation $(0_c \Leftrightarrow 1_m)$, the position $y^+_m = 4$ based on the origin at LCD scale (0_c) will correspond to position $y^+_m = 5$ since we consider the wall to be at the origin of LMD scale and $(0_c = 1_m)$ as shown in Fig.8. Such relation between the unity and the zero of adjacent scales within the hierarchy, Fig.7, is necessary for description of statistical fields from cosmic to photonic scales as discussed earlier [48].

The predicted velocity profiles for the turbulent boundary layer calculated from (62) and (63) are shown in Fig.9 along with the experimental data from various sources in the literature [37, 50, 51].

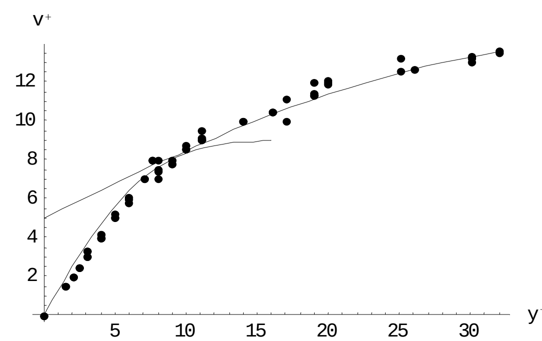

Fig.9 Comparison between the predicted velocity profile from (62) and (63) and experimental data in the literature [37, 50, 51].

As shown in Fig.9, the agreement between the theory and the experimental data is good. It is important to emphasize that the results of the present theory suggests that the classical and well-known *logarithmic law of the wall* introduced by *von Karman-Prandtl* [26, 37, 51] requires a closer examination and should be modified. Indeed, recently it has been suggested [52] that a power law rather than logarithmic law could be a closer representation of the velocity field in turbulent boundary layers. However, according to the present theory the velocity profiles in turbulent boundary layers should satisfy error-function type solutions such as presented in (62) and (63).

7 Concluding Remarks

The scale-invariant model of statistical mechanics was applied to describe a statistical theory of turbulence. The invariant modified form of the equation of motion was solved for the classical problem of turbulent boundary layer over a flat plate. The predicted velocity profile was found to be in good agreement with the large body of experimental data reported in the literature.

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