Weighted Moments Based Identification of Continuous-Time Systems

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Abstract: - In this paper we present an algorithm for continuous-time model identification from sample data using the weighted power moments of the output signal of a linear, time-invariant system. While most of the latest methods used in identification utilize a discrete-time model, the moments method is an alternative approach to directly identify a continuous-time model from discrete-time data. The method defines a set of relationships between the power series coefficients of a stable transfer function and the power moments of the output signal of this system. Based on these relations, an algorithm for off-line parameter identification is developed. The method is applied to identify the parameters of a real experimental platform.

Key-Words: - off-line identification, weighted power moments, sampled data

1 Introduction

Due to the success of digital computers and the availability of digital data acquisition boards, most system identification schemes usually aim at identifying the parameters of discrete-time models based on sampled input-output data. Over the last few years there has been strong interest in continuous-time approaches for system identification from sampled data. Identification of continuous-time models is indeed a problem of considerable importance in various disciplines such economics, signal processing and control [3]. A simplistic way of estimating the parameters of continuous-time model by an indirect approach is to use the sampled data to first estimate a discrete-time model and then convert it into an equivalent continuous-time model. However, the second step, i.e. obtaining an equivalent continuous-time model from the estimated discrete-time model, is not always easy. Difficulties are encountered whenever the sampling time is either too large or too small [11].

Whereas a large sampling interval may lead to loss of information, making it very small may create numerical problems due to the fact that the poles are constrained to lie in a very small area of the z-plane close to the unit circle. Some conversion methods use the matrix logarithm which may produce complex arithmetic when the matrix has negative eigenvalues. Moreover, the zeros of the discrete-time model are not as easily transformable to continuous-time equivalents as the poles are [9, 10].

In every tuning algorithm, the most difficult phase is the identification one, the whole control design depending on it. We can underline two approaches of identification algorithms: on-line identification algorithms and off-line identification algorithms. In on-line identification approach, the result is obtained in the same moment with a new observation data acquisition. The on-line identification deals with parametric methods (deterministic or stochastic), which identify the parameters of a mathematical model with a structure apriori known. The main on-line methods can be found in [2], [5], [8].

In off-line identification approach it is possible to identify both the structure of linear time invariant systems and the parameters of the mathematical model using observations over a larger time interval, including the steady state. The moments method presented in this paper is an off-line integral method.

2 Moments Definition

The moment problem is a classical one in the functional analysis [1], [6]. The order j power moment [4] (denoted m_j) of a function $y(t)$ is a characteristic of an original function with index of convergence $\sigma_0 \le 0$ with the following Laplace transform:

$$Y(s) = L\{y(t)\} = \int_0^\infty y(t)e^{-st}dt,\ \operatorname{Re}(s) > \sigma_0 \quad (1)$$

Denoting by $\varepsilon(t)$ deviation of $y(t)$ from steady state

$\varepsilon(t) = y(\infty) - y(t),\, t \ge 0$ one define the order *j*-power moment [7] (that we will call the classical moment) for function $y(t)$:

$$m_j = \int_0^\infty \frac{(-t)^j}{j!}(y(\infty) - y(t))dt = \int_0^\infty \frac{(-t)^j}{j!}\varepsilon(t)dt \qquad (2)$$

The Laplace transform of error $\varepsilon(t)$ is:

$$E(s) = L\{\varepsilon(t)\} = L\{y(\infty) - y(t)\} = \int_0^\infty \varepsilon(t)e^{-st}dt,\ \operatorname{Re}(s) > \sigma_0 \qquad (3)$$

The complex function e^{-st} is holomorphic in the complex plane *s* and it has an ordinary point to infinity then, the Taylor series around *s*=0 is uniform convergent with radius ∞, so:

$$e^{-st} = \sum_{j=0}^\infty \left(\frac{(-t)^j}{j!}\right)s^j \Leftrightarrow e^{-st} = \sum_{j=0}^\infty \left(\frac{(-s)^j}{j!}\right)t^j \qquad (4)$$

Because $\varepsilon(t)$ is bounded, multiplying (4) with $\varepsilon(t)$ one get an uniform convergent series in respect with *t*,

$$\varepsilon(t)e^{-st} = \sum_{j=0}^\infty s^j \frac{(-1)^j}{j!}t^j \varepsilon(t) \qquad (5)$$

which can be integrated term by term on the interval $t \in [0, \infty]$,

$$\int_0^\infty \varepsilon(t)e^{-st}dt = \sum_{j=0}^\infty m_j s^j = E(s) = L\{y(\infty) - y(t)\},\ \forall t \ge 0,\ \forall s \in C \qquad (6)$$

But

$$L\{y(\infty) - y(t)\} = \frac{y(\infty)}{s} - Y(s) \Rightarrow$$

$$Y(s) = \frac{y(\infty)}{s} - \sum_{j=0}^\infty m_j s^j \qquad (7)$$

3 Moments Method Based Identification

Suppose first that $y(\infty)$ is known and bounded. The transfer function that we want to identify is a stable rational function with non-minimum phase with unknown order and parameters: *n*, *m*, a_k, b_k and $a_0 \ne 0$, $b_0 \ne 0$.

$$H(s) = \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n} = \frac{M(s)}{L(s)}, \quad b_0 \ne 0;\ a_0 \ne 0;\ n \ge m \qquad (8)$$

If the input signal is a step function $U(s) = \frac{\Delta u}{s}$ then, if $y(\infty)$=finite $\Rightarrow a_0 \ne 0$ and, if $y(\infty) \ne 0 \Rightarrow b_0 \ne 0$, there are no poles and zeros in the origin of the complex plane.
In these conditions one can normalize the transfer function coefficients:

$$H(s) = K\frac{1 + b_1' s + \ldots + b_m' s^m}{1 + a_1' s + \ldots + a_n' s^n}, \quad b_k' = \frac{b_k}{b_0};\ a_k' = \frac{a_k}{a_0};\ k \ge 1;\ K = \frac{b_0}{a_0} \qquad (9)$$

Developing in power series of s the function $1/H(s)$:

$$\frac{1}{H(s)} = \frac{L(s)}{M(s)} = \sum_{k=0}^\infty C_k s^k,\ |s| < \sigma_z = R \qquad (10)$$

So

$$a_0 + a_1 s + \ldots + a_n s^n = (b_0 + b_1 s + \ldots + b_m s^m)(C_0 + C_1 s + \ldots)$$

and by identification term by term and normalising, one obtain:

$$a_k' = C_k' + b_1' C_{k-1}' + \ldots + b_j' C_{k-j}' + \ldots + b_k', \quad 0 \le k \le n \qquad (11)$$

$$0 = C_k' + b_1' C_{k-1}' + \ldots + b_j' C_{k-j}' + \ldots + b_m' C_k', \quad k \ge n+1 \qquad (12)$$

From

$Y(s)=H(s)U(s)$, $U(s)=\frac{\Delta u}{s}$ and

$$\frac{y(\infty)}{s} - H(s)\frac{\Delta u}{s} = \sum_{j=0}^\infty m_j s^j$$

one get

$$\frac{y(\infty)}{s} - H(s)\frac{\Delta u}{s} = \sum_{j=0}^\infty m_j s^j \qquad (13)$$

From (10) and (13) one obtain:

$$y(\infty)\sum_{k=0}^\infty C_k s^k - \Delta u = \left(\sum_{j=0}^\infty m_j s^{j+1}\right)\left(\sum_{k=0}^\infty C_k s^k\right) \qquad (14)$$

and by identification term by term:

$$C_0 = \frac{\Delta u}{y(\infty)} = \frac{1}{K} = \frac{a_0}{b_0}$$

$$C_k' = m_0' C_{k-1}' + m_1' C_{k-2}' + \ldots + m_{k-1}' \qquad (15)$$

Now, one can construct an identification algorithm in three steps:

Step 1: From input-output sampled data one calculate the moments m_j using the relation (2)

Step 2: From relations (15) one computes the coefficients C'_k. If there is k=n such as for $\forall k \geq n$, the system is compatible, then n is the denominator degree and the number of unknowns m is the nominator degree of the transfer function. This can be express in the following algebraic form:

$$D_k = \begin{vmatrix} C'_{k-m} & C'_{k-m+1} & \dots & -C'_k \\ C'_{k-m+1} & C'_{k-m+2} & \dots & -C'_{k-1} \\ \dots & \dots & & \dots \\ C'_k & C'_{k+1} & \dots & -C'_{k+m} \end{vmatrix} = 0, \qquad (16)$$

$\forall k \geq n+1$

Step 3: One computes the coefficients of the transfer function with the relations (11), (12) (firstly the coefficients $b'_k, 1 \leq k \leq m$, then $a'_k, 1 \leq k \leq n$).

Example: One consider a system described by the following transfer function:

$$H_1(s) = \frac{1}{4s^2 + 8s + 1}$$

The experimental step response is presented in Fig. 1.

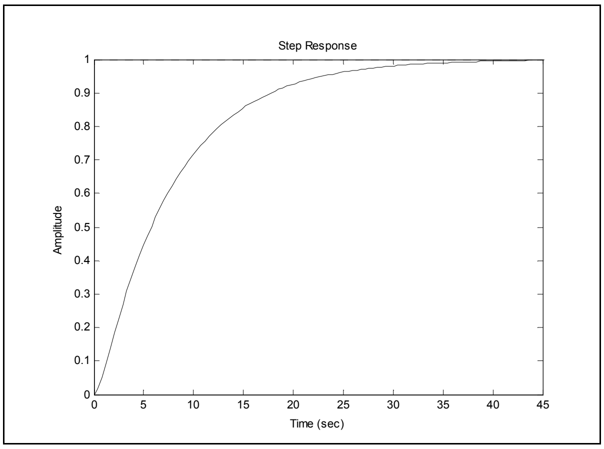

Fig. 1 Step response of $H_1(s)$

It is easy to observe that $y(\infty) = 1$. The values of the moments m_j and coefficients c'_k are:

$m_1 = 7.9999$ $\qquad c'_1 = 7.9999$

$m_2 = -60.0787$ $\qquad c'_2 = 3.9899$

$m_3 = 448.5854$ $\qquad c'_3 = -0.6795$

$m_4 = -3.3483e + 003$ $\qquad c'_4 = -0.6281$

Neglecting the small terms due to the approximation error, one obtain the following identified transfer function:

$$H_1(s) = \frac{1}{3.98s^2 + 7.99s + 1}$$

4 Weighted Moments Method Based Identification

As it was presented in previous sections, the *j*-power moment of an original function $f(t):[0,\infty) \rightarrow R$ is given by the following relation:

$$m_j = m_j(f) = \int_0^\infty \psi_j(t) f(t) dt \qquad (17)$$

where the function

$$\psi_j(t) = \frac{(-t)^j}{j!} \qquad (18)$$

represent a weight applied to the function $f(t)$ in integration.

In identification, the moments m_j calculus using relation (17) when value of j is great, creates a series of problems like:

- the weighting function $\psi_j(t)$ unbound;
- the integration is effectuated on a finite time interval;
- the f(t) function represents the overall response of a system which contains both the free component due to the initial conditions and the forced component on which the moments method is based.

In Fig. 2 are presented time evolution for few $\psi_j(t)$ functions. One observe that for small values of *t*, the f(t) signal in relation (17) is less amplified, but for big values of *t*, big weightings appear which can exceed the precision possibilities of the numerical calculus.

The convergence of the integrals from relation (17) is done by the condition $\sigma_0 < 0$. For slow processes the convergence radius is very small and the transient response is very long so, in the integrals evaluation components of the form $\psi_j(t) f(t)$, that is a product between a very large number $\psi_j(t),\ j \geq 2$, and a finite value f(t).

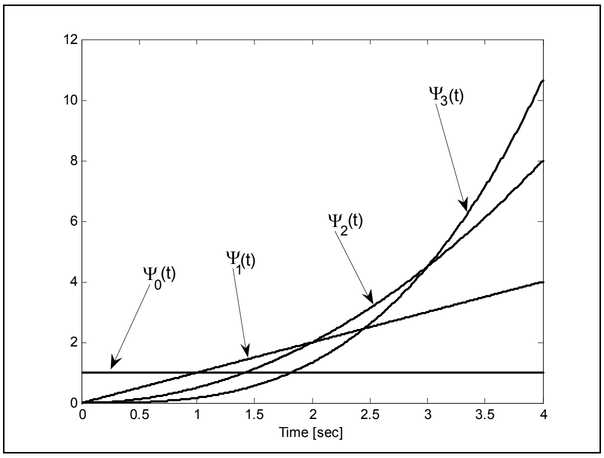

Fig. 2 Weighting functions for classical moments

On the other hand, if the integration is stopped to a finite moment t_2, the *j*-power moment of f(t) is approximated by:

$$m_j(t_0,t_2)=\int_{t_0}^{t_2}\psi_j(t)f(t)dt \quad (19)$$

and finite interval evaluation error:

$$\delta m_j(t_0,t_2)=m_j(0,\infty)-m_j(t_0,t_2) \quad (20)$$

is big even for a big value of t_2.
For these reasons one uses the weighted power moments that are defined by the following relation:

$$m_j^\alpha(f)=\int_0^\infty\frac{(-t)^j}{j!}w(t,\alpha)f(t))dt= \int_0^\infty\frac{(-t)^j}{j!}f^\alpha(t)dt \quad (21)$$

$w(t,\alpha)$ is an original functions family defined par rapport with t in $[0,\infty)$ interval, with parameter $\alpha\ge 0$ and the convergence radius $\sigma_1=\sigma_0$.
The time function $f^\alpha(t):[0,\infty)\to R$, defined by:

$$f^\alpha(t)=f(t)w(t,\alpha) \quad (22)$$

is an original function also, with convergence radius:

$$\sigma_2=\sigma_1+\sigma_0 \quad (23)$$

The order *j* weighted moment appear as a real functional of function *f*:

$$m_j^\alpha(f)=\int_0^\infty\psi_j(t,\alpha)f(t))dt \quad (24)$$

with the following kernel:

$$\psi_j(t,\alpha)=\frac{(-t)^j}{j!}w(t,\alpha) \quad (25)$$

The Laplace transform $F^\alpha(s)$ of $f^\alpha(t)$ function admits a power series expansion:

$$F^\alpha(s)=\sum_{j=0}^\infty f_j^\alpha s^j,\quad |s|<R_2 \quad (26)$$

uniform convergence and holomorphic inside of a circle with the radius:

$$R_2=|\sigma_2|=|\sigma_0+\sigma_1(\alpha)| \quad (27)$$

If $\sigma_0\le 0$ and $\sigma_1=\sigma_1(\alpha)\le 0,\ \forall\alpha$, then:

$$R_2=|\sigma_0+\sigma_1|=|\sigma_0|+|\sigma_1(\alpha)|=R+|\sigma_1(\alpha)|>R \quad (28)$$

So, choosing a weighting $w(t,\alpha)$ with $\sigma_1=\sigma_1(\alpha)\le 0$ one obtain a series with a larger convergence radius, that has obvious advantages for numerical calculus.
The weighted moments contain the same information about f function as the classical moments but they have a better numerical robustness due to the condition imposed to convergence radius.
There are many possibilities for choosing the weighting function $w(t,\alpha)$ using efficiency and applicability criterions. In this work we consider:

$$w(t,\alpha)=e^{-\alpha t},\ \alpha\ge 0,\ \sigma_1=-\alpha\le 0 \quad (29)$$

For this weighting function, one get:

$$R_2=|\sigma_0-\alpha| \quad (30)$$

$$m_j^\alpha(f)=\int_0^\infty\frac{(-t)^j}{j!}e^{-\alpha t}f(t))dt \quad (31)$$

$$\psi_j(t,\alpha)=\frac{(-t)^j}{j!}e^{-\alpha t} \quad (32)$$

$$F^\alpha(s)=F(s+\alpha),\ \mathrm{Re}(s)>\sigma_0-\alpha \quad (33)$$

Unlike of $\psi_j(t)$ presented in Fig. 2, the kernel $\psi_j(t,\alpha)$ presented in Fig. 3 for some values of *j*, realize a weighting at the moment *t*:

$$t_j=\frac{j}{\alpha},\ j\ge 0 \quad (34)$$

with the amplitude

$$\psi_j(t_j,\alpha)=\frac{j!}{j!\alpha^j}e^{-j} \quad (35)$$

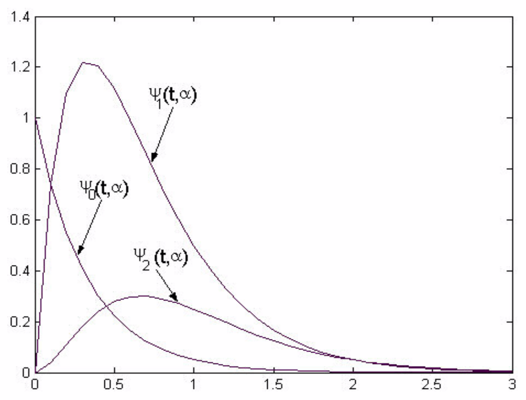

Fig. 3 The kernel $\psi_j(t,\alpha)$

Depending on value of α parameter, the weighted moment give more importance to some time evolutions. For a big α, the attention is focus on

initial evolution and for a small α, towards the final evolution. By choosing many values for α in identification procedures one get the essential aspects of time evolution of the signal *f(t)*.
The identification algorithm remains the same, the only modification is that we will process the weighting function $f^{\alpha}(t)$ instead of *f(t)*. Also, instead of the original transfer function:

$$H(s) = \frac{b_0 + b_1 s + ... + b_m s^m}{a_0 + a_1 s + ... + a_n s^n} = \frac{Y(s)}{U(s)} \quad (36)$$

characterized by n, m, b_k, a_k, we will obtain the following transfer function:

$$H^{\alpha}(s) = \frac{b_0^{\alpha} + b_1^{\alpha} s + ... + b_m^{\alpha} s^m}{a_0^{\alpha} + a_1^{\alpha} s + ... + a_n^{\alpha} s^n} = \frac{Y^{\alpha}(s)}{U^{\alpha}(s)} \quad (37)$$

characterized by $n, m, a_k^{\alpha}, b_k^{\alpha}$.
When $w(t,\alpha) = e^{-\alpha t}$, then

$$H^{\alpha}(s) = H(s + \alpha) \quad (38)$$

Example: One considers a system described by the following transfer function:

$$H_2(s) = \frac{s}{4s^2 + 2s + 1}$$

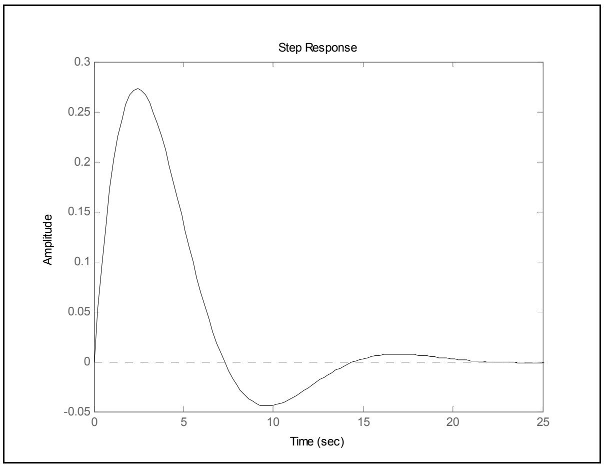

Fig. 4 Step response of $H_2(s)$

As it can be seen, $y(\infty) = 0$. Using the identification algorithm one get: $K^{\alpha} = 0.0483$, $a_1^{\alpha} = 0.6147$, $a_2^{\alpha} = 0.0992$.

One obtains the following estimated transfer function:

$$H_2(s) = \frac{s}{4.083 s^2 + 2.0515 s + 1}$$

5 Experimental Results

To illustrate the performance of the proposed identification algorithms, one a real Quanser experiment using a DC servomotor with built in gearbox, is provided in this section. The "rotational series" that we have is the SRV-02ET (E-encoder, T-tachometer), and the DC servo is shown in the Figure 3. A high quality DC servo motor is mounted in a solid aluminum frame. The motor drives a built-in Swiss-made 14:1 gearbox whose output drives an external gear. The motor gear drives a gear attached to an independent output shaft that rotates in a precisely machined aluminum ball bearing block.

The output shaft is equipped with an encoder. This second gear on the output shaft drives an anti-backlash gear connected to a precision potentiometer. The potentiometer is used to measure the output angle. The external gear ratio can be changed from 1:1 to 5:1 using various gears. Two inertial loads are supplied with the system in order to examine the effect of changing inertia on closed loop performance. In the high gear ratio configuration, rotary motion modules attach to the output shaft using two 8-32 thumbscrews. The square frame allows for installations resulting in rotations about a vertical or a horizontal axis.

Fig. 5 SRV02ET

The system is interfaced by means of a data acquisition card and driven by Matlab/Simulink based real time software. The model of this system can be found from physical considerations. One considers as input U of the system the voltage applied to the motor armature and as output Y of the system the angle of the output shaft. As is described in [12] the transfer function of the system has the following the form:

$$\frac{Y(s)}{U(s)} = \frac{K_p}{s(T_p s + 1)} \quad (39)$$

Clearly, the open loop position response of the DC motor is unstable due to the pole at the origin. A proportional controller in closed loop is used in order to stabilize the system and to perform the identification experiments. The closed loop transfer function is:

$$\frac{Y(s)}{V(s)} = \frac{K_{cl}}{T_p s^2 + s + K_{cl}} \quad (40)$$

or, equivalently:

$$\frac{Y(s)}{V(s)} = \frac{1}{a_2 s^2 + a_1 s + 1} \quad (41)$$

where:

$$a_1 = \frac{1}{K_{cl}},\ a_2 = \frac{T_p}{K_{cl}}, K_{cl} = K_p \cdot K_c \quad (42)$$

K_c - controller gain.

In Fig. 6 the experimental step response of the closed-loop system with controller gain $K_c=0.2$ is presented.

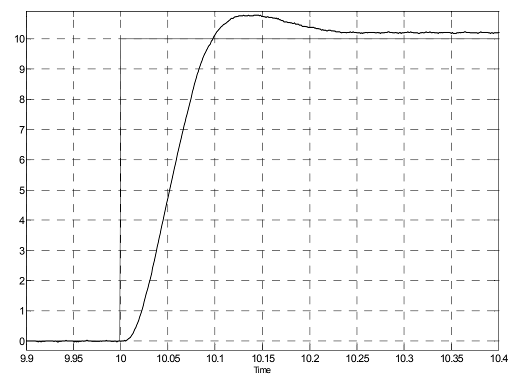

Fig. 6 Step response of the closed-loop system with controller gain $K_c=0.2$

Using the algorithm described above one can identify the a_1 and a_2 parameters and then one can deduce T_p and K_p. One obtains the following values:

$$a_1 = 0.0412; a_2 = 0.00095$$

and from relations (42) one get:

$$K_{cl} = 24.2584,$$
$$K_p = 121.2919,$$
$$T_p = 0.0232,$$

and, the identified DC motor transfer function:

$$H(s) = \frac{121.2919}{s(0.0232s + 1)}$$

6 Conclusions

In this paper we presented a novel method for continuous-time invariant system identification. The method has two main advantages related to most methods found in the specialty literature: first, there is no need of *apriori* information about the structure of the identified system and second, the continuous-time model is obtained direct from the input-output sampled data. The algorithm determines the order of the system from the relation between the power moments and the transfer function coefficients. The method was applied to identify the parameters of a real experimental platform consisting of a DC servomotor with built in gearbox. Because the open loop system is unstable, the closed loop identification was performed.

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