Bifurcations, transition to turbulence and development of chaotic regimes for double-diffusive convection.

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Abstract: - For double-diffusive convection the sequence of bifurcations from stationary motion to stochastic motion was investigated. Attractor has the structure of a Mobius band in chaotic regimes. With the help of Poincare sections and Poincare maps modification of the attractor were illustrated. First, Poincare map can be represented as a one-valued function, then with the growth of supercriticality Poincare map remains one-dimensional but with many minima and self-intersections so it can't be approximated with some function. Relative residual was calculated for all the calculations, so we can affirm that the properties of the solutions presented practically exactly correspond to the Navier-Stokes equations (third range of accuracy).

Key-Words: - turbulence, convection, double-diffusive, stochastic regimes, strange attractors, bifurcations

1 Introduction

At present time there is a great many studies devoted to transition to chaos in dynamical systems. A string impulse to this domain was made by fundamental works by Ruelle and Takens [1] and famous example of Lorenz [2], which is obtained by approximation of Navier-Stockes equations with first harmonics after application of Bubnov-Galerkin method. Despite the beauty and importance of Lorenz attractor, it has nothing to do with the initial physical problem, all the properties where thrown out with the higher harmonics. Till today we don't have many investigations of attractors of hydrodynamical systems with full Navier-Stockes equation. In the present work for double-diffusive convection properties of the attractors are investigated. During the calculations we estimated the relative residual after substitution of the solution to the initial system of equations. The relative residual was about 10^{-3}.

2 Problem Formulation

Let the temperature and the salinity to be supported constant on the boundaries of the plane layer $z=0$ and $z=H$, the acceleration of gravity is directed along axis z. Let H and H^2/k_T be considered as the units of length and time, where k_T is the coefficient of the thermal diffusion. To describe the dependence of the density of liquid on the temperature T and salinity the S let us take linear law $\rho=\rho_0(1-\alpha T+\beta S)$. In two-dimensional problem for the components of the velocity one can introduce the nondimensional stream function ψ: $v_x = k_T\psi_z/H,\ v_z=-k_T\psi_x/H$. Hereafter coordinates x and z and time t are supposed to be nondimensional. Instead of the temperature and the salinity let us introduce nondimensional variables τ, s: $T=T_0+(T_1-T_0)(1-z+\tau),\ \ S=S_0+(S_1-S_0)(1-z+s)$, where $T_0,T_1;\ S_0,S_1$ mean temperature and salinity on the layer boundaries (for $z=0$ and $z=1$ accordingly). Closed system of equations for double-diffusive convection in plane case has the following form

$$\Delta\psi_t+\sigma(R_T\tau_x-R_S s_x-\Delta^2\psi)=J(\psi,\Delta\psi);$$

$$\tau_t+\psi_x-\Delta\tau=J(\psi,\tau);$$

$$s_t+\psi_x-k\Delta s=J(\psi,s).$$

In this system four nondimensional parameters are introduced - the Rayleigh number for temperature $R_T\equiv\alpha g(T_1-T_0)H^3/(k_T\nu)$, the Rayleigh number for salinity $R_S=\beta g(S_1-S_0)H^3/(k_T\nu)$, the ratio of the coefficients of salt diffusion and temperature K_S/K_T, the Prandtl number $\sigma\equiv\nu/k_T$. Jacobian $J(f,g)$ is determined by formula $J(f,g)\equiv f_x g_z - f_z g_{,x}$. For $z=0$ and $z=1$ the Rayleigh boundaries conditions of absence of tangential viscosity forces are taken. So, the full system of boundary conditions is:

$\psi=\Delta\psi=\tau=s=0$ при $z=0,1$.

Results of some numerical computations of the boundary problems, periodic with respect to x can be found in fundamental work [3].

2.1 Method of investigation.

Unlike the work [3] let us seek the solution of the initial system by Bubnov-Galerkin method in the form, satisfying the boundary condition:

$$\psi_N = \sum_{i,j} \psi_{ij}(t) \sin(i\pi x/\sqrt{2}) \sin j\pi z,$$

$$\tau_N = \sum_{i,j} \tau_{ij}(t) \cos(i\pi \frac{x}{\sqrt{2}}) \sin j\pi z,$$

$$s_N = \sum_{i,j} s_{ij}(t) \cos(i\pi \frac{x}{\sqrt{2}}) \sin j\pi z.$$

Here the summing over all pares (i,j) of integer non-negative numbers with even sum $i+j=2n$, and $i^2+j^2 \le (2N)^2$ is done. The length of the period along x axis corresponds to the length of the wave, for which the static solution loses stability with minimal Rayleigh number, is $R_T = 27\pi^4/4$. For $N=1$ we have an approximation of the solution with two space harmonics.

Corresponding dynamic systems (accordingly to the terminology of the authors [4]) belong to the systems of the hydrodynamic type possessing the property of decreasing of the phase volume with time growth.

2.2 Evaluation of the convergence of Boubnov-Galerkin approximation.

For the evaluation of the convergence of the solutions of the system, where functions $\psi_{ij}, \tau_{ij}, s_{ij}$ satisfy the systems of the hydrodynamic type, for the solutions of the original system let us introduce an energetic norm of the solution $E(t)$, which is proportional to kinetic energy of the
liquid of a cell of periodicity $0 < z < 1;\ 0 < x < 1/\sqrt{2}$.
For the approximation with $2N$ space harmonics we have the following approximate expression for $E(t)$:

$$E_N(t) = \sum_{i,j} \psi_{ij}^2(t)(\frac{i^2}{2} + j^2), \quad i^2 + j^2 \le (2N)^2.$$

For the evaluation of the convergence in Sobolev space $W_2^1 = \sqrt{<E>} + \sqrt{<D>}$ (symbol <> means time average along fixed trajectory of dynamic system) the norm, which is proportional to the rate of viscous dissipation of the energy is useful:

$$D_N(t) = \sum_{i,j} \psi_{ij}^2(t)(\frac{i^2}{2} + j^2)^2; \quad i^2 + j^2 \le (2N)^2.$$

3 Investigation of properties of the solution

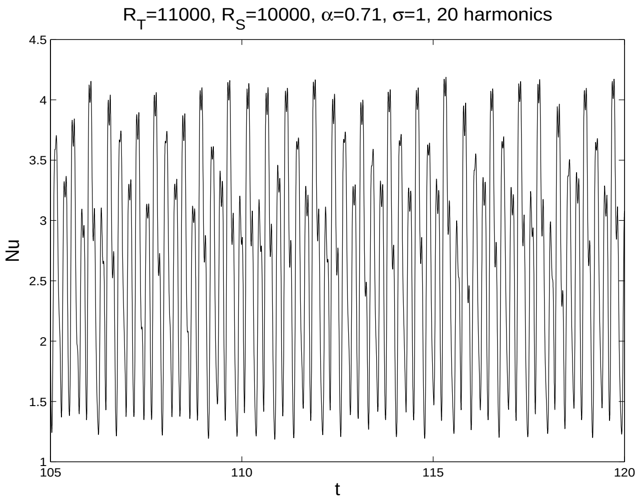

Fig.1

In figure 1 stochastic regimes for moderate supercriticality are demonstrated using Nusselt number (that is flow of temperature through the surface). This figure and figure 2 for higher supercriticality are given here to provide insight for what kind of chaotic motion we will present pictures of attractors.

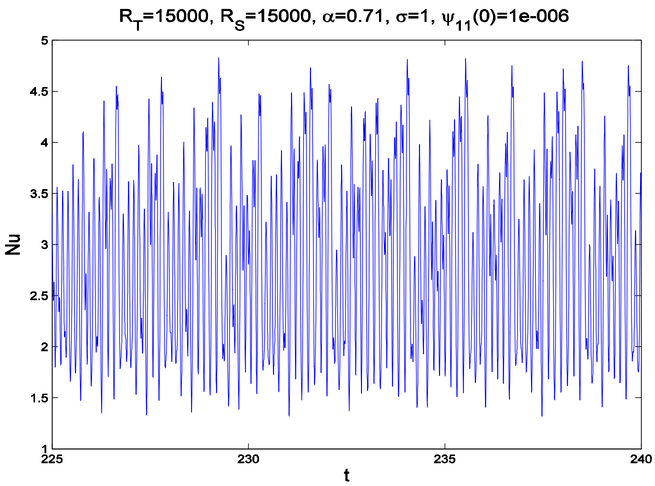

Fig.2

One of the most important characteristics of turbulent motions and fundamental notion in dynamical systems is sensitivity on the initial values. Figure 3 shows haw two close trajectories diverge in stochastic regime (actually tow close trajectories on the attractor).

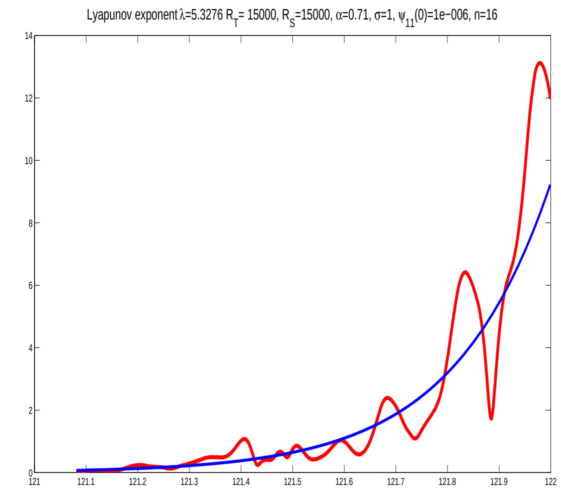

Fig.3

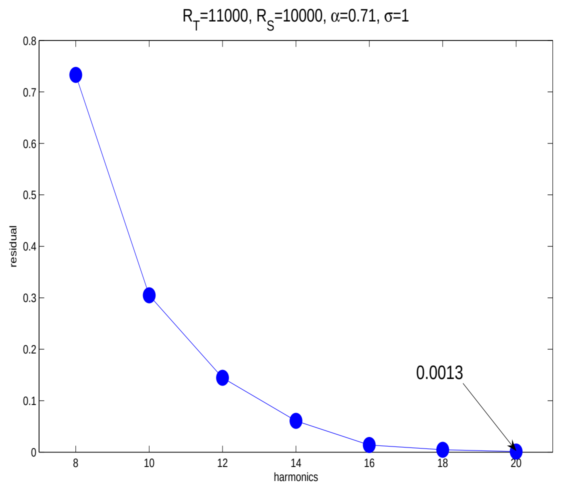

Fig.4

Method of Bubnov-Galerkin has a very agreeable benefit over finite element method for this type of problems. We can estimate relative residual of the initial model after the substitution of the numerical solution. For finite elements method this is not easy because we must calculate time derivatives for evaluation of the residual. In figure 4 you can see the dependency of the residual on the number of harmonics.

Figures 5 and 6 give insight on the structure of the attractor. By investigation of the transversal section we saw that it always has structure of Mobius band. (Attractor of cause is not exactly two-dimensional and can have only fractal dimension) With the growth of Rayleigh number after loosing of stability of stationary motion limit cycle appears. Then, with further increase of supercriticality, periodical motion loses stability.

The projection of limit cycle on plane (ψ_{11}, τ_{11}) appears to be closed no self-intersected curve, diffeomorphic to circle up to values of parameter $R_T = 8870$. For this value of parameter R_T bifurcation of period doubling of limit cycle takes place. The cascade of bifurcations can be understood better with the help of succession mapping (mapping of Poincare). Let us transversally (locally) intersect the limit cycle by hyperplane and points of intersection denote M_j, $j = 1, 2, \ldots$. For the small perturbations of the dynamic system around the limit cycle we will get linear system with periodic coefficients. Accordingly to the Floquet theorem there exist a linear substitution of variables with periodic matrix which reduces the system with periodic coefficients to the system of linear equations with constant coefficients. Roots of corresponding characteristic equation will be eigenvalues (multiplicators) of the matrix of monodromy of system with periodic coefficients. If all the multiplicators are different and less than one in absolute value, then limit cycle is stable. If one of the multiplicators with the parameter growth traverses -1, then locally monodromy transformation in transversal plane (using the possibility of linear transformation of parameter and variable x) may be represented as follows $x_{j+1} = -x_j(1+\lambda) + x_j^2 + \beta x_j^3$.

For $\lambda < 0$ mapping of succession converges to zero, corresponding to stable limit cycle. For $\lambda > 0$ sequence converges to cycle of two points $x_{1,2} = \pm\sqrt{\lambda/(1+\beta)}$. So for $\lambda > 0$ stable limit cycle of doubled period appears.

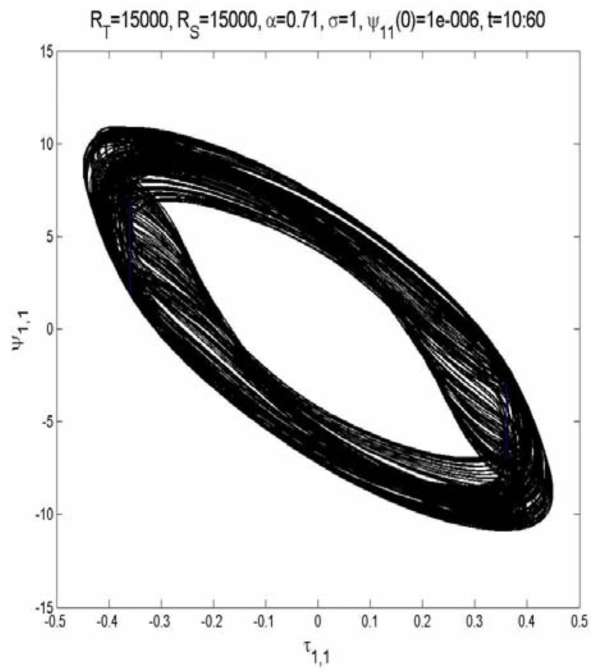

Fig. 5

Figure 7 represents Poincare map after two bifurcations of period doubling. In figure 8 we have Poincare map for stochastic regime. In [5] you can find more details on sequence of bifurcations. With the further increase of R_T the reverse process begins, the Mobius band is cut along itself (by cutting the Mobius band we again obtain the Mobius band due to non-orientability of the surface). Finally we arrive to periodic solution. Appearance of periodic solutions is close to the assertion of the Schakowsky theorem (see the discussion in [6]). After loosing of stability of this periodic solution new sequence of bifurcations begins which again leads to attractor in the form of Mobius band, but this time Poincare map can't be represented in the form of a one-valued function and has many minima and self-intersection, so beautiful theory of Feigebaum [7] can't be applicable any more.

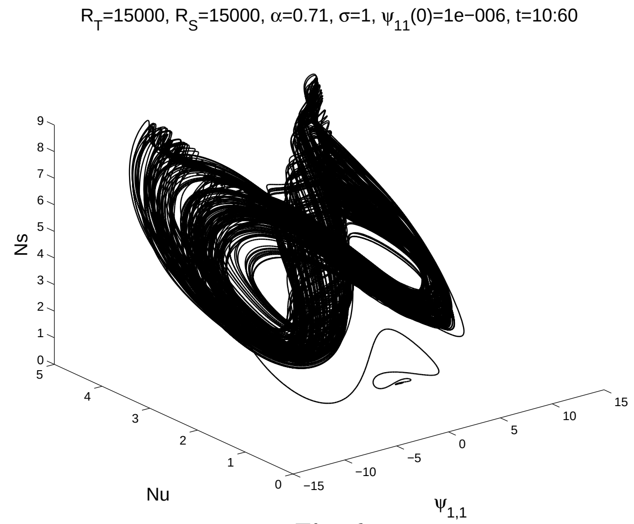

Fig. 6

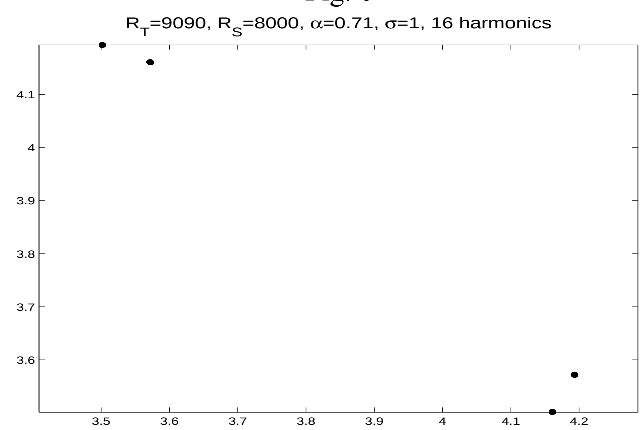

Fig. 7

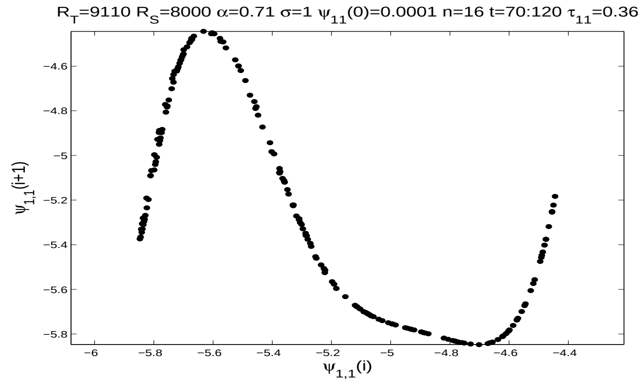

Fig. 8

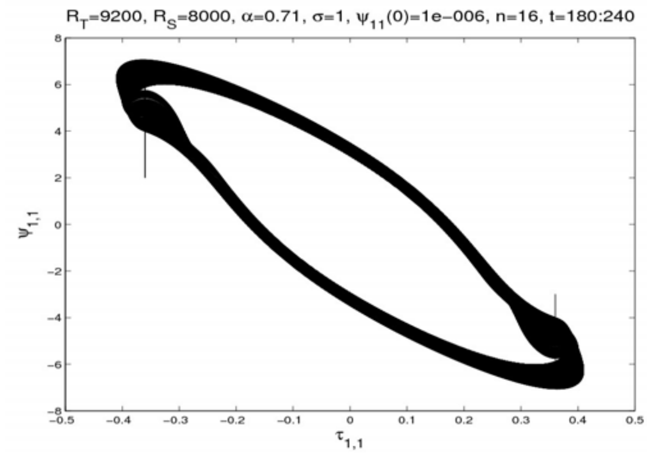

Fig. 9

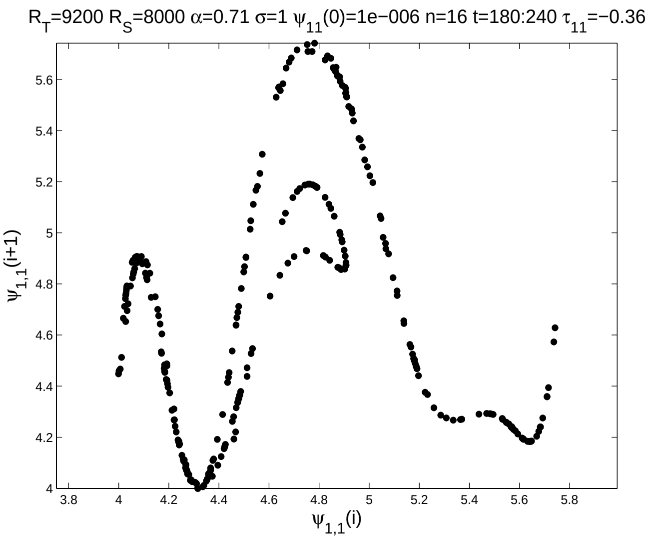

Fig. 10

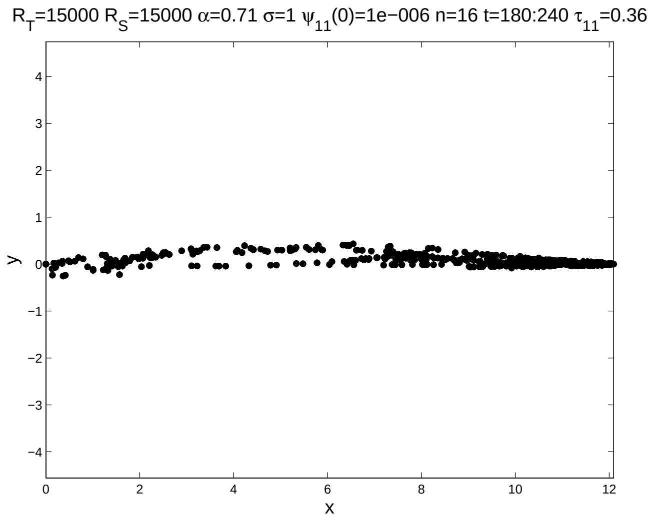

Fig. 11 Poincare section in projection to the plane (x,y) where x and y are direction with the greatest scattering of points. Figure 12 corresponds to this section.

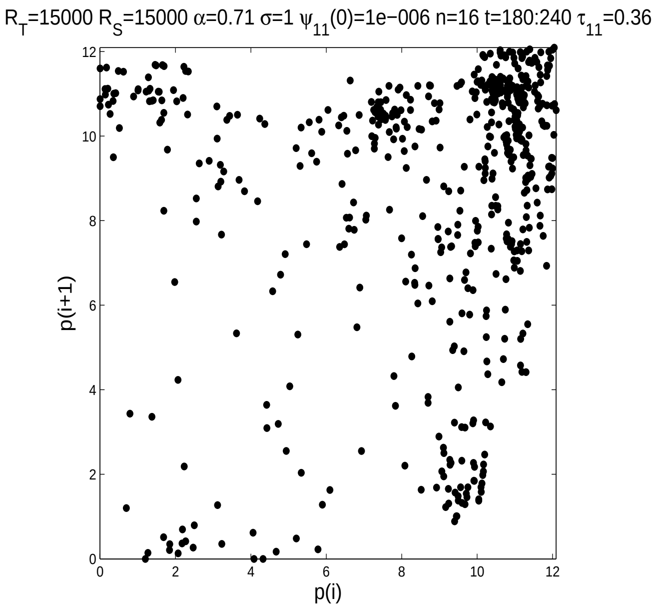

Fig. 12

Figure 12 shows Poincare map for higher supercriticality. From the first glance it may appear that here there is no structure, just a cloud of points. But more accurate consideration with higher number of points allows to see that the structure of Poincare map remains one-dimensional.

4 Conclusion

Structure and development of attractors for double-diffusive convection were demonstrated. Attractor has the form of a Mobius band. Positive Liapunov exponent on the attractors was demonstrated. Modifications of Poincare map with the growth of supercriticality were shown. The convergence of Bubnov-Galerkin method was demonstrated in norms of kinetic energy and dissipation function. Dependence of relative residual on the number of harmonics was shown. So the described properties of the solution adequately correspond to the initial model of double-diffusive convection.

References:

[1] Ruelle, D., Takens, F. On the Nature of Turbulence // *Comm. Math. Phys.*, Volume 20, p. 167-192

[2] Lorentz E.N. Deterministic Nonperiodic Flow // *J. Atmos. Sci.* 1963. 20.130–141.

[3] Huppert E. N., Moore P. R. Nonlinear double-diffusive convection // *J. Fluid Mechanics*. 1976.78. 821-854.

[4] F.V. Dolzhanskii, V.I. Kliatskin, A.M. Obuhov, A.M. Chusov The nonlinear systems of hydrodynamical type// Nauka, Moscow, 1974

[5] Sibgatullin, I. N.; Gertsenstein, S. Ja.; Sibgatullin, N. R. Some properties of two-dimensional stochastic regimes of double-diffusive convection in plane layer // *Chaos*, Vol. 13, Issue 4, p.1231

[6] G.M. Zaslavski, R.Z. Sagdeev Introduction to nonlinear physics. From a pendulum to turbulence and chaos// Nauka, Moscow, 1988.

[7] Feigenbaum M.J. Quantitative universality for a class of nonlinear transformations // *J. Statist. Phys.* 1978.21.25-52.