# A Sudden Explosion in a Cylindrical Cavity

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*Abstract*: A fundamental solution, given by Kadioglu and Ataoglu, has been revised to be used in reciprocal theorem for the solutions of axially symmetric transient problems of elastodynamics. And using this in reciprocal identity, the boundary values of a sample problem, representing a sudden explosion in a cylindrical cavity, have been obtained. To calculate the inner values of the displacement field of any axially symmetric problem a second elastodynamic state has been also derived. And, this new elastodynamic state has been used in reciprocity theorem to compute the inner values of the displacement of the sample problem.

*Key-Words* - reciprocity theorem, transient load, axially symmetric, fundamental elastodynamic state, sudden explosion

## 1 Introduction

The dynamic reciprocal identity, sometimes also referred to as the dynamic Betti-Rayleigh theorem presents an integral equation between two elastodynamic states of the same body. The definition of elastodynamic state was introduced by Wheeler and Sternberg [1]. The first elastodynamic state in the expression of the reciprocal theorem represents a problem to be solved as the second one represents the displacement and stress field in an unbounded medium due to a sudden application of a time-dependent point load. Second state is also named as the fundamental solution or a singular state. These definitions have also been given by Achenbach [2]. The resulting integral equation mentioned above is also the starting point of the boundary element method. An analytical solution based on the reciprocity theorem is presented for both spherically and axially symmetric problems by Kadioglu and Ataoglu [3], [4]. Here, the fundamental state given by Kadioglu and Ataoglu [4] has been revised eliminating some zero terms. And, using this as the first elastodynamic state in reciprocity theorem, an integral equation has been obtained for a sample problem, representing a sudden explosion in a cylindrical cavity. The unknown of this integral equation is boundary value of the radial displacement. Solving this integral equation, numerically, the variation of radial displacement versus time have been determined. But, the first elastodynamic state mentioned above cannot be used to find out the radial displacement at any point inside the region. Then, a second elastodynamic state has also been derived for this purpose. The inner values of the sample problem has been obtained writing the reciprocity theorem between this second elastodynamic state and the sample problem.

## 2 Basic Formulations

If a region with interior volume $V$ and boundary $S$ is considered, the ordered triple $\mathcal{S}[\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{f}]$ defines an elastodynamic state on $(\overline{V} \times \mathcal{T})$, where $\overline{V}$ is the closure of $V$ and $\mathcal{T}$ is an arbitrary interval of time. $\boldsymbol{u}(\boldsymbol{x}, t)$ is the displacement vector and $\boldsymbol{x}$, $t$, $\boldsymbol{f}$ denote the position vector of an arbitrary point, time, body-force density, respectively. They satisfy the following relations.

$$\tau_{ij,j} + \rho f_i = \rho \ddot{u}_i \tag{1}$$

$$\tau_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) \tag{2}$$

where $\rho$, $\lambda$ and $\mu$ denote the density of the material filling $V$ and Lamé' s elastic constants, respectively. $\ddot{\boldsymbol{u}}$ is the second time derivative of $\boldsymbol{u}$ and $\delta_{ij}$ represents Kronecker' s delta. Summation convention is valid. The expression of the dynamic reciprocal identity which is written between two elastodynamic state $\mathcal{S}[\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{f}]$ and $\mathcal{S}^*[\boldsymbol{u^*}, \boldsymbol{\tau^*}, \boldsymbol{f^*}]$ is

$$\int_S \boldsymbol{T} * \boldsymbol{u^*}\, dS + \rho \int_V \boldsymbol{f} * \boldsymbol{u^*}\, dV$$

$$= \int_S \boldsymbol{T^*} * \boldsymbol{u}\, dS + \rho \int_V \boldsymbol{f^*} * \boldsymbol{u}\, dV \tag{3}$$

.

$$T_i = \tau_{ij} n_j, \quad T_i^* = \tau_{ij}^* n_j \tag{4}$$

$\boldsymbol{T}$ and $\boldsymbol{T^*}$ are surface traction vectors in two states, respectively, and $\boldsymbol{n}$ is the outward normal of the surface $S$. Sign $*$ represents Riemann convolution as follows:

$$\boldsymbol{f}(\boldsymbol{x}, t) * \boldsymbol{g}(\boldsymbol{x}, t)$$

$$= \int_0^t f_i(\boldsymbol{x}, t-\tau) g_i(\boldsymbol{x}, t-\tau) d\tau \tag{5}$$

From now on, it is accepted that $\mathcal{S}[\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{f}]$ represents a first boundary value problem to be solved. And the body force $\boldsymbol{f}$ will be neglected. The second elastodynamic state $\mathcal{S}^*(\boldsymbol{u^*}, \boldsymbol{\tau^*}, \boldsymbol{f^*})$ represents the displacement and stress fields due to a sudden application of a time-dependent load $\boldsymbol{f^*}$ in an infinite medium. This state is also named as a fundamental solution. There are two steps in finding the solution of an elastodynamic problem by reciprocal identity. At first, boundary values of the displacement vector and the unknown stress component must be determined. After this, using boundary values, displacement and stress fields at interior points are calculated. In general, a singular body force is considered for the construction of necessary fundamental solution. But, here, for both steps, a different fundamental elastodynamic state is used for the solutions of axially symmetric problems via reciprocity theorem. The first state which is proper for finding boundary values in an axial symmetric problem via reciprocity theorem is given by Kadioglu and Ataoglu [4]. But this state cannot be used for determining the displacement and stress fields at interior points. Then here a second fundamental state is also constructed for this purpose.

## 3 The First Elastodynamic State

This state $\mathcal{S}'[\boldsymbol{u'}, \boldsymbol{\tau'}, \boldsymbol{f'}]$ will be used to determine the boundary values of an axial symmetric, first boundary value problem under transient loads. For convenience cylindrical coordinates $(R, \theta)$ [5] have been used. $\boldsymbol{e_R}$ and $\boldsymbol{e_\theta}$ denote the base vectors in this coordinate system. In an infinite medium, a distributed body force in the direction of $\boldsymbol{e_R}$ at every point over the cylindrical surface with zero radius is considered. At any point $\boldsymbol{x}(R, \theta)$, the nonzero components of the displacement vector, strain tensor $\boldsymbol{\epsilon'}$ and stress tensor due this body force acting at the origin are

$$u'_R = \frac{1}{4\pi}\{H(t - \frac{R}{c_1})\frac{1}{c_1}(\frac{-R}{\sqrt{c_1^2 t^2 - R^2}^3})$$

$$+ \delta(t - \frac{R}{c_1})\frac{1}{c_1^2}(\frac{1}{\sqrt{c_1^2 t^2 - R^2}})\} \tag{6}$$

$$\epsilon'_{RR} = \frac{1}{4\pi}\{H(t - \frac{R}{c_1})\frac{1}{c_1}(-\frac{1}{\sqrt{c_1^2 t^2 - R^2}^3}$$

$$- \frac{3R^2}{\sqrt{c_1^2 t^2 - R^2}^5})$$

$$+ \delta(t - \frac{R}{c_1})\frac{1}{c_1^2}(\frac{2R}{\sqrt{c_1^2 t^2 - R^2}^3})\}$$

$$+ \dot{\delta}(t - \frac{R}{c_1})\frac{1}{c_1^3}(-\frac{1}{\sqrt{c_1^2 t^2 - R^2}})\} \tag{7}$$

$$\epsilon'_{\theta\theta} = \frac{1}{4\pi}\{H(t - \frac{R}{c_1})\frac{1}{c_1}(-\frac{1}{\sqrt{c_1^2 t^2 - R^2}^3})$$

$$+ \delta(t - \frac{R}{c_1})\frac{1}{c_1^2}(\frac{1}{R\sqrt{c_1^2 t^2 - R^2}})\} \tag{8}$$

$$\tau'_{RR} = \lambda(\epsilon'_{\theta\theta}) + (\lambda + 2\mu)\epsilon'_{RR} \tag{9}$$

$$\tau'_{\theta\theta} = \lambda(\epsilon'_{RR}) + (\lambda + 2\mu)\epsilon'_{\theta\theta} \tag{10}$$

where $H(t)$, $\delta(t)$ are the Heaviside' s unit step function and Dirac delta function, respectively. They satisfy the following relations.

$$\int_a^b \delta(x)f(x)dx =$$

$$\begin{cases} [H(a) - H(b)]f(0) & x \in [b,a] \\ 0 & x \notin [b,a] \end{cases} \tag{11}$$

$$H(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases} \tag{12}$$

$$\frac{dH(x)}{dx} = \delta(x), \;\; \dot{\delta}(x) = \frac{d\delta(x)}{dx} \tag{13}$$

$$\delta(x)f(x) = 0 \;\; for \;\; f(0) = 0 \tag{14}$$

$$\dot{\delta}(x)f(x) = 0 \;\; for \;\; \dot{f}(0) = 0 \tag{15}$$

And $c_1$ is the velocity of $P$ wave, defined as

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{16}$$

Eqs. (6), (7) and (8) have been taken from Kadioglu and Ataoglu [4] but the terms being equal to zero have been ignored. Besides $\dot{f}$ represents the derivative of the function $f$ with respect to the argument of $f$.

## 4 A Second Elastodynamic State

After finding the boundary values, the first elastodynamic state $S'(\boldsymbol{u'}, \boldsymbol{\tau'})$ cannot be used for calculations of the displacement and stress components on the inner points. For this another elastodynamic state $S''(\boldsymbol{u''}, \boldsymbol{\tau''})$ state must be constructed. If in an infinite plane region, a body force, having the magnitude $\delta(t)$, acting at a specific point $\boldsymbol{y}$, in the direction of the Cartesian base vector $\boldsymbol{e_k}(k = 1, 2)$, exists, the displacement vector $\boldsymbol{u^k}$ due to this body force at an arbitrary point $\boldsymbol{x}$ and at the time $t$ can be expressed as [2]

$$\boldsymbol{u^k}(\boldsymbol{x}, t) = \nabla\nabla.\boldsymbol{F^k} - \nabla \wedge \nabla \wedge \boldsymbol{G^k} \tag{17}$$

where

$$\boldsymbol{F^k} = \frac{1}{4\pi}\{H(t - \frac{\rho}{c_1})\,[t\,ln(\frac{c_1 t + \sqrt{c_1^2 t^2 - \rho^2}}{\rho})$$

$$- \frac{\sqrt{c_1^2 t^2 - \rho^2}}{c_1}\;]\}\boldsymbol{e_k} \tag{18}$$

$$\boldsymbol{G^k} = \frac{1}{4\pi}\{H(t - \frac{\rho}{c_2})\,[t\,ln(\frac{c_2 t + \sqrt{c_2^2 t^2 - \rho^2}}{\rho})$$

$$- \frac{\sqrt{c_2^2 t^2 - \rho^2}}{c_2}\;]\}\boldsymbol{e_k} \tag{19}$$

here $\rho$ is the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ points and $c_2$ denote the velocity of $S$ wave as given below:

$$\rho = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{20}$$

$$c_2 = \sqrt{\frac{\mu}{\rho}} \tag{21}$$

Since this new elastodynamic state $S''[\boldsymbol{u''}, \boldsymbol{\tau''}, \boldsymbol{f''}]$ is suitable for axial symmetric problems, from now on, cylindrical coordinates $(R, \theta)$ will be used. The base vector $\boldsymbol{e_{R_1}}$ at the point $\boldsymbol{y}(R_1, \theta_1)$ can be expressed in terms of the base vectors $\boldsymbol{e_R}$ and $\boldsymbol{e_\theta}$ at an arbitrary point $\boldsymbol{x}(R, \theta)$ as:

$$\boldsymbol{e_{R_1}} = \boldsymbol{e_R}\,cos(\theta_1 - \theta) + \boldsymbol{e_\theta}\,sin(\theta_1 - \theta) \tag{22}$$

To replace $\boldsymbol{e_k}$ to $\boldsymbol{e_{R_1}}$ is enough in Eqs. (18) and (19) to determine the displacement and stress fields due to a time dependent point load at $\boldsymbol{y}(R_1, \theta_1)$ in the $R$ direction. Performing this operation $\boldsymbol{F^{R_1}}(\boldsymbol{x}(R, \theta), t)$ function corresponding to this load can be expressed in cylindrical coordinates as

$$\boldsymbol{F^{R_1}} = \frac{1}{4\pi}\{H(t - \frac{\rho}{c_1})\frac{1}{c_1}[f(\rho, t)]\}$$

$$\{\boldsymbol{e_R}\,cos(\theta_1 - \theta) + \boldsymbol{e_\theta}\,sin(\theta_1 - \theta)\} \tag{23}$$

where

$$f(\rho,t) = c_1 t\, ln(\frac{c_1 t + \sqrt{c_1^2 t^2 - \rho^2}}{\rho})$$

$$- \sqrt{c_1^2 t^2 - \rho^2} \quad (24)$$

$$\rho = \sqrt{R^2 + R_1^2 - 2RR_1 cos(\theta_1 - \theta)} \quad (25)$$

Now a distributed load with magnitude $\delta(t)/2\pi R_1$ over the circle with radius $R_1$ will be considered. The direction of this load is $\boldsymbol{e_{R_1}}$ at every point. Dividing Eq. (23) by $2\pi R_1$ and integrating over the circle of radius $R_1$ is enough to determine the $\boldsymbol{F}''(\boldsymbol{x},t)$ function corresponding to this load. Performing these the first form of this function becomes

$$\boldsymbol{F}''(\boldsymbol{x},t) = \frac{1}{4\pi^2}\{\int_{\theta-\beta}^{\theta+\beta}[H(t - \frac{\rho}{c_1})\frac{1}{c_1}\, f(\rho,t)$$

$$][\boldsymbol{e_R}\, cos(\theta_1 - \theta) + \boldsymbol{e_\theta}\, sin(\theta_1 - \theta)]\}d\theta_1 \quad (26)$$

The vectors $\boldsymbol{e_R}$ and $\boldsymbol{e_\theta}$ are constants in this integration and $\beta$ is a constant angle in the interval of $[0, 2\pi]$. Changing $\theta_1$ variable to $\varphi + \theta$ and eliminating the terms including $sin(\varphi)$ the second form of $\boldsymbol{F}''(\boldsymbol{x},t)$ can be expressed as

$$\boldsymbol{F}''(\boldsymbol{x},t) = \boldsymbol{F}''(R,R_1,t)$$

$$= \boldsymbol{e_R}\,\frac{1}{2\pi^2}\int_0^{\beta}\{[H(t - \frac{\rho}{c_1})\frac{1}{c_1}\, f(\rho,t)$$

$$]\, cos(\varphi)\}d\varphi \quad (27)$$

where $f(\rho,t)$ has been given in Eq. (24) and $\rho$ takes the following forms.

$$\rho = \sqrt{R^2 + R_1^2 - 2RR_1 cos(\varphi)}$$

$$= \sqrt{(R_1 - R)^2 + 4RR_1 sin^2(\varphi/2)} \quad (28)$$

But at that point a restriction on $R_1$ is necessary. This will be expressed as:

$$R_1 > R \quad (29)$$

But the integral in Eq. (26) cannot be calculated analytically. To simplify this, some mathematical manipulations are necessary. Defining a new variable $\phi$ as $\varphi = 2\phi$ this integral can be expressed as

$$I_1 = 2\int_0^{\beta/2} F(\rho)cos(2\phi)d\phi \quad (30)$$

where

$$F(\rho) = H(t - \frac{\rho}{c_1})\frac{1}{c_1}\, f(\rho,t) \quad (31)$$

$$\rho = \sqrt{(R_1 - R)^2 + 4RR_1 sin^2(\phi)} \quad (32)$$

Then $\boldsymbol{F''}(R,R_1,t)$ is found as below:

$$\boldsymbol{F''}(R,R_1,t) = \boldsymbol{e_R}\frac{1}{\pi^2}\int_0^{\beta/2} F(\rho)cos(2\phi)d\phi \quad (33)$$

The body force corresponding to the function $\boldsymbol{F''}(R,R_1,t)$ is

$$\boldsymbol{f''} = \delta(t)\delta(R - R_1)\,\frac{1}{2\pi R}\,\boldsymbol{e_R} \quad (34)$$

The $\boldsymbol{G''}(R,R_1,t)$ function corresponding to this loading can also be calculated inverting $c_1$ to $c_2$ in the expression of $\boldsymbol{F''}(R,R_1,t)$. But this term has not been written since it has a zero curl.
Now the components of the displacement vector $\boldsymbol{u''}$ due to this body force can be constructed in cylindrical coordinates as

$$u''_R(R,R_1,t) = (\nabla\nabla.\boldsymbol{F''})_R$$

$$= \frac{1}{\pi^2}\int_0^{\beta/2}\{\frac{\partial^2 F(\rho)}{\partial R^2}$$

$$+ \frac{1}{R}\frac{\partial F(\rho)}{\partial R} - \frac{F(\rho)}{R^2}\}cos(2\phi)d\phi \quad (35)$$

$$u''_\theta(R,R_1,t) = (\nabla\nabla.\boldsymbol{F''})_\theta = 0 \quad (36)$$

The integrand in Eq. (35) can be expressed as

$$(\frac{\partial^2 F(\rho)}{\partial R^2} + \frac{1}{R}\frac{\partial F(\rho)}{\partial R} - \frac{F(\rho)}{R^2})cos(2\phi)$$

$$= \{\frac{\partial^2 F(\rho)}{\partial \rho^2}(\frac{\partial \rho}{\partial R})^2 + \frac{\partial F(\rho)}{\partial \rho}(\frac{\partial^2 \rho}{\partial R^2})$$

$$+\frac{1}{R}\frac{\partial F(\rho)}{\partial \rho}(\frac{\partial(\rho)}{\partial R})-\frac{F(\rho)}{R^2}\}cos(2\phi) \quad (37)$$

By the way, there are following relations between partial $R$ and $\phi$ derivatives of $\rho$:

$$\frac{\partial \rho}{\partial R}=\frac{-(R_1-R)+2R_1 sin^2\phi}{\rho} \quad (38)$$

$$\frac{\partial^2 \rho}{\partial R^2}=\frac{1}{\rho}-\frac{1}{\rho}(\frac{\partial \rho}{\partial R})^2 \quad (39)$$

$$\frac{\partial \rho}{\partial \phi}=\frac{4RR_1\, sin\phi\, cos\phi}{\rho} \quad (40)$$

$$\frac{\partial^2 \rho}{\partial \phi^2}=\frac{4RR_1(cos^2\phi-sin^2\phi)}{\rho}-\frac{1}{\rho}(\frac{\partial \rho}{\partial \phi})^2 \quad (41)$$

$$(\frac{\partial \rho}{\partial R})^2+\frac{1}{4R^2}(\frac{\partial \rho}{\partial \phi})^2=1 \quad (42)$$

$$\frac{\partial^2 \rho}{\partial R^2}+\frac{1}{R}\frac{\partial \rho}{\partial R}+\frac{1}{4R^2}\frac{\partial^2 \rho}{\partial R^2}=\frac{1}{\rho} \quad (43)$$

Then, using Eqs. (38) to (43) in Eq. (37), Eq. (35) can be simplified as

$$u''_R(R,R_1,t)=\frac{1}{\pi^2}\int_0^{\beta/2}\{\frac{\partial^2 F(\rho)}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial F(\rho)}{\partial R}$$

$$-\frac{1}{4R^2}\frac{\partial^2 F(\rho)}{\partial \phi^2}-\frac{F(\rho)}{R^2}\}cos(2\phi)d\phi \quad (44)$$

The third term in the integral can be integrated by partial integration as follows:

$$\int_0^{\beta/2}\{-\frac{1}{4R^2}\frac{\partial^2 F(\rho)}{\partial \phi^2}\}cos(2\phi)d\phi=\{$$

$$\Big|-\frac{1}{4R^2}\frac{\partial F(\rho)}{\partial \rho}\frac{\partial \rho}{\partial \phi}cos(2\phi)-\frac{2F(\rho)}{4R^2}sin(2\phi)\Big|_0^{\beta/2}$$

$$+\int_0^{\beta/2}\frac{F(\rho)}{R^2}cos(2\phi)d\phi\} \quad (45)$$

Both $(\partial\rho/\partial\phi)$ and $sin(2\phi)$ vanish for $\phi=0$. Substituting these in Eq. (45), and using the new expression of Eq. (45) in Eq. (44),

$$u''_R(R,R_1,t)=$$

$$\Big|-\frac{1}{4R^2}\frac{\partial F(\rho)}{\partial \rho}\frac{\partial \rho}{\partial \phi}cos(2\phi)-\frac{2F(\rho)}{4R^2}sin(2\phi)\Big|_{\phi=\beta}$$

$$+\frac{1}{\pi^2}\int_0^{\beta/2}\{\frac{\partial^2 F(\rho)}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial F(\rho)}{\partial \rho}\}cos(2\phi)d\phi \quad (46)$$

is found. $F(\rho)$ had been defined by Eqs. (31) and (24), this function and the first and second $\rho$ derivatives of it can be expressed as

$$F(\rho)=H(t-\frac{\rho}{c_1})\frac{1}{c_1}\{c_1 t ln(\frac{c_1 t+\sqrt{c_1^2t^2-\rho^2}}{\rho})$$

$$-\sqrt{c_1^2t^2-\rho^2}\} \quad (47)$$

$$\frac{\partial F(\rho)}{\partial \rho}=H(t-\frac{\rho}{c_1})\frac{1}{c_1}\{-\frac{\sqrt{c_1^2t^2-\rho^2}}{\rho}\} \quad (48)$$

$$\frac{\partial^2 F(\rho)}{\partial \rho^2}=H(t-\frac{\rho}{c_1})\frac{1}{c_1}\{\frac{c_1^2t^2}{\rho^2\sqrt{c_1^2t^2-\rho^2}}\} \quad (49)$$

During these derivations, the property of Dirac delta function given in Eq. (14) has been used. Substituting Eqs. (48) and (49) in Eq. (46) the new form of $u''_R(R,R_1,t)$ becomes

$$u''_R(R,R_1,t)=\frac{1}{\pi^2}\Big\{\frac{1}{4R^2}[H(t-\frac{\rho(\beta)}{c_1})\frac{1}{c_1}]$$

$$[\frac{\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{(\rho(\beta))^2}2RR_1 sin\beta cos\beta$$

$$-2\{c_1 t\, ln(\frac{c_1 t+\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{\rho(\beta)})$$

$$-\sqrt{c_1^2\,t^2-(\rho(\beta))^2}\}sin\beta]$$

$$\int_0^{\beta/2}\{H(t-\frac{\rho}{c_1})\frac{1}{c_1}[\frac{1}{\sqrt{c_1^2t^2-\rho^2}}]\}cos(2\phi)d\phi\Big\} \quad (50)$$

where

$$\rho(\beta) = \sqrt{(R_1 - R)^2 + 4RR_1 sin^2(\phi)} \quad (51)$$

If a new variable $z$ is defined as

$$z = t - \frac{\rho}{c_1}$$

$$= t - \frac{\sqrt{(R_1 - R)^2 + 4RR_1 sin^2(\phi)}}{c_1} \quad (52)$$

the new boundaries of the integral in Eq. (50) are

$$z = z_1 = t - \frac{(R_1 - R)}{c_1} \quad for \quad \phi = 0 \quad (53)$$

$$z = z_2 = t - \frac{\rho(\beta)}{c_1} \quad for \quad \phi = \beta/2 \quad (54)$$

and the integral in Eq. (50) takes the following form:

$$I_1 = \frac{1}{\pi^2}\Big\{ \int_0^{\beta/2} \{H(t - \frac{\rho}{c_1})\frac{1}{c_1}\frac{cos(2\phi)}{\sqrt{c_1^2 t^2 - \rho^2}}\}d\phi\Big\}$$

$$= \frac{1}{\pi^2}\int_{z_1}^{z_2} \{H(z)[-\frac{\partial k(z)}{\partial z}]\}dz$$

$$= \frac{1}{\pi^2}\int_{z_2}^{z_1} \{H(z)[\frac{\partial k(z)}{\partial z}]\}dz \quad (55)$$

where

$$\frac{\partial k(z)}{\partial z} = (\frac{R_1^2 + R^2 - c_1^2(t-z)^2}{2RR_1\sqrt{t^2 - (t-z)^2}})$$

$$(\frac{(t-z)}{\sqrt{c_1^2(t-z)^2 - (R_1 - R)^2}})$$

$$(\frac{1}{\sqrt{(R_1 + R)^2 - c_1^2(t-z)^2}}) \quad (56)$$

By partial integration Eq. (55) is reduced to

$$I_1 = \frac{1}{\pi^2}\{\,|H(z)k(z)|_{z_2}^{z_1}$$

$$- \int_{z_2}^{z_1} \delta(z)\,k(z)\,dz\} \quad (57)$$

and performing the integral in Eq. (57) using Eq. (11),

$$I_1 = \frac{1}{\pi^2}\{H(z_1)k(z_1) - H(z_2)k(z_2)$$

$$- \lim_{z\to 0}\{[H(z_1 - z) - H(z_2 - z)]k(z)\}\}$$

$$= \frac{1}{\pi^2}\{H(z_1)[k(z_1) - k(0)]$$

$$- H(z_2)[k(z_2) - k(0)]\}$$

$$= \frac{1}{\pi^2}\{H(z_1)\int_0^{z_1}\frac{\partial k(z)}{\partial z}dz$$

$$- H(z_2)\int_0^{z_2}\frac{\partial k(z)}{\partial z}dz\}$$

$$= \frac{1}{\pi^2}\{H(z_1)\int_0^{z_1}\frac{\partial k(z)}{\partial z}dz$$

$$- H(z_2)\,[\int_0^{z_1}\frac{\partial k(z)}{\partial z}dz - \int_{z_2}^{z_1}\frac{\partial k(z)}{\partial z}dz]\}$$

$$= \frac{1}{\pi^2}\{[H(z_1) - H(z_2)]\int_0^{z_1}\frac{\partial k(z)}{\partial z}dz$$

$$+ H(z_2)\int_{z_2}^{z_1}\frac{\partial k(z)}{\partial z}dz\} \quad (58)$$

is obtained. Then returning back to $\phi$ variable from $z$ in the integrals and substituting this form of Eq. (58) in Eq. (50), the new form of $u''_R(R, R_1, t)$ is found as follows:

$$u''_R(R, R_1, t) = \frac{1}{\pi^2}\Big\{[H(t - \frac{R_1 - R}{c_1})\frac{1}{c_1}$$

$$- H(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1}]$$

$$\int_0^{\alpha}\frac{cos(2\phi)d\phi}{\sqrt{c_1^2 t^2 - (R_1 - R)^2 - 4RR_1 sin^2(\phi)}}$$

$$+ H(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1}\{\frac{1}{4R^2}$$

$$[\frac{\sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{(\rho(\beta))^2} 2RR_1 sin\beta cos\beta$$

$$- 2\{c_1 t ln(\frac{c_1 t + \sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{\rho(\beta)})$$

$$- \sqrt{c_1^2 t^2 - (\rho(\beta))^2}\} sin\beta +$$

$$\int_0^{\beta/2} \frac{cos(2\phi)d\phi}{\sqrt{c_1^2 t^2 - (R_1 - R)^2 - 4RR_1 sin^2(\phi)}}]\}\Big\} \quad (59)$$

Where angle $\alpha$, corresponding to the $\phi$ value for $z = 0$, is defined as:

$$sin(\alpha) = \sqrt{\frac{c_1^2 t^2 - (R - R_1)^2}{4RR_1}} \quad (60)$$

Eq. (59) is not convenient for derivation since the first integral is singular for $\phi = \alpha$. To bring the upper boundary to $\pi/2$ in the first integral, another variable $\psi$ will be defined as

$$sin(\phi) = sin(\alpha) sin(\psi) \quad (61)$$

Using this variable and Eq. (60) in the first integral of Eq. (59), the last form of $u''_R(R, R_1, t)$ is

$$u''_R(R, R_1, t) = \frac{1}{\pi^2}\Big\{[H(t - \frac{R_1 - R}{c_1})\frac{1}{c_1}$$

$$- H(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1}]$$

$$\int_0^{\pi/2} \frac{(1 - 2sin^2(\alpha)sin^2(\psi))d\psi}{\sqrt{4RR_1}\sqrt{1 - sin^2(\alpha)sin^2(\psi)}}$$

$$+ H(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1}\{\frac{1}{4R^2}$$

$$[\frac{\sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{(\rho(\beta))^2} 2RR_1 sin\beta cos\beta$$

$$- 2\{c_1 t ln(\frac{c_1 t + \sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{\rho(\beta)})$$

$$- \sqrt{c_1^2 t^2 - (\rho(\beta))^2}\} sin\beta] +$$

$$\int_0^{\beta/2} \frac{cos(2\phi)d\phi}{\sqrt{c_1^2 t^2 - (R_1 - R)^2 - 4RR_1 sin^2(\phi)}}\}\Big\} \quad (62)$$

Now the nonzero components of the strain tensor $\boldsymbol{\epsilon''}(R, R_1, t)$ and the stress tensor $\boldsymbol{\tau''}(R, R_1, t)$ related to this displacement field $\boldsymbol{u''}(R, R_1, t)$ are calculated in the cylindrical coordinates as follows:

$$\epsilon''_{\theta\theta}(R, R_1, t) = \frac{u''_R(R, R_1, t)}{R} \quad (63)$$

$$\epsilon''_{RR}(R, R_1, t) = \frac{\partial u''_R(R, R_1, t)}{\partial R} = \frac{1}{\pi^2}\Big\{$$

$$[\delta(t - \frac{R_1 - R}{c_1})\frac{1}{c_1^2} + \delta(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1^2}\frac{\partial \rho(\beta)}{\partial R}]$$

$$\int_0^{\pi/2} \frac{(1 - 2sin^2(\alpha)sin^2(\psi))d\psi}{\sqrt{4RR_1}\sqrt{1 - sin^2(\alpha)sin^2(\psi)}}$$

$$- \delta(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1^2}\frac{\partial \rho(\beta)}{\partial R}\{\frac{1}{4R^2}$$

$$[\frac{\sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{(\rho(\beta))^2} 2RR_1 sin\beta cos\beta$$

$$- 2\{c_1 t ln(\frac{c_1 t + \sqrt{c_1^2 t^2 - (\rho(\beta))^2}}{\rho(\beta)})$$

$$- \sqrt{c_1^2 t^2 - (\rho(\beta))^2}\} sin\beta] +$$

$$\int_0^{\beta/2} \frac{cos(2\phi)d\phi}{\sqrt{c_1^2 t^2 - (R_1 - R)^2 - 4RR_1 sin^2(\phi)}}\}$$

$$+ [H(t - \frac{R_1 - R}{c_1})\frac{1}{c_1} - H(t - \frac{\rho(\beta)}{c_1})\frac{1}{c_1}]$$

$$\frac{1}{\sqrt{4RR_1}}\Big[E(\alpha)[\frac{1}{R}(\frac{1}{4sin^2\alpha cos^2(\alpha)}-2)$$

$$+\frac{1}{R_1}(-\frac{1}{4sin^2(\alpha)}+\frac{1}{4cos^2(\alpha)})]$$

$$+F(\alpha)[\frac{1}{R}(1-\frac{1}{4sin^2(\alpha)})+\frac{1}{R_1}(\frac{1}{4sin(\alpha)})]\Big]$$

$$+H(t-\frac{R_1+R}{c_1})\frac{1}{c_1}\{$$

$$\int_0^{\pi/2}\frac{(R\,cos2\varphi-R_1\,cos^2 2\varphi)d\varphi}{\sqrt{c_1^2t^2-(R_1-R)^2-4RR_1 sin^2\varphi}^{\,3}}$$

$$-\frac{1}{2R^3}[\frac{\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{(\rho(\beta))^2}2RR_1 sin\beta cos\beta$$

$$-2\{c_1 t ln(\frac{c_1 t+\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{\rho(\beta)})$$

$$-\sqrt{c_1^2\,t^2-(\rho(\beta))^2}\}sin\beta]+$$

$$\frac{1}{4R^2}[\frac{\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{(\rho(\beta))^2}2R_1 sin\beta cos\beta$$

$$+\frac{\partial\rho(\beta)}{\partial R}($$

$$\frac{(\rho(\beta))^2-2c_1^2t^2}{(\rho(\beta))^3\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}2RR_1 sin\beta cos\beta$$

$$+\frac{2\sqrt{c_1^2\,t^2-(\rho(\beta))^2}}{\rho(\beta)}sin\beta)]\}\Big\} \qquad (64)$$

Where $F(\alpha)$ and $E(\alpha)$ represent the first and the second kind of complete elliptical integrals, respectively. These integrals are

$$F(\alpha)=\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-sin^2\alpha\, sin^2\psi}} \qquad (65)$$

$$E(\alpha)=\int_0^{\pi/2}\sqrt{1-sin^2\alpha\, sin^2\psi}\,d\psi \qquad (66)$$

The third kind complete elliptical integral $\Pi(\alpha)$ is also come out during the $R$ derivation of $u''_R(R,R_1,t)$. But this integral has been eliminated using following relation [6]

$$\Pi(\alpha)=\int_0^{\pi/2}\frac{d\psi}{\sqrt[3]{1-sin^2\alpha\, sin^2\psi}}=\frac{E(\alpha)}{cos^2\alpha} \qquad (67)$$

In Eq. (64), the multiplier of $\delta(t-(R_1-R)/c_1)$ can be simplified as follows:

$$\frac{1}{\pi^2}\{\delta(t-\frac{R_1-R}{c_1})\frac{1}{c_1^2}$$

$$\int_0^{\pi/2}\frac{(1-2sin^2(\alpha)sin^2(\psi))d\psi}{\sqrt{4RR_1}\sqrt{1-sin^2(\alpha)sin^2(\psi)}}\}$$

$$=\frac{1}{\pi^2}\Big\{\delta(t-\frac{R_1-R}{c_1})\frac{1}{c_1^2}\frac{\pi}{4\sqrt{RR_1}}\Big\} \qquad (68)$$

And the nonzero stress components are

$$\tau''_{RR}=(\lambda+2\mu)\epsilon''_{RR}+\lambda\,\epsilon''_{\theta\theta} \qquad (69)$$

$$\tau''_{\theta\theta}=(\lambda+2\mu)\epsilon''_{\theta\theta}+\lambda\,\epsilon''_{RR} \qquad (70)$$

## 5 Sample Problem

An infinite medium, without body forces, having a cylindrical cavity with radius $a=50cm$, under a variable pressure $p\,(t)$ is considered. The variation of pressure function has been given as

$$p\,(t)=\begin{cases} p_o(1+40\,sin(200t)\,e^{-200t}) & (t>0)\\ 0 & (t\le 0)\end{cases} \qquad (71)$$

where $t$ has been given in seconds, and the multiplier 200 has the dimension of $1/sec$. $p_o$ is initial and final value of the pressure function representing a sudden explosion. The effect of explosion is ended nearly in 30 milliseconds. For numerical calculations Poisson' s ratio and wave velocity have been selected as $\nu=0.3$ and $c_1=2000m/s$, respectively. The variation of the non-dimensional pressure $p(t)/p_o$ versus time is plotted in Fig. 1.

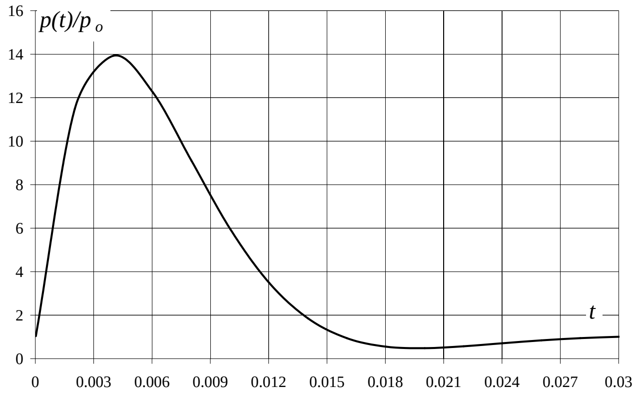


Fig.1. Variation of dimensionless pressure versus time (s)

The boundary is defined as $R = a$. Where $a$ is the radius of the cavity. In this problem, the stress tensor and displacement vector are as follows:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{RR} & 0 & 0 \\ 0 & \tau_{\theta\theta} & 0 \\ 0 & 0 & \tau_{zz} \end{bmatrix} \tag{72}$$

$$\boldsymbol{u} = u_R \boldsymbol{e}_R \tag{73}$$

where all the components are the functions of $R$ and $t$ only. Body force has been ignored. Then, $S[\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{0}]$ forms an elastodynamic state. The surface tractions on the boundary for $S$ and $S'$ defined in Sect. 3, are

$$\boldsymbol{T}(a,t) = p(t)H^+(t)\boldsymbol{e}_R \tag{74}$$

$$\boldsymbol{T'}(a,t) = -\tau'_{RR}(a,t)\boldsymbol{e}_R \tag{75}$$

And further, for every $R \in [0, \infty)$

$$u_R(R,0) = 0 \tag{76}$$

where $H^+(t)$ has been chosen to be equal to unity in the interval $[0, \infty)$ with zero derivatives. The expression of the dynamic reciprocal identity Eq. (3), which is written between $S$ and $S'$ states, is

$$\int_S \boldsymbol{T'} * \boldsymbol{u}\, dS = \int_S \boldsymbol{T} * \boldsymbol{u'} dS \tag{77}$$

In Eq. (3), the second integral vanishes because of absence of the body force $\boldsymbol{f}$ in $S$. The body force in $S'$ exists out of the volume $V$; then, the fourth integral in Eq. (3) is also eliminated. Before constructing the integral equation, the loading time of $\boldsymbol{f'}$ will be changed to $t = a/c_1$ from $t = 0$ in $S'$ to eliminate the singularities at $R = a$. After these, using Eq. (5) in Eq. (77), substituting Eqs. (6) to (9) and Eqs. (74) to (75), an integral equation is obtained as follows:

$$\begin{aligned} &- \int_0^t \Big\{ \Big[\lambda + 2\mu\Big] \Big[ H(t-\tau)\frac{1}{c_1}[-\frac{1}{K_1^3} - \frac{3a^2}{K_1^5}] \\ &+ \delta(t-\tau)\frac{1}{c_1^2}[\frac{2a}{K_1^3}] + \dot{\delta}(t-\tau)\frac{1}{c_1^3}[-\frac{1}{K_1}]\Big] \\ &+ \Big[\lambda\Big]\Big[H(t-\tau)\frac{1}{c_1}[-\frac{1}{K_1^3}] \\ &+ \delta(t-\tau)\frac{1}{c_1^2}[\frac{1}{aK_1}]\Big]\Big\} u_R(a,\tau)d\tau \\ &= \int_0^t p(\tau)\Big\{\Big[H(t-\tau)\frac{1}{c_1}[-\frac{a}{K_1^3}] \\ &+ \delta(t-\tau)\frac{1}{c_1^2}[\frac{1}{K_1}]\Big]\Big\} d\tau \end{aligned} \tag{78}$$

where

$$K_1 = \sqrt{c_1^2(t - \tau + \frac{a}{c_1})^2 - a^2} \tag{79}$$

$u_R(a,t)$ is the dependent variable of the integral equation given in Eq. (78) and the kernel of this integral equation is strongly singular. The following equalities can be used to reduce this singularity to a weak form.

$$\begin{aligned} &\int_0^t \delta(t-\tau)f(t-\tau)u(\tau)d\tau = \\ &- H(t-\tau)f(t-\tau)u(\tau)\Big|_{\tau=0}^{\tau=t} - \\ &\int_0^t H(t-\tau)[\dot{f}(t-\tau)u(\tau) - f(t-\tau)\dot{u}(\tau)]d\tau \end{aligned} \tag{80}$$

$$\begin{aligned} &\int_0^t \dot{\delta}(t-\tau)f(t-\tau)u(\tau)d\tau = \\ &- \delta(t-\tau)f(t-\tau)u(\tau)\Big|_{\tau=0}^{\tau=t} - \end{aligned}$$

$$\int_0^t \delta(t-\tau)[\dot f(t-\tau)u(\tau) - f(t-\tau)\dot u(\tau)]d\tau \quad (81)$$

Using these relations, the terms involving $\delta(t-\tau)$ and $\dot\delta(t-\tau)$ in the integrals in Eq. (78), can be eliminated. Then the new form of the Eq. (78) becomes

$$\frac{(\lambda+2\mu)}{c_1}\Big\{\Big| - \delta(t-\tau)\frac{u_R(\tau)}{c_1^2}\frac{1}{K_1}$$

$$+ H(t-\tau)\frac{1}{c_1}[(\frac{a-c_1(t-\tau)}{K_1^3})u_R(\tau)$$

$$-\frac{1}{K_1}\dot{u_R}(\tau)]\Big|_0^t +$$

$$\int_0^t H(t-\tau)[u_R(\tau)\frac{3c_1^2(t-\tau)^2}{K_1^5}$$

$$+\frac{\dot{u_R}(\tau)}{c_1}\frac{2c_1(t-\tau)}{K_1^3} + \frac{\ddot u_R(\tau)}{c_1^2}\frac{1}{K_1}]d\tau\Big\}$$

$$+\frac{\lambda}{c_1}\Big\{\Big|H(t-\tau)\frac{1}{c_1}\frac{1}{aK_1^3}\Big|_0^t$$

$$+\int_0^t H(t-\tau)[u_R(\tau)(-\frac{c_1(t-\tau)}{aK_1^3})$$

$$+\frac{\dot u_R(\tau)}{c_1}(-\frac{1}{aK_1})]d\tau\Big\}$$

$$= \Big| - H(t-\tau)\frac{1}{c_1^2}\frac{p(\tau)}{K_1}\Big|_0^t +$$

$$\frac{1}{c_1}\int_0^t \{H(t-\tau)[\frac{p(\tau)c_1(t-\tau)}{K_1^3} + \frac{\dot p(\tau)}{c_1K_1}]\}d\tau \quad (82)$$

Substituting the end values, some of the terms out of the integrals becomes zero, but some of them take the form of $0/0$. Calculating necessary derivatives Eq. (82) takes the following form.

$$H(t)\frac{1}{c_1^2}\frac{1}{\sqrt{c_1^2(t-\frac{a}{c_1})^2-a^2}}[\frac{(\lambda+2\mu)}{c_1}\dot{u_R}(0) - ap(0)]$$

$$+ H(t)\int_0^t \{[\frac{\lambda+2\mu}{c_1}][u_R(\tau)\frac{3c_1^2(t-\tau)^2}{K_1^5}$$

$$+\frac{\dot u_R(\tau)}{c_1}\frac{2c_1(t-\tau)}{K_1^3} + \frac{\ddot u_R(\tau)}{c_1^2}\frac{1}{K_1}]$$

$$+[\frac{\lambda}{c_1}][-u_R(\tau)\frac{c_1(t-\tau)}{aK_1^3} - \frac{\dot{u_R}(\tau)}{c_1}\frac{1}{aK_1}]\}d\tau$$

$$= \frac{1}{c_1}H(t)\int_0^t \{p(\tau)\frac{c_1(t-\tau)}{K_1^3} + \frac{\dot p(\tau)}{c_1}\frac{1}{K_1}\} \quad (83)$$

Equating the terms out of the integrals the initial velocity is found as below:

$$\dot u_R(0) = c_1\frac{p(0)}{\lambda+2\mu} \quad (84)$$

The remaining terms give a integro-differential equation with weak singularity. Before solving this, two new dimensionless quantities will be defined as

$$t^* = \frac{c_1 t}{a} \quad , \quad U(t^*) = \frac{2\mu}{ap_o}\, u_R(a,t) \quad (85)$$

and substituting these variables the new form of the integral equation in terms of $U$ and $t^*$ becomes

$$\frac{\lambda+2\mu}{2\mu}\int_0^{t^*}\{U(\tau^*)\frac{3(t^*-\tau^*)^2}{K_2^5}$$

$$+\dot U(\tau^*)\frac{2(t^*-\tau^*)}{K_2^3} + \ddot U(\tau^*)\frac{1}{K_2}\}d\tau^*$$

$$-\frac{\lambda}{2\mu}\int_0^{t^*}\{U(\tau^*)\frac{(t^*-\tau^*)}{K_2^3} + \dot U(\tau^*)\frac{1}{K_2}\}d\tau^*$$

$$= \int_0^{t^*}\{P(\tau^*)\frac{(t^*-\tau^*)}{K_2^3} + \dot P(\tau^*)\frac{1}{K_2}\}d\tau^* \quad (86)$$

where

$$K_2 = \sqrt{(t^*-\tau^*)}\sqrt{(t^*-\tau^*+2)} \quad (87)$$

$$P(t^*) = \frac{p(t^*)}{p_o} = 1 + 40\sin(200\frac{at^*}{c_1})\,e^{-200\frac{at^*}{c_1}} \quad (88)$$

Variation of nondimensionless pressure, $p(t^*)$, versus dimensionless time, $t^*$, has been given in Fig. 2.

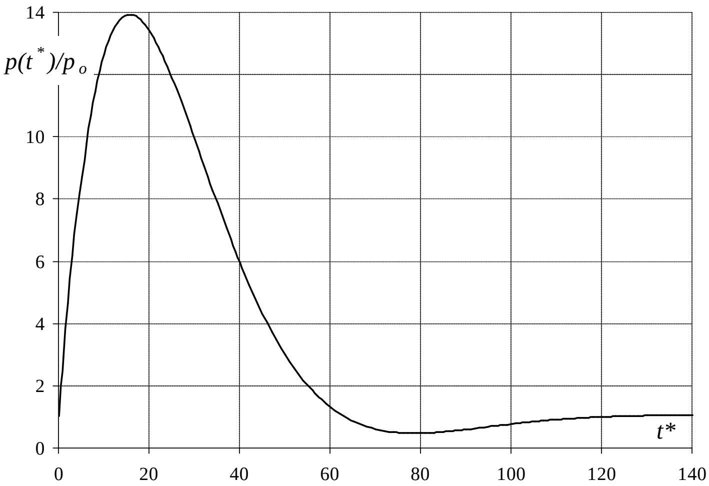


Fig. 2. Variation of dimensionless pressure versus dimensionless time

To solve the integral equation given in Eq. (86), the following numerical method is introduced: The dimensionless time is divided to intervals with constant length $\Delta t^* = t^*_{k+1} - t^*_k$. It is accepted that the function $U(t^*)$ and its first derivative are continuous while passing from an interval to another. Besides instead of using original form of $P(t^*)$ function, this function is also represented by a third order polynomial on each interval. This function and its first derivative are also continuous at end points of an interval. With these assumptions, the functions $U(t^*)$ and $P(t^*)$ can be expressed in the interval between $t^*_k$ and $t^*_{k+1}$ as follows:

$$U(t^*) = U(k)\Big[1 - \frac{(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)^2}\Big]$$

$$+ U(k+1)\Big[\frac{(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)^2}\Big]$$

$$+ \dot{U}(k)\Big[(t^* - t^*_k) - \frac{(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)}\Big] \quad (89)$$

$$P(t^*) = P(k)\Big[1 + \frac{2(t^* - t^*_k)^3}{(t^*_{k+1} - t^*_k)^3} - \frac{3(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)^2}\Big]$$

$$+ P(k+1)\Big[\frac{3(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)^2} - \frac{2(t^* - t^*_k)^3}{(t^*_{k+1} - t^*_k)^3}\Big]+$$

$$\dot{P}(k)\Big[(t^* - t^*_k) + \frac{(t^* - t^*_k)^3}{(t^*_{k+1} - t^*_k)^2} - \frac{2(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)}\Big]$$

$$+ \dot{P}(k+1)\Big[\frac{(t^* - t^*_k)^3}{(t^*_{k+1} - t^*_k)^2} - \frac{(t^* - t^*_k)^2}{(t^*_{k+1} - t^*_k)}\Big] \quad (90)$$

$P(k)$, $P(k+1)$, $\dot{P}(k)$, $\dot{P}(k+1)$ values will be calculated from original $P(t^*)$ function given by Eq. (88). It is noted that

$$t^*_1 = 0 \ , \ U(1) = 0 \ , \ \dot{U}(1) = \frac{2\mu}{\lambda + 2\mu} \quad (91)$$

Substituting Eqs. (89) and (90) in Eq. (86) the following equation is obtained between the end values of $U(t^*)$ till $t^*_{j+1}$

$$U(j+1)A(j,j) + U(j)B(j,j) + \dot{U}(j)C(j,j)+$$

$$\sum_{k=1}^{j-1}\Big\{U(k+1)A(k,j) + U(k)B(k,j) + \dot{U}(k)C(k,j)\Big\}$$

$$= \sum_{k=1}^{j}\Big\{P(k+1)A1(k,j) + P(k)B1(k,j)$$

$$+ \dot{P}(k+1)C1(k,j) + \dot{P}(k)D1(k,j)\Big\} \quad (92)$$

where

$$A(k,j) = \{\frac{\lambda + 2\mu}{2\mu}[3I_2(k,j) + 4I_4(k,j)$$

$$+ 2I_6(k,j)] - \frac{\lambda}{2\mu}[I_7(k,j) + 2I_8(k,j)]\}\frac{1}{(\Delta^*)^2} \quad (93)$$

$$B(k,j) = -A(k,j) + \frac{\lambda + 2\mu}{2\mu}[3I_1(k,j)]$$

$$- \frac{\lambda}{2\mu}[I_5(k,j)] \quad (94)$$

$$C(k,j) = -A(k,j)\Delta^* + \frac{\lambda + 2\mu}{2\mu}[I_3(k,j)$$

$$+ 2I_5(k,j)] - \frac{\lambda}{2\mu}[I_4(k,j) + I_6(k,j)] \quad (95)$$

$$A1(k,j) = [-2I_9(k,j) - 6I_{10}(k,j)]\frac{1}{(\Delta^*)^3}$$

$$+ [3I_7(k,j) + 6I_8(k,j)]\frac{1}{(\Delta^*)^2} \quad (96)$$

$$B1(k,j) = -A1(k,j) + I_5(k,j) \quad (97)$$

$$C1(k,j) = [I_9(k,j) + 3I_{10}(k,j)]\frac{1}{(\Delta^*)^2}$$

$$+ [-I_7(k,j) - 2I_8(k,j)]\frac{1}{(\Delta^*)} \quad (98)$$

$$D1(k,j) = [I_9(k,j) + 3I_{10}(k,j)]\frac{1}{(\Delta^*)^2}$$

$$+ [-2I_7(k,j) - 4I_8(k,j)]\frac{1}{(\Delta^*)}$$

$$+ [I_4(k,j) + 4I_6(k,j)] \quad (99)$$

$$I_1(k,j) = \int_0^{\Delta^*} \frac{(t^*(j+1) - t^*(k) - \tau)^2}{K^5} d\tau \quad (100)$$

$$I_2(k,j) = \int_0^{\Delta^*} \frac{\tau^2(t^*(j+1) - t^*(k) - \tau)^2}{K^5} d\tau \quad (101)$$

$$I_3(k,j) = \int_0^{\Delta^*} \frac{\tau(t^*(j+1) - t^*(k) - \tau)^2}{K^5} d\tau \quad (102)$$

$$I_4(k,j) = \int_0^{\Delta^*} \frac{\tau(t^*(j+1) - t^*(k) - \tau)}{K^3} d\tau \quad (103)$$

$$I_5(k,j) = \int_0^{\Delta^*} \frac{(t^*(j+1) - t^*(k) - \tau)}{K^3} d\tau \quad (104)$$

$$I_6(k,j) = \int_0^{\Delta^*} \frac{1}{K} d\tau \quad (105)$$

$$I_7(k,j) = \int_0^{\Delta^*} \frac{\tau^2(t^*(j+1) - t^*(k) - \tau)}{K^3} d\tau \quad (106)$$

$$I_8(k,j) = \int_0^{\Delta^*} \frac{\tau}{K} d\tau \quad (107)$$

$$I_9(k,j) = \int_0^{\Delta^*} \frac{\tau^3(t^*(j+1) - t^*(k) - \tau)}{K^3} d\tau \quad (108)$$

$$I_{10}(k,j) = \int_0^{\Delta^*} \frac{\tau^2}{K} d\tau \quad (109)$$

$$K = \sqrt{(t^*(j+1) - t^*(k) - \tau + 1)^2 - 1} \quad (110)$$

Calculating the integrals, the values of $U(j+1)$ is obtained from Eq. (92) for $j = 1, 2, ..., n$. Using these, the variation of dimensionless displacement $U(t^*)$ versus dimensionless time $t^*$ is given in Fig. 3. And using the $U(t^*)$ function the dimensionless unknown stress component $\tau_{\theta\theta}/p_o$ can also be calculated and the variation of this versus $t^*$ is given in Fig. 4.

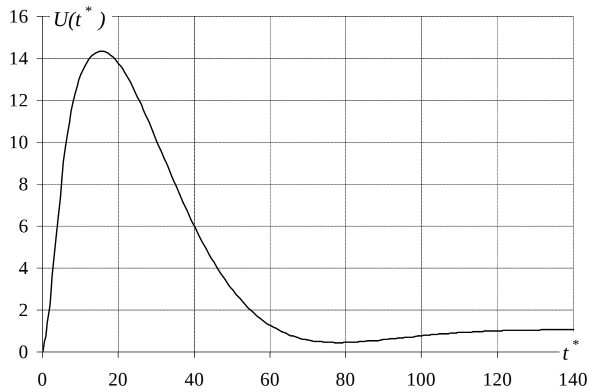


Fig. 3. Variation of dimensionless radial displacement, $U(t^*)$, on the surface of circular cavity versus dimensionless time, $t^*$

Now the displacement component $u_R(R_1, t)$ at any $(R_1, \theta_1)$ point will be calculated writing reciprocal identity (Eq. (3)) between $S(\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{f})$ and $S''(\boldsymbol{u''}, \boldsymbol{\tau''}, \boldsymbol{f''})$ elastodynamic states. In that case, in Eq. (3), second integral vanishes since $\boldsymbol{f} = \boldsymbol{0}$. $\boldsymbol{u}$ has been calculated on the boundary, $\boldsymbol{T}$ and $\boldsymbol{f''}$ have also been given by Eq. (74) and Eq. (34), respectively. $\boldsymbol{T''}$ can be written similarly to $\boldsymbol{T'}$ (Eq. (75)) as

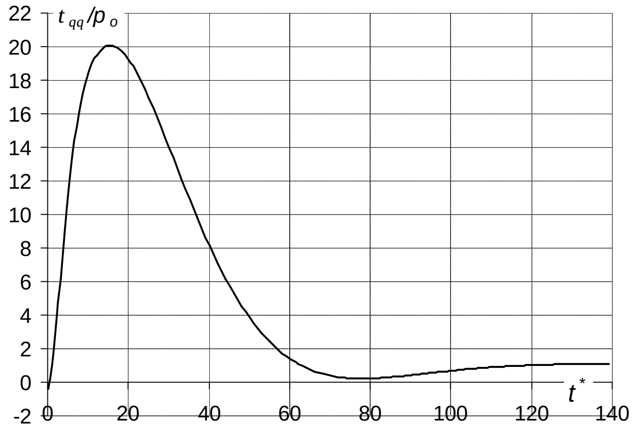


Fig. 4. Variation of $\tau_{\theta\theta}/p_o$ on the surface of circular cavity versus $t^*$

$$\boldsymbol{T''} = -\tau''_{RR}(a, R_1, t)\boldsymbol{e_R} \tag{111}$$

Using these, reciprocal identity takes the following form.

$$\int_0^{2\beta} p(t) * u''_R(a, R_1, t) a d\theta =$$

$$\int_0^{2\beta} (-T''_{RR}(a, R_1, t) * u_R(t)) a d\theta$$

$$+ \rho \int_V [\delta(t)\delta(R - R_1)\frac{1}{2\pi R} * u_R(R, t) dV] \tag{112}$$

where $dV = R\, dR\, d\theta$. Performing these integrals

$$u_R(R_1, t) = \{\frac{1}{\rho} p(t) * u''_R(a, R_1, t)$$

$$+ \frac{1}{\rho}\tau''_{RR}(a, R_1, t) * u_R(a, t)\} 2\beta a \tag{113}$$

is found. The signals propagating from whole $R = a$ boundary cannot reach to a point $(R_1, \theta_1)$. Only the effects coming from the interval, $[\theta(1) = \theta_1 - \beta\ ,\ \theta(2) = \theta_1 + \beta]$, can reach to this point and angle $\beta$ can be calculated from Fig. 5 as follows:

$$cos\beta = \frac{R}{R_1} \tag{114}$$

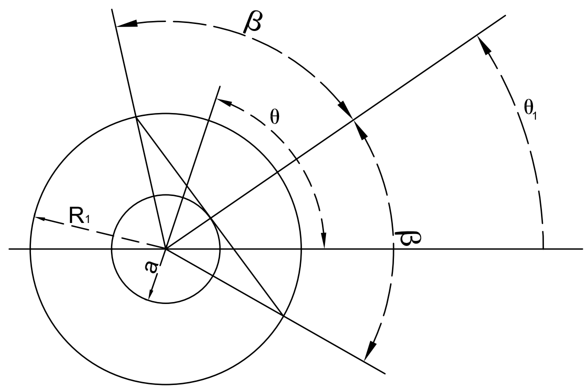


Fig. 5. Representation of angle $\beta$

Depending upon to this value of $\beta$, the terms, including $\partial\rho(\beta)/\partial R$ in Eq. (64), vanish. $u_R(R_1, t)$ can be found by direct integration substituting Eqs. (62) to (66), (68), (69) and (71) and the boundary values of $u_R(a, t)$ which has been found as the solution of Eq. (86) in Eq. (113). But, instead of this a new dimensionless quantity, defined below, has been calculated.

$$U(R_1, t^*) = \frac{2\mu}{ap_o} u_R(R_1, a, t^*) \tag{115}$$

Result has been given in Fig. 6 for $R_1 = 2a$.

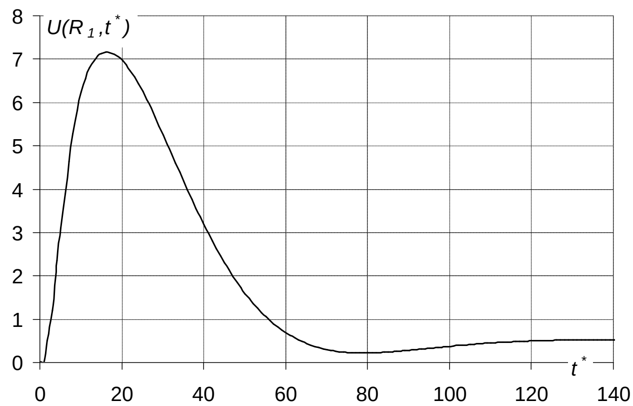


Fig. 6. Variation of dimensionless component of radial displacement, $U(R_1, t^*)$, versus dimensionless time, $t^*$, at $R_1 = 2a$

## 6 Conclusions

A fundamental elastodynamic state, which has been presented by Kadioglu and Ataoglu [4] before, was simplified. This state will be used as the fundamental solution in the reciprocity theorem for the solutions of axially symmetric transient problems of elastodynamics. The reciprocity theorem will give an integral equation, if the first state mentioned above is used for any problem. The unknown of this integral equation is the radial component of the displacement vector. But, this first elastodynamic state cannot be used to determine the radial displacement at any point inside the region. For this purpose, a second fundamental state has also been constructed. The form of this state is really interesting and there is not any similarity with the known classical elastodynamic state. Specially, an angle $\beta$, which has been taken to be equal to $\pi$ in classical formulation, is introduced. It is physically seen that the upper limit of this angle $\beta$ is $\pi/2$ for hole problems. Furthermore, only one of the two different expressions of this second state is valid for different values of time. To check the formulation, a sample problem, representing a sudden explosion in a cylindrical cavity, has been solved. The body forces has been neglected and this problem can be represented by an elastodynamic state. If the reciprocal identity is written between this state and the first fundamental state mentioned above, an integral equation, whose kernel is strongly singular, is obtained. The resulting integral equation of the sample problem has been reduced to another form whose kernel is weakly singular and has been solved using a simple numerical technique. The solution of this integral equation gives the boundary values of the radial displacement component for the sample problem. After determining the boundary values, the radial displacement at a point in the region can be calculated writing the reciprocity theorem between the second elastodynamic state mentioned above and the elastodynamic state representing the sample problem.

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