Optimal designing scalar quantizers using a hybrid quantization method for the Laplacian source

[1]ZORAN PERIC, [1]JELENA NIKOLIC, [2]DRAGOLJUB POKRAJAC
[1]Telecommunication, [2]Computer and Information Sciences Department and Applied Mathematics Research Center
[1]Faculty of Electronic Engineering Nis, [2] Delaware State University
[1]Aleksandra Medvedeva 14, 18000 Nis, [2]1200 N DuPont Hwy, Dover, Delaware 19901
[1]SERBIA, [2]USA

Abstract: - This paper proposes new method for designing scalar quantizers. The proposed method, denoted as hybrid method, combines two quantization techniques, the companding technique and the Lloyd-Max's algorithm. In this paper an exact and complete analysis of the hybrid quantization method considering the Laplacian input signals is carried out. Furthermore, two approaches to the problem of finding the sets of parameters are considered. It is demonstrated that by using both approaches the hybrid quantization method provides optimal scalar quantizer design. Moreover, the designing complexity of the proposed method is considered and compared with the complexity of the other models in use. It is shown that, in case of average and large number of quantization levels, scalar quantizers designed by using the hybrid quantization method have less complexity than optimal Lloyd-Max's scalar quantizers. Also, it is demonstrated that the proposed quantization method has a little bit greater complexity than the method based on the companding technique, but provides optimal quantizer's performance.

Keywords: Companding technique, Hybrid method, Lloyd-Max’s algorithm, optimal scalar quantizer design

1 Introduction

The problem of finding an optimal scalar quantizer is not only important but is also an intriguing one. The lack of any straightforward design solution is a result of the difficulty in dealing with the highly nonlinear nature of quantization [1]. There are several methods for designing nonuniform scalar quantizers, but there has not been much theoretical or even quantitive comparison among them. Consequently, much work is still need in order to determine which method provides the best performances versus complexity trade off and in gaining an understanding of why certain complexity-reducing methods are better than others. Complexity is itself a difficult thing to quantify. In this paper we consider the designing complexity that is defined with the computation promptitude when computing the optimal parameters of scalar quantizers. Consequently, the goal of this paper is to find method that provides optimal quantizer design attaining as less as possible the designing complexity.

Here, we begin with a brief review of the quantization methods (or techniques) and their complexities. The principal goal of scalar quantzer design is to select the representation levels and the partition or cells so as to provide the minimum possible average distortion for a fixed number of quantization levels N. In general, this problem does not have any explicit, closed form solution. Lloyd [2] and Max [3] independently proposed an algorithm to compute optimal quantizers using mean-square error distortion measure. Particularly, Lloyd-Max’s algorithm is an iterative algorithm, which in each iteration performs calculation of all representation levels (N parameters) and decision thresholds (N+1 parameters) of the N levels scalar quantizer. Hence, the values of $2N+1$ parameters should be computed and memorized in each iteration. The amount of the necessary computation and the number of iterations are the deficiencies of this algorithm, and they grow with the number of quantization levels N. However, when analyzing and optimizing nonuniform scalar quantizers with average and large number of levels it is advisable to use the companding technique [4, 5]. Namely, a nonuniform quantization can also be achieved by compressing the input signal, than quantizing it with a uniform quantizer and expanding the quantized version of the compressed signal using a nonuniform transfer characteristic inverse to that of the compressor. The described quantization technique is called the companding technique [4, 5].

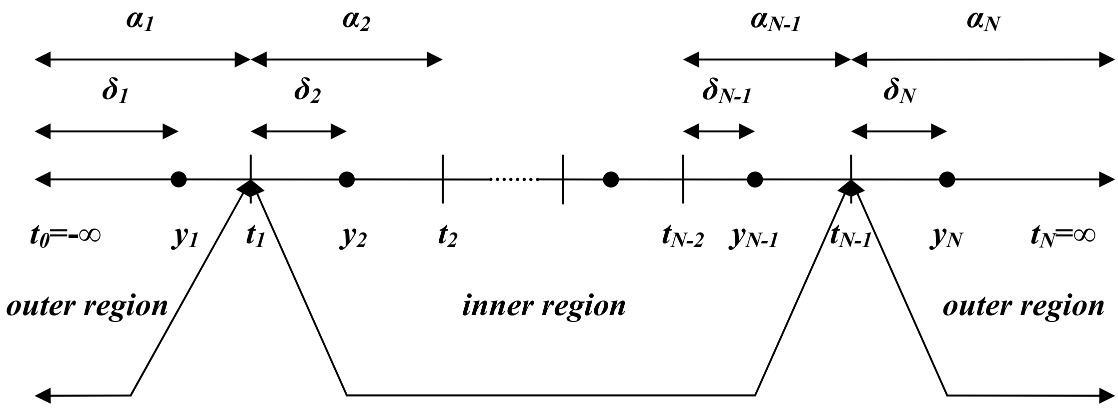

Fig. 1 Illustration of the inner region and the outer region of the scalar quantizer for L=1

Choosing smaller cells where the probability of occurance of the input random variable x is high,i.e., where the pdf $p(x)$ is comparable high and choosing larger cells otherwise results in rough approximation of the input signal in the region of high amplitudes. This is an essential deficiency of the companding technique.

The afore mention issues were insighted in [6], and resolved in [6, 7] by giving the heuristic solution of the problem for L=1, (L«N). Namely, Sangsin Na and David L. Neuhoff applied the companding technique to N-2 inner cells and Lloyd-Max's algorithm when computed the values of the first and the last representation levels. Namely, they considered the special case for L=1, of the proposed hybrid method and provided solution that are not optimal regarding to distortion optimlity [8].

In order to reduce the amount of the necessary computation in comparison to that for Lloyd-Max's algorithm, as well as to improve the deficiency introduced by using the companding technique, we propose one quantization method, denoted here as the *hybrid method*. The proposed method generalizes the method considered in [6, 7]. It is based on the combination of the companding technique and the Lloyd-Max's algorithm of designing scalar quantizers. Namely, applying the companding technique to N-2L inner cells, (L«N), and Lloyd-Max's algorithm to 2L outer cells it is possible to design N-levels scalar quantizer. The proposed method is very simple for analysis. Analyzing it we demonstrate that for particular value of L the hybrid method provides an optimal scalar quantizer design. Moreover, we derive the expression for determining the support region of the observed scalar quantizer, ranging ($-t_{N-1}$, t_{N-1}). Furthermore, we consider the performances of the quantizers designed by using the hybrid method. We demonstrate that these performances are arbitrarily close to those of optimal scalar quantizers, that are given in [5] for $R\geq 5$ ($R=\log_2 N$).

2. Scalar quantization methods

Let us consider an N-level nonuniform scalar quantizer Q for the Laplacian input signals. Scalar quantizer Q can be defined with Q: $R\rightarrow C$, as a functional mapping of the set of real numbers R onto the set of the output representation. The set of the output representation constitutes the code book:

$$C \equiv \{y_1, y_2, y_3, \cdots y_N\} \subset R \qquad (1)$$

that has the size $|C|=N$. The output values, y_j, are called the representation levels. Here, we define the nonuniform scalar quantizer Q with the set of output values and with the partition of the input range of values onto N cells, i.e., intervals α_j, j=1,2,...,N. Cells α_j are defined with the decision thresholds $\{t_0, t_1, \ldots, t_N\}$, such that $\alpha_j=(t_{j-1}, t_j]$, j=1,2,...,N. A quantized signal has value y_j when the original signal belongs to the quantization cell α_j. Hence, N-level scalar quantizer is defined as a functional mapping of an input value x onto an output representation, such that:

$$Q(x) = y_j, \quad x \in \alpha_j. \qquad (2)$$

Due to symmetry of the Laplacian distribution we can consider that the negative thresholds and the representation levels are symmetric to their nonnegative counterparts. Therefore, the considered scalar quantizer can be depicted by using only positive values of the decision thresholds, $0=t_{N/2}<t_{N/2+1}<\ldots<t_{N-1}<t_N=\infty$, and representation levels, $y_{N/2+1}<y_{N/2+2}<\ldots<y_N$. Let us denote the distances from the representative levels to the nether decision thresholds, i.e., reconstruction offsets with δ_j, j=1,…,N. The values of the reconstruction offsets are necessary when computing the parameters of scalar quantizers.

Let us consider the region of the scalar qantizers as a union of inner and outer region, as depisted in Fig. 1 in caes of L=1. The cells $\alpha_{L+1},\ldots,\alpha_{N-L}$, are reffered to as the inner cells, and they comprise the inner region ranging ($-t_{N-L}$, t_{N-L}) interval. The set of cells $\alpha_1,\ldots,\alpha_L$ and $\alpha_{N-L+1},\ldots,\alpha_N$ are reffered to as the outer cells and they comprise the outer region, ranging the union of intervals (t_0,$-t_{N-L}$) and (t_{N-L}, t_N). Therefore, the threshold denoted with t_{N-L} presents the boundary between the inner and the outer region. In order to achieve the optimal quantizer design the observed quantizer should have the optimal value of the boundary t_{N-L}. This condition of optimality is considered in the Section 2.3 and the expression for the boundary t_{N-L}, in case of the Laplacian source is derived. In the following subsections we consider quantization method based on the companding technique, quantization method based on the Lloyd-Max's algorithm and the mixed quantization method, proposed in this paper, and denoted as hybrid quantization method.

2.1. Quantization method based on the Companding technique

Nonuniform quantization can be achieved by the following procedure:

Step 1. Compress the signal x using a nonlinear compressor characteristic $c(\cdot)$.

Step 2. Quantize the compressed signal $c(x)$ with a uniform quantizer.

Step 3. Expand the quantized version of the compressed signal using a nonlinear transfer characteristic $c^{-1}(\cdot)$ inverse to that of the compressor.

The corresponding structure of a nonuniform quantizer consisting of a compressor, a uniform quantizer, and expandor in cascade is called the compandor. There are several definitions of the compressor functions. In [9] we provided the procedure of finding the optimal compressor function. Furthermore, we demonstrated that the following compressor function $c(x)$ is optimal:

$$c(t_j) = -t_{\max} + 2t_{\max}\frac{\int_{-t_{\max}}^{t_j} p^{1/3}(x)dx}{\int_{-t_{\max}}^{+t_{\max}} p^{1/3}(x)dx}, \quad (3)$$

where $p(x)$ is probability density function of continuous random variable (corresponding to the source signal), t_{max} is maximum amplitude of the input signal and $c(t_j)$ are the values of the compressor function $c(x)$ for the decision thresholds t_j, j=0,1,...,N.

The compandor distortion consists of two components, inner and outer distortion, symbolically,

$$D = D_i + D_o\,. \quad (4)$$

The inner distortion of quantizers based on the companding technique can be expresed by using Bennett's integral [10] ranging [$-t_{N-1}$,+ t_{N-1}]. Hence, the total distortion can be defined as:

$$D = \frac{1}{12(N-2)^2}\left(\int_{-t_{N-1}}^{t_{N-1}} p^{1/3}(x)dx\right)^3 + 2\int_{t_{N-1}}^{\infty}(x-y_N)^2 p(x)dx\,. \quad (5)$$

In this paper we consider the Laplacian source with memoryless property. In such a case, assuming unit variance, the probability density function of continuous random variable can be expressed by the Laplacian distribution defined as [5]:

$$p(x) = \frac{\sqrt{2}}{2}e^{-|x|\sqrt{2}}\,. \quad (6)$$

Hence, the total distortion of the compandor may be rewritten such as:

$$D = \frac{9}{2(N-2)^2}\left(1-\exp\left(-\frac{\sqrt{2}t_{N-1}}{3}\right)\right)^3 + \exp\left(-\sqrt{2}t_{N-1}\right)\left(t_{N-1}^2+\sqrt{2}t_{N-1}+1-y_N\left(2t_{N-1}+\sqrt{2}\right)+y_N^2\right). \quad (7)$$

2.2 Quantization method based on the Lloyd Max algorithm

Lloyd and Max proposed an algorithm to compute optimum quantizers using mean-square error distortion measure [2, 3, 12]. They provided the nonlinear quantization procedure in order to minimize the quantization noise. The Lloyd-Max's algorithm is widely used in practice because of its easily implementation. It is frequently called in literature the Lloyd-Max's I algorithm to distinguish it from the second algorithm. Namely, Lloyd developed the second algorithm for the scalar quantizers, known as Lloyd-Max's II algorithm [11, 12]. These two algorithms differ in regard to stopping criterion that interrrupts the algorithm. Namely, the Lloyd-Max's I algorithm stops when further iteration no longer produce any changes in distortion or changes are below the suitable threshold, while Lloyd-Max's II algorithm stops when suggested absolute accuracy of the last representation level is achieved. Let us consider Lloyd-Max's I algorithm. The algorithm consists of following steps:

Step 1. Begin with an initial codebook C_1. Set m=1.
Step 2.Gven the codebook, C_m, perform the Lloyd iteration to generate the improved codebook C_{m+1}.
Step 3. Compute the average distortion for C_{m+1}. If it has changed by a small enough amount since the last iteration, stop. Otherwise set m+1→m and go to Step. 2.
When using the quantization method based on the Lloyd-Max's I algorithm, the total distortion of the scalar quantizer may be defined as:

$$D = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \left(x - y_{j+1}\right)^2 p(x) dx \qquad (8)$$

2.3. Hybrid method

In this paper we combine two quantization methods (techniques). One of them is based on the companding technique and the other one on the Lloyd-Max's algorithm. Namely, for L«N, applying the companding technique to the range ($-t_{N-L}$, t_{N-L}) (inner region), i.e., to N-2L inner cells $\alpha_{L+1},\ldots, \alpha_{N-L}$, and Lloyd-Max's algorithm to union of ranges (t_0,$-t_{N-L}$) and (t_{N-L}, t_N) (outer region), i.e., 2L outer cells, $\alpha_1,\ldots,\alpha_L$ and $\alpha_{N-L+1},\ldots, \alpha_N$, it is possible to design the N-levels scalar quantizer. It is demonstrated in [5, 6] that the widths of the outer cells $\alpha_1,\ldots,\alpha_L$ and $\alpha_{N-L+1},\ldots, \alpha_N$ are constant and independent of the number of quantization levels N. When designing N-level scalar quantizer, based on the hybrid method, it is required to know the boundary between the inner and the outer region, t_{N-L}, and the set of L values of the reconstruction offsets $\delta_{N-L+1},\ldots,\delta_N$, i.e., the set of L+1 values. Thus, sparing the memory space in comparison to those, that are necessary for Lloyd-Max's scalar quantizers, simpler solution of hardware can be achieved. This is particularly of interest when designing scalar quantizers with average and large number of quantization levels N. The main contribution of this paper is finding for the values of L that provides optimal scalar quantizer design.

Considering the definition of the optimal compresor function given by Eq. (3) we can define the optimal compressor function (for the values of the decision thresholds t_j j=0,1,...,N) that is applied by the observed hybrid quantization method as follows:

$$c(t_j) = -t_{N-L} + 2t_{N-L} \frac{\int_{-t_{N-L}}^{t_j} p^{1/3}(x)dx}{\int_{-t_{N-L}}^{+t_{N-L}} p^{1/3}(x)dx}, \qquad (9)$$

where t_{N-L} is boundary between the inner and the outer region. When the values of the input signal x are within the ($-t_{N-L}$, t_{N-L}) range the values of $c(t_j)$ are copied into the [$-t_{N-L}$, t_{N-L}] range by using thus defined compressor function. Considering the Eqs. (7) and (8) the distortion of the obsreved quantizer can be expresed as a sum of the inner and the outer distortion:

$$D = \frac{1}{12(N-2L)^2} \left(\int_{-t_{N-L}}^{t_{N-L}} p^{1/3}(x)dx \right)^3 + 2 \sum_{j=N-L}^{N-1} \int_{t_j}^{t_{j+1}} \left(x - y_{j+1}\right)^2 p(x)dx . \qquad (10)$$

The optimal value of the boundary t_{N-L} can be find by minimizing the total distortion. Consequently, setting the first derivative of the total distortion, $\frac{\partial D}{\partial t_{N-L}} = 0$, to zero leads to the following equation for the determining of the optimal boundary t_{N-L}:

$$t_{N-L} = \frac{3}{\sqrt{2}} \ln\left(1 + \frac{\sqrt{2}}{3}(N-2L)\delta_{N-L+1} \right). \qquad (11)$$

3. Parameters of the hybrid model

We can consider two approahes in order to compute the values of the decision thresholds and the representation levels of the outer region. Namely we can use the values of the reconstruction offsets $\delta_{N-L+1},\ldots,\delta_N$ that were iterively computed in case of optimal Lloyd-Max's scalar quantizers [5] or we can use the values of the reconstruction offsets $\delta_{N-L+1},\ldots,\delta_N$ obtained by using the linearizing method, provided in [13]. These values are necessary when computing the values of the decision thresholds and the representation levels of the outer region that can be defined with:

$$y_{N-i+1} = t_{N-i} + \delta_{N-i+1}, \quad i = 1,\ldots, L, \qquad (12)$$

$$t_{N-i+1} = y_{N-i+1} + \delta_{N-i+2}, \quad i = 2,\ldots, L . \qquad (13)$$

Now, it is easy to define the support region of observed scalar quantizer ranging ($-t_{N-1}$,t_{N-1L}) interval, where t_{N-1} can be defined with:

$$t_{N-1} = t_{N-L} + 2 \sum_{k=N-L+1}^{N} \delta_k - \delta_{N-L+1} - \delta_N . \qquad (14)$$

The values of the decision thresholds and representation levels of the compandors are not optimal [5]. The goal of the proposed hybrid method is to make, as much as possible, the decision thresholds and the representation levels to be optimal. Also, when designing N-level Lloyd-Max's scalar quantizer it is necessary to know all the values of the decision thresholds and the representation levels. Hence, in such a case, considering the symmetry of the scalar quantizers' parameters, N+1 values should be memorized. The proposed quantization method is generale quantization method which for $L=N/2$ presents the quantization method used when designing Lloyd-Max's quantizers, while in case of L=0 presents the quantization method based on the companding technique. Increasing the values of L it is possible to arbitrarily approach to the optimal solution of the scalar quantizer designing problem.

	t_{31} (N=32)	t_{63} (N=64)	t_{127} (N=128)
L=1	5.0864	6.5244	7.9784
L=2	5.1119	6.5483	8.0015
L=3	5.1194	6.5552	8.0081
L=4	5.1184	6.5531	8.0055

Table 1. Support region thresholds t_{N-1}

	N=32	N=64	N=128
t_{N-1}^{opt}	5.1259	6.5604	8.0125

Table 2. Optimal reference of the support region thresholds t_{N-1}^{opt}

SNRQ	N=32	N=64	N=128
L=0	23.5709	29.5915	35.6121
L=1	23.8382	29.7261	35.6797
L=2	23.8523	29.7333	35.6833
L=3	23.8582	29.7363	35.6848
L=4	23.8615	29.7380	35.6856

Table 3. Numerical values of the *SNRQ*

	N=32	N=64	N=128
$SNRQ^{opt}$	23.87	29.74	35.69

Table 4. Optimal reference of the *SNRQ*

δ	N=32	N=64	N=128
L=0	0.0713	0.0348	0.0181
L=1	0.0074	0.0032	0.0024
L=2	0.0041	0.0015	0.0015
L=3	0.0027	0.0008	0.0012
L=4	0.0020	0.0004	0.0010

Table 5. Relative distortion error δ

4. The quantizer performances

In analyzing the behavior of the quantizer, it is preferable to use relative quantities, like signal to quantization noise ratio and relative distortion error instead of absolute quantities, such as distortion. Relative parameters portray the behavior of the quantizer in a way that is independent of the signal level and hence is more general. The performance of a quantizer is often specified in terms of *SNRQ* (signal to quantization noise ratio), given by [14]:

$$SNRQ = 10\log_{10}\left(\frac{\sigma^2}{D}\right), \tag{15}$$

measured in decibels, with σ^2denoting the variance of x. Here we assume the unit variance input signal, therefore *SNRQ* can be given by:

$$SNRQ = 10\log_{10}\left(\frac{1}{D}\right) \tag{16}$$

and it can be calculated for the proposed quantization method combining with (10). Therefore, using *SNRQ* we could assess the performance of a particular model of scalar quantizer. Let us define the relative distortion error δ such as:

$$\delta = \frac{D - D^{opt}}{D^{opt}}, \tag{17}$$

where D^{opt} is the optimal distortion value. Also, let us denote the optimal value of *SNRQ* with $SNRQ^{opt}$. Introducing the relation:

$$\Delta SNRQ = SNRQ - SNRQ^{opt} \tag{18}$$

the Eq. (17) becomes:

$$\delta = 10^{-\frac{\Delta SNRQ}{10}} - 1. \tag{19}$$

4.1 Analyse of numerical results

Table 1 provides numerical values of the support region thresholds t_{N-1}, computed for L=1,2,3,4, when the number of quantization levels varies (N=32, 64, 128). Then, Table 2 depicts optimal reference of the support region thresholds t_{N-1}^{opt} for optimal Lloyd-Max's scalar quantizers [5, 6, 7]. Assimilating the appropriate values from Table 1 and Table 2, one can notice that when the value of L grows it is approximately possible to approach to optimal values of the support region thresholds. Summary of the numerical values for *SNRQ*, computed for L=0,1,2,3,4, when the number of quantization levels are N=32, 64, 128, is given by Table 3. Furthermore, Table 4 provides the optimal reference of *SNRQ*, denoted as $SNRQ^{opt}$ [5]. Moreover, Table 5 provides the values of relative distorion error δ, computed for L=0,1,2,3,4, when the number of quantization levels varies (N=32, 64, 128). We can introduce the criterion of optimality which will be satisfied if the value of relative distorion error is less than small constant ε=0.005 [8]. The introduced criterion provides the choice of the minimum value of L that gives the optimal scalar quantizer design. It is obvious that for L=2, the appointed criteria is satisfied for all considered values of the number of levels N. Therefore, by choosing L=2, the proposed hybrid mehod provides optimal scalar quantizer design.

5. Conclusion

The proposed hybrid quantization method makes possible simpler design of optimal scalar quantizers demanding less memory space to store parameter values in comparison to Lloyd-Max's quantizers. It is very important to point out that for fixed values of L, by using the proposed hybrid quantization method, when the number of quantization levels N varies, the amount of the necessary computation of scalar quantizers' parameters is constant. Furthermore, the required memory space remains constant. The results demonstrate that by choosing the value of L=2 the introduced criterion of optimality is satisfied. Analysis presented in this paper has the practical importance since it could be of great help to engineers. Particularly, it provides fast and efficient design of scalar quantizers that are used for source coding of images [12] and speech [14].

6. Acknowledgement

D. Pokrajac has been partially supported by NIH-funded Delaware IDeA Network of Biomedical Research Excellence (INBRE) Grant, DoD HBCU/MI Infrastructure Support Program (45395-MA-ISP Department of Army), National Science Foundation (NSF) Infrastructure Grant (award # 0320991) and NSF grant "Seeds of Success: A Comprehensive Program for the Retention, Quality Training, and Advancement of STEM Student" (award #HRD-0310163).

References

[1] Robert M. Gray and David L. Neuhoff, Quantization, *IEEE Transactions on Information Theory*, Vol. 44, No. 6, October 1998., pp. 2325-2383.

[2] S. P. Lloyd, Least squares quantization in PCM, unpublished memo., Bell Lab.,1957; *IEEE Transactions on Information Theory*, Vol. IT-28, Mar. 1982, pp. 129-137.

[3] J. Max, Quantizing for minimum distortion, *IRE, Transactions on Information Theory*, Vol. IT-6, Mar. 1960., pp. 7-12.

[4] Neil Judell and Louis Scharf, A Simple Derivation of Lloyd's Classical Result for the Optimum Scalar Quantizer, *IEEE Transactions on Information Theory*, Vol. 32, No. 2, March 1986., pp. 326-328.

[5] N.S. Jayant, Peter Noll, *Digital coding of waveforms*, Prentice-Hall, New Jersey, 1984, Chapter 4, pp. 129-139.

[6] Sangsin Na and David L. Neuhoff, On the Support of MSE-Optimal, Fixed-Rate, Scalar Quantizers, *IEEE Transactions on Information Theory*, Vol. 47, No. 7, November 2001., pp. 2972-2982.

[7] Sangsin Na, On the Support of Fixed-Rate Minimum Mean-Squared Error Scalar Quantizers for a Laplacian Source, *IEEE Transactions on Information Theory*, Vol. 50, No. 5, May 2004., pp. 937-944.

[8] Robert Gray, *Quantization and data compression*, Lecture notes, Stanford University, 2004.

[9] Zoran Peric and Jelena Nikolic, Analysis of compressor functions for Laplacian source's scalar compandor construction, *Data Recording, Storage and Processing*, Vol. 8, No. 2, Jun 2006 pp. 15-24.

[10] Sangsin Na and David L. Neuhoff, Bennett's Integral for Vector Quantizers, *IEEE Transaction on Information Theory*, Vol.41, July 1995., pp. 886-900.

[11] X. Wu, On initialization of Max's algorithm for optimum quantization, *IEEE Transactions on Communication,* Vol. 38, No.10, pp.1653-1656, October 1990.

[12] Allen Gersho and Robert M. Gray, *Vector Quantization and Signal Compression*, Kluwer, Academ. Pub, 1992, Chapter 6, pp. 173-202.

[13] Zoran Peric, Jelena Nikolic and Dragoljub Pokrajac, New method for construction of optimal scalar quantizers for Laplacian source, accepted and will be published in *International Journal of Computing*, Vol. 5, No. 2, August 2006.

[14] Wai C. Chu, *Speech coding algorithms*, John Wiley & Sons, New Jersey, 2003, Chapter 6, pp. 161-165.