## Model for Determining of MTTF for Safety Related Electronical Systems by Monte Carlo Simulation by means of a 2004-System

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Abstract: - The Monte Carlo Simulation is one possibility to calculate safety parameters like MTTF for safety related systems. With this simulation the real world of failure on demand of a safety function is simulated using random numbers. The MTTF-value can be calculated directly via Monte Carlo Simulation or with the help of distributions like  $\chi^2$  - or Student-distribution. Important is the fact, that a high number of simulation cycles and/or a large simulation time is used.

*Key-Words:* - Simulation – Monte-Carlo-Method – Random Numbers – 2004 System – MTTF –  $\chi^2$ distribution – Student distribution – Frequency distribution

### **1** Introduction

Monte Carlo Simulation (MCS), a numeric experimental procedure, is applied to tasks that can be described as statistical problems. Typical statistical tasks are the determination of mean values, which can also be determined by a MCS approximation. One uses random values, which follow particular distributions, to solve the problem. A random variable is a variable, which in case of a repeated experiment or respectively a simulation can have different values from the space of possible values with certain probability. The values themselves are characterized as random values. From a mathematical point of view the sentence of Glivenco-Cantelli gives the basis for a MCS:

A function of random variants Z, which are defined by a distribution  $Z = F(x_1, x_2, ..., x_n)$  contains by experiment for the values  $x_i$  from its distribution a random solution  $Z_j$ .

The distribution of  $Z_i$  converges to the true distribution Z if one carries out a number of *n* simulations. It is important for simulations with random variables that the single results  $Z_i$  stay random. Therefore, the random value to be used is chosen by equally distributed random numbers. If a computer carries out the simulation, as is today the case, then the random numbers are derived from a random number generator of the computer. The random numbers are not purely random by nature, as the numbers with roulette, because they are calculated from a calculation algorithm. In this paragraph it is shown how failure time calculations of a system can be carried out with help from MCS.

#### 2.1 Mathematical Basis

A random value is defined as a value, which will have a different value (from the space of possible values) depending on probabilities when repeating the same experiment. They are described with the Greek alphabet  $\tau$ ,  $\xi$ ,  $\zeta$  etc. The random value, e. g.  $\tau$ , is characterized by the distribution function

$$F(x) = P\{\tau \le x\}.$$

This means that the value of F(x) represents the probability with which the random value  $\tau$  takes a value  $\leq x$ . The probability that  $\tau$  takes a value in the interval (a, b] can be described with

$$P\{a < \tau \le b\} = F(b) - F(a)$$

The distribution density (or probability density or just density), of  $\tau$  is characterized as

$$f(x) = \frac{dF(x)}{dx}$$
  
with

$$F(x) = \int_{-\infty}^{x} f(y) dy$$
 and  $\int_{-\infty}^{\infty} f(x) dx = 1$ .

The failure free operational time of a unit is often characterized in the reliability theory as  $\tau$ . Here,  $\tau$  represents a positive random value with F(0) = 0. The reliability function R(t) represents the probability that the considered unit operates failure free in the interval (0, t]. Thus  $P\{\tau \setminus t\}$ R(t)

$$R(t) = P\{t > t\},$$
  
and therefore for  $R(t)$ :  
$$R(t) = 1 - F(t).$$
  
Also the failure rate is defined as  
$$\lambda(t) = \frac{f(t)}{1 - F(t)} = -\frac{dR(t)/dt}{R(t)}.$$

In summary it applies, for a continuous random value  $\tau$ with density f(t), that the expected value (mean value) is

calculated as  $E[\tau] = \int_{-\infty}^{\infty} tf(t)dt$ . The above-mentioned equation is reduced for positive random values (t > 0) to

 $E[\tau] = \int_{0}^{\infty} tf(t)dt$ , and it can be shown that for a positive

value the following applies:

$$E[\tau] = \int_{0}^{\infty} (1 - F(t))dt = \int_{0}^{\infty} R(t)dt .$$

In general it is assumed here that R(0) = 1. The mean value calculated in this way is therefore identical to the mean value of the failure free operational time MTTF. Therefore, one gets the equation for the calculation of the *MTTF* values

$$MTTF = E[\tau] = \int_{0}^{\infty} R(t)dt .$$

It also applies here that this equation is valid for single components as well as for systems.

Two distributions are of major importance when calculating the MTTF via Monte Carlo Simulation. First of all, the  $\chi^2$  distribution. Here, the upper and lower limits are determined with the  $\alpha$  value, where the MTTF values to be simulated need to be 95 % of the time. The 95 % value means that 5% of the simulated value is allowed to be outside these limits. This means that for the same number of simulations a wider confidence interval is required if one wants to reduce this value. The number of simulations needs to be increased if a smaller confidence interval -and at the same time higher safetyis desired, which, on the other hand, leads to a significant increase in simulation time. It is known that a  $\chi^2$ -distribution becomes a normal distribution for the range against infinity.

On the other hand, the student distribution (also known as t-distribution) converges faster or less fast to a normal distribution when the number of simulations increases. For the  $\chi^2$  distribution the statements apply concerning the convergence interval.

In order to get a first estimate it is possible to get useful interval limits by considering the relative frequency. For this, one selects the frequency at determined times of the

simulation, for which corresponding specific MTTF values are available, and derives the so-called average MTTF value. Afterwards the deviation of the individual values can be determined compared to the mean value. Also here a deviation limit of 5 % is chosen.

If the random values  $X_1, X_2, ..., X_N$  are independent and identically distributed and a further random variable  $Y_N$ is represented by the function

$$Y_N = \frac{1}{N} \cdot (X_1 + X_2 + \dots + X_N)$$
 then the limit value

 $\lim_{N \to \infty} Y_N = c \text{ is developed with } c = \sum_i^{j} i \cdot P(X_1 = i).$ 

If *j* represents the maximum possible number of elementary events then the value c is represented as mean and represents the expected value for the random variable  $X_1$ . The random variable  $Y_N$ , on the other hand, is characterized as mean time and converges for Nattempts to the mean c for each individual attempt [10] [13] [14] [15] [16] [17] [18] [19].

#### 2.1.1 Random numbers

A number of real phenomena can be represented as stochastical processes. In analytical models randomness is seized by setting distribution types and the corresponding parameters for the calculations. One of the most important questions when considering random numbers that are computer-generated, is the formulation of the program's algorithm for the generation and derivation of the ordered distribution of the generated numbers.

In principle the following requirements are formulated for this set of tasks. The computer system needs to derive a sequence of real numbers that behave as if they were independent and randomly chosen from the interval between 0 and 1 and equally distributed.

The uniform distribution in the interval [0, 1] belongs to the continuous distributions and can be described as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Often this is also characterized as pdf (probability density function). The expected value of the distribution can then be determined with

$$E(x) = \int_{0}^{1} x \cdot f(x) \cdot dx = \int_{0}^{1} x \, dx = \left[\frac{x^{2}}{2}\right]_{0}^{1} = \frac{1}{2}$$

The variance is determined by

$$V(x) = \int_{0}^{1} [x \cdot E(x)]^{2} \cdot f(x) \, dx = \left[\frac{x^{3}}{3} - \frac{x^{2}}{2} + \frac{x}{4}\right]_{0}^{1} = \frac{1}{12}$$

The distribution function is

$$F(x) = \int_{0}^{x} f(t) \cdot dt$$

Here we need to consider what affects the quality of the random generator. Also here a lot of research has taken place in the past, which resulted in criteria that can be used to classify the quality of random number generators. One criterion is the independence of the random generated sequence. Furthermore, the uniform distribution is used for the evaluation. The empirical distribution of the random numbers must as much as possible represent a constant process for the interval [0, 1] and an often-increasing density function is not permissible. Furthermore, the population density is a criterion for the quality of the generator and therefore also the total simulation. There are always gaps between the characteristics, which in principle cannot be closed, as algorithmically based operating random number generators can only derive a limited amount of different numbers. The gaps may not significantly influence the uniform distribution criterion, and a sufficient different amount of numbers must be produced. The statement often mentioned in literature about the efficiency (memory and program efficiency) is nowadays, because of the possible memory space and the dramatically increased calculation power, of no significant meaning any more.

The methods most often used today are taken from the class of methods of the linear congruence.

Next, one of the most used algorithms for the derivation of random numbers is presented:

 $x_{i+1} = (a \cdot x_i + c) \mod m$ 

With this recursive algorithm it is to be ascertained that the parameters are selected favourably. In general *m* is chosen as double power based on the efficient processing in computer systems. The choice of parameters has also a direct influence on the potential cycle length. An algorithm is particularly promising when the largest common denominator of *m* and *c* is not larger than 1 and *a* equals a value of  $1 + 4 \cdot k$ , where *k* is an integer.

These requirements need to be considered when choosing a mathematical program when developing a program for this specific problem. The program chosen for the following simulations meets these requirements and works with the briefly mentioned linear congruence method.

For the Monte Carlo Simulation of safety related systems it is required to generate random numbers for the cases where within one path a component failure can occur. One falls back on the following equation for the calculation of the new value:

$$\lambda_i = -\frac{1}{\lambda} \ln(x)$$

Here, x represents the generated uniformly distributed random value in the interval between 0 and 1, and  $\lambda$  the primary failure rate of a block [11] [18] [21].

### 2.1 Monte-Carlo-Algorithm

The Monte Carlo method represents a possibility to generate random systems by simulation. In this way one can realize the *MTTF* value of a system. The following requirements need to be met for the used Monte Carlo algorithm:

The system consists of n components. All components and their states are independent of each other.

The individual components and the system exist in the states "failure", "operational" or "not operational".

All components have an exponentially distributed probability of failure, i. e. the failure rate  $\lambda$  or its reciprocal value respectively, the lifetime  $\tau_{LT}$ , is constant. The lifetime  $\tau_{LT}$  is known for each component. If a reliability block diagram of the system is present, then at least one path must exists between input and output of the system (which consists of those components) that carries out the function of the system. The path description is in literature known as "success path" method. The method is comparable with the Lee algorithm [11]. Also with this algorithm associated components are combined in paths. A path is not allowed to be a loop, i. e. a component is only allowed to be present in a path once.

The components of a path cannot be used any more unless they are used by another path. If the path of the system fails, the failed path is taken into operation again. After a system failure, a random generator derives a new lifetime for the failed component.

The repair time for each component is neglect able. Repair or exchange of components has no influence on the way the system functions and are not considered with the developed program.

The following notations are used in the calculation algorithm:

- *i* Component index
- *j* Path index
- *k* Index of failed systems
- $m_i$  Number of paths that contain component *i*
- $Pf_{ij}$  Path *j*, which contains component *i*, described by a vector
- $q^k$  System operational time until failure k
- $t_i^k$  Remaining lifetime of component *i* in case of system failure k 1
- $L_{ij}^k$  Lifetime of the path  $Pf_{ij}$  with component *i* of the path *j*
- $S_i^k$  Maximum path lifetime (= max<sub>j</sub>{ $L_{ij}^k$ }) of a component *i*, which exists in the different paths *j*.

If for example  $Pf_{ij}$  represents the path vector of a series system then

$$L_{ij}^k = \min_i \left\{ t_i^k \right\}$$

it applies for all components *i* with  $i \in Pf_{ij}$  for the  $k^{th}$  failure of the system.

If the system fails with the last operation path then

 $q^k = \max_{ij} \left\{ L_{ij}^k \right\}$ 

applies over all paths j of a component i and for all components i with the k<sup>th</sup> failure of the system.

If for the remaining lifetime of the component i in comparison to the maximum path lifetime the following applies

$$t_i^k \leq S_i^k$$

the component *i* lays in between the operational time  $(q^{k-1})$  with the (k-1)<sup>th</sup> failure and the operational time  $(q^k)$  with the k<sup>th</sup> failure.

If for the remaining lifetime of the *i*<sup>th</sup> component in comparison to the maximum path life time the following applies  $t_i^k > S_i^k$  the component causes within the interval  $\left[q^{k-1}, q^k\right]$  not the failure within the path, in which the component lays, and in the interval  $\left[q^k, q^{k+1}\right]$  the remaining lifetime of the *i*<sup>th</sup> component is  $t_i^{k+1} = t_i^k - S_i^k$  [4] [5] [6] [11].

# 2.1.1 Calculation algorithm of the MTTF with Monte Carlo simulation

The determination of the mean value takes place by determining for each simulation cycle the number of failures k for the considered time frame  $t = T_x$ . Per simulation cycle the MTTF value can be calculated from the number of failures k at a certain time  $T_x$  with the formula

$$MTTF_{for \ simulation \ i} = \frac{T_x}{k_i + 1}.$$

Attention should be paid to raising the derived number of failure k to one, as it is assumed that in between the last registered failure and  $t \rightarrow \infty$  another failure will occur. The *MTTF* mean value for all simulations is obtained if the mean value is now calculated from the *MTTF* values to be calculated. The formula for this is:

$$MTTF_{mean for j simulations} = \frac{\sum_{i=1}^{J} MTTF_{for simulation i}}{i}$$

or with using above mentioned formula:

$$MTTF_{mean for j simulations} = \frac{\sum_{i=1}^{j} \frac{T_x}{k_i + 1}}{j}.$$

If one wants to calculate the mean time for *j* simulations at time  $t = T_x$  hours, then the number of all failures  $k_s$  that occurred at all simulations needs to be determined.

The sum of all failures  $k_s$  is  $k_s = \sum_{i=1}^{J} k_i$ 

The median number of all failures for j simulations is

calculated with 
$$\overline{k_S} = \frac{\sum_{i=1}^{J} k_i}{j}$$

For all simulations the same time frame  $0 < t < T_x$  is considered, where for *j* simulations the failures  $k_i$  with  $I = 1 \dots j$  have occurred in order to determine the *MTTF* mean time of all simulations. The formula for the mean time is:

$$MTTF_{meantime for j simulations} = \frac{T_x}{\overline{k_s}}$$
$$= \frac{T_x}{\frac{\sum_{i=1}^{j} k_i}{\sum_{i=1}^{j} k_i}}$$

The mean as well as the mean time for the *MTTF* value is calculated according to the above formula. The result is presented as a vector, where the first element represents the mean time, the second element the mean, and the third element the sum of all failures per simulation cycle.

The values  $MTTF_{for simulation i}$  are required to obtain an approximated distribution density for the random variable  $MTTF_{Mean}$  with help of the relative frequency. From these values a minimum and a maximum can be determined. The distribution of the searched random value MTTF<sub>Mean</sub> is obtained if the interval is divided between the minimum and the maximum in n equal intervals and one adds how many values MTTF for simulation i fall in each interval. The relative frequency is obtained by dividing the corresponding number of values in each interval by the number of all values. A histogram can be created with the relative frequencies. It can serve as a first approximation of the unknown density for the random value  $MTTF_{Mean}$ . Further histograms need to be created at different times  $T_x$  to make a statement about which density it possibly is. If one derives the area below the histogram, then this value is equal to the probability that a, from the random sample taken, MTTF<sub>mean</sub> falls within the confidence limits, which are determined by the left and right limits of the area. Mathematically this is formulated by the equation:

$$P\{x_u < MTTF_{mean} < x_o\} = 0.95$$

Here,  $x_u$  and  $x_o$  are the lower and upper limit respectively, where the random value  $MTTF_{Mean}$  exists within a probability of 95 %.

Another possibility to calculate the confidence interval is with help from the  $\chi^2$ -distribution. Also here a lower and

upper limit where the random value  $MTTF_{Mean}$  exists with 95 % probability must be given. The interval limits at time  $T_x$  result in

$$[2 \cdot \frac{sim \cdot T_x}{chi\_u}, 2 \cdot \frac{sim \cdot T_x}{chi\_o}].$$

Here, the number of simulations for the time  $T_x$  must be set for the variable *sim*. *chi\_u* and *chi\_o* are calculated as follows:

The corresponding variables  $chi_u$  and  $chi_o$  can be calculated with help from the inverse  $\chi^2$  distribution and as basis the formulas

$$P\{\chi^2 - Dist.\} = \int_{0}^{chi_{-}o} \frac{e^{-\frac{x}{2}} \cdot \left(\frac{x}{2}\right)^{\frac{n_{chi}}{2} - 1}}{2 \cdot \Gamma\left(\frac{n_{chi}}{2}\right)} \cdot dx = 1 - \frac{\alpha}{2} = 0,975$$

with *chi\_o* as upper integrand and the formula

$$P\{\chi^{2} - Dist.\} = \int_{0}^{chi_{-}u} \frac{e^{-\frac{x}{2}} \cdot \left(\frac{x}{2}\right)^{\frac{hcm_{-}}{2}}}{2 \cdot \Gamma\left(\frac{n_{chi}}{2}\right)} \cdot dx = \frac{\alpha}{2} = 0,025$$

with  $chi\_u$  as lower integrand. Besides the  $\alpha$  value also the so-called error probability is required, which gives the probability that the desired random number does not exist within the desired limits, nor the desired degree of freedom  $n_{chi}$ . For the upper limit the degree of freedom  $n_{chi\_o}$  results in  $n_{chi\_o} = 2 \cdot n$ .

For the calculation of the  $MTTF_{Mean}$  the variable *n* represents the number of all failures over all simulations at time  $T_x$ . For the determination of the lower limit the following approach is chosen:

$$n_{chi}$$
  $_{u} = 2 \cdot n + 2$ 

In literature this approach is justified because these random samples concerning time are cancelled. The t distribution or student distribution is the third possibility to determine the confidence limits. Also here a lower and upper limit must be given where the random value  $MTTF_{Mean}$  lays within with a probability of 95 %.

The interval limits at time  $T_x$  are determined by

$$\left[\overline{x}-t\right]_{f-1;\frac{\alpha}{2}}\cdot\frac{s}{\sqrt{f}}, \overline{x}+t\right]_{f-1;\frac{\alpha}{2}}\cdot\frac{s}{\sqrt{f}}$$

Here,  $\overline{x}$  is the *MTTF* mean value, s the standard deviation of the given simulation and f the degree of freedom for the t distribution.

For the t-distribution the following approach is chosen:

The density 
$$p$$
 of the t-distribution is  
 $p\{t - Distribution\} = \Gamma\left(\frac{f+1}{2}\right) \cdot \frac{\left(1 + \frac{x^2}{f}\right)^{-\frac{f+1}{2}}}{\sqrt{\pi \cdot f} \cdot \Gamma\left(\frac{f}{2}\right)}$ 

where  $\Gamma$  means the Gamma function and *f* the degrees of freedom.

The number of degrees of freedom is obtained from f = sim - 1

$$P\{t \text{ distribution}\} = \int_{-x}^{+x} p\{t \text{ distribution}\} \cdot dx = 1 - \alpha = 0.95$$

the integrants -x and +x are calculated with help from the inverse t-distribution function  $qt(\alpha, f)$ , where applies

$$qt(\frac{\alpha}{2}, f) = -qt(1 - \frac{\alpha}{2}, f).$$

Next to the  $\alpha$  value the so-called error probability and degree of freedom *f* are required [7] [8] [9] [12] [14] [20] [22].

#### **3** 2004 System, an Example

The 2004 system, as presented in consists of twelve functional blocks, four input circuits (AI), four safe logic solvers (L) and four output circuits (AO), where the output circuits are divided into two individual blocks.

The failure rates listed in Table 1 for these functional blocks are in size comparable with values in practice and are also used in the calculations with the Markov models. These failures rates are the starting values for the failure rates of the Monte Carlo program. Before the start of the program the individual path combinations need to be determined beside the determination of the failure rates of the individual functional blocks.



Fig. 1: Block diagram 2004 system

In one path all those functional blocks of a system are in series (for example of a safety system) that are required to confirm an incoming signal by the logic solver at the input of the system and that require an appropriate response at the output of the system. Here counts that if a functional block fails also the path fails. The block diagram of the possible path combinations for a 2004 system is presented in Fig. 2. The matrix required for the simulation program:

Path combination matrix :

1	Z	5	
4	5	6	
7	8	9	
10	11	12	

 $^{\neg}$ 

 Table 1: 2004 system; Component failure rates

Components Failure rate $\lambda$	AI	L	AO
Channel 1	1600 FIT	2500 FIT	4200 Fit
Channel 2	1600 FIT	2500 FIT	4200 Fit
Channel 3	1600 FIT	2500 FIT	4200 Fit
Channel 4	1600 FIT	2500 FIT	4200 Fit

For the simulation program the following data are required:

Start time :	t_start
End time:	Т
Time increment:	t_inkrement
Simulation cycles per point in time:	sim.

The start values for these variables can be taken from Table 2. The results of the simulation for the *MTTF* values of the current 2004 system are presented in

Fig. 3. Here the upper curve shows the mean time of the *MTTF* values, while the lower curve represents the mean. At time t = T = 500000 hours for 100 simulations the curves in

Fig. 3 show different values for the mean as well as the mean time. Still, also here counts: for  $t \rightarrow \infty$  the mean time converges to the mean. The value is then identical to the *MTTF* value from the calculated Markov model.



Fig. 2: Path combinations for a 2004 system

The values presented in Table 3 for the mean time and the mean were obtained from Fig. 3. One can determine a deviation of 13 % for the mean and a deviation of 2,5 % for the mean time if these values are compared to the theoretical *MTTF* value from the Markov model. Fig. 3 shows that the simulated values converge to the theoretical Markov values for increasing *T*.

Table 2: 2004 system; Start values for	the
Monte-Carlo-Simulation	

t_start [hours ]	T [hours ]	t_inkrement [hours ]	sim
1000	500000	10000	1000



Fig. 3: Simulation MTTF 2004; Mean time, mean for 100 simulations per point in time

Table 3: 2004 system; Mean time andmean with deviation from the theoretical*MTTF* value for 100 simulations per timeof point

Theoretical MTTF- Wert from Markov model in hours	on ne at time T Fig. 3	on mean at t om Fig. 3	Deviation MTTF <sub>Mar</sub> kov - MTTF <sub>Mea</sub> n tkme (T) in %	Deviation MTTF <sub>Markov</sub> - MTTF <sub>Mean</sub> (T) in %
134119	137500	116670	2,52	-13,01

The following simulation determines how large the deviations from the theoretical values are when the simulation cycles per point in time differ. The number of simulation cycles is 10, 100 and 1000. All other start conditions are equal to the listed values in Table 2.

The result of a first simulation of a mean for the *MTTF* value was obtained from the curves of Fig. 4 and is summarized in Table 4. From the table it can be taken that at time T = 500000 hours the deviations between  $MTTF_{\text{Markov}}$  and  $MTTF_{\text{Mean}}$  for 100 simulations is smaller than for ten simulations. For 1000 simulations the smallest deviation exists between  $MTTF_{\text{Markov}}$  and  $MTTF_{\text{Mean}}$  at time T. The result for 1000 simulations corresponds with the expected results. For 1000 simulations one would have expected a value between the value for 10 and 1000 simulations.

However, one should note the following: The results are only interpreted as so-called instant results. Especially for limited simulation cycles the results can clearly vary. It is important that the trend is in the right ranges. The trend can be confirmed based on Fig. 4. For ten simulations, presented by curve 1, very large fluctuations can exist between the individual *MTTF* values for two sequential points in time. The *MTTF* value randomly obtained from the simulation can come very close to the theoretical value. The curve for 100 simulations, curve 2, shows in general fewer fluctuations as the curve for ten simulations. If, on the other hand, the curve for 1000 simulations is considered, curve 3, then this shows the least fluctuations between two sequential points in time.



Fig. 4: Simulation *MTTF* 2004; Mean for 10, 100 and 1000 simulations per point in time

Table 4: 2004 system; first simulation; Mean with deviation of the theoretical value for 10, 100, and 1000 simulations per point in time

Theoretical MTTF value from Markov model in hours	Number of simulation cycles	Simulation mean at time <i>T</i> in hours from Fig. 4	Deviation MTTF <sub>Markov</sub> - MTTF <sub>Mean</sub> (T) in %
134119	10	108330	-19,23
134119	100	116670	-13,01
134119	1000	116670	-13,01

#### **3.1 Frequency distribution of MTTF values**

On the basis of Fig. 5 it is obvious that the emphasis of the MTTF value for a 2004 system with the chosen failure rates Table 1 is around  $1 \cdot 10^{-5}$ . At the interval on the right hand side, thus at  $T = 5 \cdot 10^5$  hours, a number of recognizable simulations ( $\approx 5$ ) exists, for which at this time no system failure is still registered.

This means that for about 0,5 % no statement can be made concerning the *MTTF* value. Nevertheless, also these values are considered when obtaining the *MTTF* value of the system. For that purpose the number of failures obtained until time *T* is increased by one in the Monte Carlo program. The smaller the number of simulations for which at time T no failure exists yet in the system, the more accurate the *MTTF* value of the system to be determined. Fluctuation in the order of percentages must be expected by determining the *MTTF* value for a 2004 system.



Fig. 5: Frequency distribution for the 2004 system at time T = 500000 hours and 1000 simulations

Two roads can be taken in order to receive a smaller deviation:

- An increase of the simulation cycles or
- An increase of the simulation time.

In order to make reliable statements concerning the values, the time T is set to 5000000 hours in the following simulations.



Fig. 6: Frequency distribution for the 2004 system, at time T = 500000 hours and 100 simulations

Fig. 6 represents the frequency distribution at time T = 5000000 hours. It is recognized that for a 2004 system with chosen failure rates according to Table 1 the focus point of the *MTTF* values exists from  $1,2 \cdot 10^{-5}$  to  $1,6 \cdot 10^{-5}$ . If one compares Fig. 6 with Fig. 5, then it is

recognized that the distribution approximates a normal distribution better at T = 5000000 hours than at T =500000 hours. This evidence shows that much better simulation values are obtained for T = 5000000 hours than for T = 500000 hours.

3.2  $\chi^2$ distribution for determining MTTF If a two-sided confidence interval is built with help from the  $\chi^2$  distribution for each determined *MTTF* value at a certain point in time t, which is chosen for the simulation, then this leads to the results in Fig. 7.



Fig. 7: Confidence interval for  $\chi^2$ distribution (10 %), 2004 system, for 100 simulations per point in time

From Fig. 7 the following values can be derived for 100 simulation cycles for T = 5000000 hours:

> Table 5: 2004 system; simulation for T =5000000 hours and 100 simulation cycles; Determination of the confidence intervals for the  $\chi^2$  distribution

Theoretical MTTF value from Markov model in hours	Number of simulation cycles	Simulation mean at time <i>T</i> in hours from Fig. 7	Lower confidence limit for 95 % certainty from Fig. 7	Upper confidence limit for 95 % certainty from Fig. 7
134119	100	129170	125000	133330

Deviation MTTF <sub>Markov</sub> MTTF <sub>Mean</sub> (T) in %	Deviation MTTF <sub>Markov</sub> - lower confidence limit for 95 % certainty in %	Deviation MTTF <sub>Markov</sub> - upper confidence limit for 95 % certainty in %	Deviation MTTF <sub>Mean</sub> - lower confidence limit for 95 % certainty in %	Deviation MTTF <sub>Mean</sub> - upper confidence limit for 95 % certainty in %
-3,69	-6,80	-0,59	3,34	3,22

The results are interpreted as follows: With a certainty of 95 % the wanted *MTTF* value at time T = 5000000 hours exists with the time interval [125000, 133330] hours. For this simulation this means a maximum deviation from the theoretical calculated MTTF value of 3,7 %. In general the following can be noted:

The  $\chi^2$  distribution is much better suited to determine failure limits, as it is possible with help from the frequency distribution.

The  $\chi^2$  distribution makes timely, much faster simulations and at the same time more accurate results possible.

These limits do not mean that a simulated value cannot exist outside the limits. Therefore the limits in this example only represent a probability of 95 % that the wanted MTTF value exists within the simulated MTTF value.

#### 3.3 Student distribution for determining MTTF

A third possibility to determine the confidence interval is the application of the student distribution. In the Monte Carlo program a certainty of 95 % was already ascertained for the determination of the limits with help from the frequency distribution. Therefore, the  $\alpha$  value was determined 0,05 as with the  $\chi^2$  distribution. With the  $\alpha$  value a lower and upper limit can be determined where 95 % of the simulated MTTF values should exist. The 95 % limit means that 5 % of the simulated values exist outside these limits. This means that if one wants to decrease this value, a wider confidence interval is created while keeping the number of simulations equal. The number of simulations needs to be increased if a very narrow confidence interval with high certainty is desired. This leads again to significantly longer simulation times.

#### It can be stated:

If one builds the two-sided confidence interval with help from the student distribution for the derived MTTF value at a certain point in time t, which is chosen for the simulation, then this results in Fig. 8.



Fig. 8: Confidence interval for student distribution (10 %), 2004 system, for 1000 simulations per point in time

It is shown that both student distribution curves (curve 1 and curve 2 in Fig. 8), which build the interval limits, approximate very well the simulated MTTF curve (curve 3) for the full time frame. Also the student distribution is considered at time T = 5000000 hours and gives the following values in Table 6 for 100 simulations per point in time:

#### Table 6: 2004 system; simulation for T =5000000 hours and 100 simulation cycles: Determination of the confidence intervals for the Student distribution

Theoreti- cal <i>MTTF</i> value from Markov model in hours	Number of simulation cycles	Simulation mean at time <i>T</i> in hours from 1 <sup>st</sup> /2 <sup>nd</sup> simulation	Lower confidence limit for 95 % certainty from 1 <sup>st</sup> /2 <sup>nd</sup> simulation	Upper confidence limit for 95 % cer- tainty from 1 <sup>st</sup> /2 <sup>nd</sup> simula tion
134119	100	130200	127900	132400
Deviation MTTF <sub>Marko</sub> v MTTF <sub>Mean</sub> (T)	Deviation MTTF <sub>Marko</sub> ,- lower confidence limit for 95 %	Deviation MTTF <sub>Markov</sub> - upper confidence limit for 95 %	Deviation MTTF <sub>Mean</sub> - lower confidence limit for 95 % certainty	Deviation MTTF <sub>Mean</sub> - upper confidence limit for 95 % certainty

certainty in

%

-1,28

% certainty

in %

1,80

in %

1.69

It is determined here that with a certainty of 95 % the wanted MTTF value at time T = 5000000 hours exists within the time intervals [128200, 133500] hours and [127900, 132400] hours. For this simulation this means a maximum deviation of approximately 3 % respectively from the theoretical calculated MTTF value. Also the deviation of about 1,8 % respectively from the confidence limits to the  $MTTF_{Mean}$  value exists in a very good range.

#### 4 Conclusion

certainty

in %

-4.64

in %

-2,92

This paper describes the calculation of MTTF-value via Monte-Carlo-Simulation. In order to minimise the deviation the following can be done:

- an increase of the simulation cycles or
- an increase of the simulation time.

Then the results are comparable with the theoretically calculated value. For a sufficient proper estimation of the MTTF-value distributions used two are for determination of a confidence interval, the  $\chi^2$ -and the Student-distribution. Also here it is important, that better values involve using a higher number of simulation cycles and/or longer simulation time. As an example a 2004-system is considered to calculate the system's MTTF value. It is obtained less deviations corresponding to simulation cycles and simulation time with both distributions. The deviations range between 2 and 4 %.

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