Proceedings of the 4th WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT, Elounda, Greece, August 21-23, 2006 (pp238-243)

Some Approximate Analytical Steady-State Solutions for Cylindrical Fin

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Abstract: - In this paper we construct some approximate analytical three-dimensional solutions for one element of cylindrical wall and fin. We assume that the heat transfer process in the wall and the fin is stationary. These solutions are obtained by the original method of conservative averaging and they are compared to some one-dimensional solutions, which are well known in literature. We give some criterions when it is possible to replace three-dimensional formulation of problem with two- or one-dimensional statement.

Key-Words: - steady-state, three-dimensional, heat exchange, cylindrical fin, analytical solutions, conservative averaging.

1 Introduction

Obtaining efficient cooling for the components of devices is a difficult challenge in modern industry. It is related to refrigerators, radiators, engines and modern electronics, etc.

Usually its mathematical modeling is realized by one dimensional steady-state assumptions [1],[5]. In our previous papers we have constructed two dimensional analytical approximate [2]-[4] and exact [3] solutions. In this paper we obtain few new approximate analytical three dimensional solutions by the original method of conservative averaging and some its simplifications (special cases).

In [1] the so-called Murray – Gardner assumptions are formulated. They are:

1) The heat flow in the fin and the temperature at any point on the fin remain constant with time;
2) The fin material is homogeneous; its thermal conductivity is the same in all directions and remains constant;
3) The heat transfer coefficient between the fin and the surrounding medium is uniform and constant over the entire surface of the fin;
4) The temperature of the medium surrounding the fin is uniform;
5) The fin width is so small compared with its height that temperature gradients *across* the fin width may be neglected;
6) The temperature at the base of the fin is uniform;
7) There are no heat sources within the fin itself;
8) Heat transfer to or from the fin is proportional to the temperature excess between the fin and the surrounding medium;
9) There is no contact resistance between fins in the configuration or between the fin at the base of the configuration and the prime surface;
10) The heat transferred through the outmost edge of the fin (the fin tip) is negligible compared to that through the lateral surfaces (faces) of the fin.

2 Mathematical Formulation of 3D Problem and reduction to 2D

We will start with accurate three-dimensional formulation of steady-state problem for one element of periodical system for cylindrical wall and fin. The one element of the wall (base) is placed in the domain $\{\tilde r \in [R_0, R], \tilde z \in [0, Z], \varphi \in [0, \phi]\}$ and we describe temperature field $\tilde V_0(\tilde r, \tilde z, \varphi)$ in the wall with the equation:

$$\frac{1}{\tilde r}\frac{\partial}{\partial \tilde r}\left(\tilde r \frac{\partial \tilde V_0}{\partial \tilde r}\right) + \frac{\partial^2 \tilde V_0}{\partial \tilde z^2} + \frac{1}{\tilde r^2}\frac{\partial \tilde V_0}{\partial \varphi^2} = 0. \qquad (1)$$

~~The c~~ylindric~~al fi~~n ~~of length~~ ~~L oc~~cupies the domain $\{\tilde r \in [R, R+L], \tilde z \in [0, Z_0], \varphi \in [0, \phi]\}$ and the temperature field $\tilde V(\tilde r, \tilde z, \varphi)$ fulfills the equation:

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{V}}{\partial\tilde{r}}\right)+\frac{\partial^2\tilde{V}}{\partial\tilde{z}^2}+\frac{1}{\tilde{r}^2}\frac{\partial\tilde{V}}{\partial\varphi^2}=0 \qquad (2)$$

And we have following boundary conditions in accordance with M-G point 5) in φ direction

$$\left.\frac{\partial\tilde{V}}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial\tilde{V}}{\partial\varphi}\right|_{\varphi=\phi}=0\,;\;\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial\tilde{V}_0}{\partial\varphi}\right|_{\varphi=\phi}=0 \qquad (3)$$

We can reduce problem (1) and (2) from 3D to 2D using following average integral for argument φ

$$\tilde{U}(\tilde{r},\tilde{z})=\frac{1}{\phi}\int_0^{\phi}\tilde{V}(\tilde{r},\tilde{z},\varphi)d\varphi$$

and $\tilde{U}_0(\tilde{r},\tilde{z})=\frac{1}{\phi}\int_0^{\phi}\tilde{V}_0(\tilde{r},\tilde{z},\varphi)d\varphi$. (4)

2.1 Description of Temperature Field in the Wall

We will use following dimensionless arguments, parameters, to transform our problem to dimensionless problem:

$$r=\frac{\tilde{r}}{Z},\, z=\frac{\tilde{z}}{Z},\, \delta_0=\frac{R_0}{Z},\; \delta=\frac{R}{Z},\delta+l=\frac{R+L}{Z},$$

$$b=\frac{Z_0}{Z},\; \beta_0=\frac{hZ}{k_0},\beta=\frac{hZ}{k},\beta_0^0=\frac{h_0Z}{k_0}.$$

And temperatures:

$$U(r,z)=\frac{\tilde{U}(r,z)-T_a}{T_b-T_a},U_0(r,z)=\frac{\tilde{U}_0(r,z)-T_a}{T_b-T_a}.$$

Here $k(k_0)$ - heat conductivity coefficient for the fin (wall), $h(h_0)$ - heat exchange coefficient for the fin (wall), Z_0 - width (thickness) of the fin, L - length of the fin, Z - thickness of the wall, T_b - the surrounding temperature on the left (hot) side (the heat source side) of the wall, T_a - the surrounding temperature on the right (cold - the heat sink side) side of the wall and the fin and $\tilde{U}(r,z)$ $\left(\tilde{U}_0(r,z)\right)$.

One element of the wall (base) placed in the domain now is $\{r\in[\delta_0,\delta],z\in[0,1]\}$ and we can describe the dimensionless temperature field $U_0(r,z)$ in the wall with the equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_0}{\partial r}\right)+\frac{\partial^2U_0}{\partial z^2}=0 \qquad (5)$$

We add needed boundary conditions as follow:

$$r\frac{\partial U_0}{\partial r}+\beta_0^0(1-U_0)=0\,,r=\delta_0\,,z\in[0,1], \qquad (6)$$

$$r\frac{\partial U_0}{\partial r}+\beta_0U_0=0\,,r=\delta\,,\; z\in[b,1]. \qquad (7)$$

And homogeneous boundary conditions

$$\left.\frac{\partial U_0}{\partial z}\right|_{z=0}=\left.\frac{\partial U_0}{\partial z}\right|_{z=1}=0\,,r=[\delta_0,\delta]. \qquad (8)$$

We assume the conjugations conditions on the surface between the wall and the fin as ideal thermal contact - there is no contact resistance:

$$U_0|_{\delta-0}=U|_{\delta+0},\;\beta\left.\frac{\partial U_0}{\partial r}\right|_{\delta-0}=\beta_0\left.\frac{\partial U}{\partial r}\right|_{\delta+0}. \qquad (9)$$

2.2 Description of Temperature Field in the Fin

The cylindrical fin of length l occupies the domain $\{r\in[\delta,\delta+l],z\in[0,b]\}$ and the temperature field $U(r,z)$ fulfills the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2U}{\partial z^2}=0 \qquad (10)$$

We have following boundary conditions for the fin:

$$\frac{\partial U}{\partial z}+\beta U=0,\; r=[\delta,\delta+l],\; z=b, \qquad (11)$$

$$r\frac{\partial U}{\partial r}+\beta U=0,\; r=\delta+l,\; z\in[0,b]. \qquad (12)$$

And homogeneous boundary conditions

$$\left.\frac{\partial U}{\partial z}\right|_{z=0}=0\,,r=[\delta,\delta+l]. \qquad (13)$$

3 Approximate Solution of 2D Problem for Periodical System

We will use the original method of conservative averaging. We will start with the case of periodical system with cylindrical fins

3.1 Reduction of the 2D Problem for the Fin

Similarly as in papers [2],[4] we will use original method of conservative averaging and approximate the 2D temperature field $U(r,z)$ for the fin in following form:

$$U(r,z)=f_0(r)+\left(e^{\rho z}-1\right)f_1(r)+ +\left(1-e^{-\rho z}\right)f_2(r),\quad \rho=\mathrm{b}^{-1} \qquad (14)$$

with three unknown functions $f_i(r), i=0,1,2$. For this purpose we introduce the integral average value of function $U(r,z)$ in the z - direction:

$$u(r)=\rho\int_0^b U(r,z)dz\,. \qquad (15)$$

This equality together with two boundary conditions (at $z=0$ and $z=b$) allow us to exclude all the unknown functions $f_i(r)$ from the representation (14). The boundary condition (13) for the function $U(r,z)$ at $z=0$ gives immediately the equality $f_1(r)=-f_2(r)$. The substitution of representation (14) in (15) gives expression:

$$f_1(r)=\frac{u(r)-f_0(r)}{2(\sinh(1)-1)} \qquad (16)$$

and representation (14) takes form:

$$U(r,z)=\frac{\cosh(\rho z)-1}{\sinh(1)-1}u(r)+ \\ +\frac{\sinh(1)-\cosh(\rho z)}{\sinh(1)-1}f_0(r) \qquad (17)$$

Finally, by the use of the boundary condition (11), we can exclude $f_0(r)$ from last expression and represent the 2D solution $U(r,z)$ for the fin in following form:

$$U(r,z)=u(r)\Phi(z). \qquad (18)$$

It is easy to check that the function $\Phi(z)$ looks like

$$\Phi(z)=\frac{\sinh(1)+\beta b(\cosh(1)-\cosh(\rho z))}{\sinh(1)+\beta b(\cosh(1)-\sinh(1))} \qquad (19)$$

The second stage for the method of conservative averaging is to transform the partial differential equation (15) for the function $U(r,z)$ to the differential equation for one arguments function $u(r)$. To realize this goal we integrate the main differential equation (10) in the z - direction, and using (15) get:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\frac{1}{b}\left.\frac{\partial U}{\partial z}\right|_{z=b}-\frac{1}{b}\left.\frac{\partial U}{\partial z}\right|_{z=0}=0 \qquad (20)$$

By using the boundary condition (13) at $z=0$ for the function $U(r,z)$ and expressing from the boundary condition (11) at $z=b$ the first derivative $\frac{\partial U}{\partial z}$ trough the function $U(r,z)$ we obtain following differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)-\left.\frac{\beta U}{b}\right|_{z=b}=0 \qquad (21)$$

It remains to express in differential equation (21) the function $U(r,z)$ through the function $u(r)$ with the help of the equality (18) and we receive the new differential equation, which describes the 1D dimensional temperature field $u(r)$ in the fin:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)-\mu^2u(r)=0. \qquad (22)$$

Here

$$\mu^2=\frac{\beta}{b}\Phi(b). \qquad (23)$$

Solving differential equation (22) we gain solution through Bessel's modified functions I_0, K_0:

$$u(r)=C_1(\mu_1 I_0(\mu r)+K_0(\mu r)), \qquad (24)$$

where

$$\mu_1=\frac{\mu K_1(\mu r)-\frac{\beta}{\delta+l}K_0(\mu r)}{\mu I_1(\mu r)+\frac{\beta}{\delta+l}I_0(\mu r)} \qquad (25)$$

And from (18) and (24) we get solution for fin which includes only one unknown constant C_1

$$U(r,z)=C_1(\mu_1 I_0(\mu r)+K_0(\mu r))\Phi(z). \qquad (26)$$

3.2 Reduction of the 2D Problem for the Wall

We will use same method of conservative averaging and approximate the 2D temperature field $U_0(r,z)$ for the fin in following form:

$$U_0(r,z)=g_0(z)+\left(\frac{1}{r}-\frac{1}{\delta}\right)g_1(z)+ \\ +\frac{\delta-r}{\delta-\delta_0}g_2(z) \qquad (27)$$

with three unknown functions $g_i(z), i=0,1,2$. For this purpose we introduce the integral average value of function $U_0(r,z)$ in the r - direction:

$$u_0(z)=\frac{2}{\delta^2-\delta_0^2}\int_{\delta_0}^{\delta} rU_0(r,z)dr. \qquad (28)$$

This equality together with equality (27):

$$u_0(z)=g_0(z)+\varphi_1 g_1(z)+\varphi_2 g_2(z) \qquad (29)$$

where $\varphi_1=\frac{\delta-\delta_0}{(\delta+\delta_0)\delta}$, $\varphi_2=\frac{\delta+2\delta_0}{3(\delta+\delta_0)}$.

Finding derivation of (27) and using boundary condition (6) we get

$$\beta_0^0(1-g_0)=\frac{(\delta-\delta_0)\beta_0^0+\delta}{\delta\delta_0}g_1(z)+ \\ +\frac{(\delta-\delta_0)\beta_0^0+\delta_0}{\delta-\delta_0}g_2(z) \qquad (30)$$

Express function $g_2(z)$ from equation (29) and put to expression (30) gives

$$K_1 g_1(z)=A_1 g_0(z)-B_1 u_0(z)+D_1, \qquad (31)$$

where

$$K_1 = \frac{(\delta-\delta_0)\beta_0^0+\delta}{\delta\delta_0} - \frac{\varphi_1((\delta-\delta_0)\beta_0^0+\delta_0)}{\varphi_2(\delta-\delta_0)},$$

$$A_1 = -\beta_0^0 - \frac{(\delta-\delta_0)\beta_0^0+\delta_0}{\varphi_2(\delta-\delta_0)},$$

$$B_1 = \frac{(\delta-\delta_0)\beta_0^0+\delta_0}{\varphi_2(\delta-\delta_0)},\ D_1 = \beta_0^0.$$

Express function $g_1(z)$ from equation (29) and put to expression (30) gives

$$K_2 g_2(z) = -A_2 g_0(z) + B_2 u_0(z) - D_2, \qquad (32)$$

where

$$K_2 = \frac{(\delta-\delta_0)\beta_0^0+\delta_0}{\delta-\delta_0} - \frac{\varphi_2((\delta-\delta_0)\beta_0^0+\delta)}{\varphi_1\delta\delta_0},$$

$$A_2 = \beta_0^0 - \frac{(\delta-\delta_0)\beta_0^0+\delta}{\delta\delta_0\varphi_1},$$

$$B_2 = -\frac{(\delta-\delta_0)\beta_0^0+\delta}{\delta\delta_0\varphi_1},\ D_2 = -\beta_0^0.$$

Equations (31) and (32) give

$$\begin{cases} g_1(z) = a_1 g_0(z) - b_1 u_0(z) + d_1 \\ g_2(z) = -a_2 g_0(z) + b_2 u_0(z) + d_2 \end{cases} \qquad (33)$$

where $a_i = \frac{A_i}{K_i}, b_i = \frac{B_i}{K_i}, d_i = \frac{D_i}{K_i}$.

Putting (33) to (27) we gain

$$U_0(r,z) = g_0(z)\left[1 + (\delta - r)\left(\frac{a_1}{\delta r} - \frac{a_2}{\delta-\delta_0}\right)\right] + \\ +(\delta - r)\left(\frac{b_2}{\delta-\delta_0} - \frac{b_1}{\delta r}\right)u_0(z) + \\ +(\delta - r)\left(\frac{d_1}{\delta r} - \frac{d_2}{\delta-\delta_0}\right) \qquad (34)$$

Now we still have two unknown functions $g_0(z)$ and $u_0(z)$. Therefore we will use different boundary and conjugations conditions on the wall to exclude these functions.

3.3.1 Solution for upper Wall

Putting (34) to boundary condition (7) we get

$$g_0(z) = b_0 u_0(z) - d_0 \qquad (35)$$

where $b_0 = \frac{B_0}{K_0}, d_0 = \frac{D_0}{K_0}, B_0 = \frac{b_2}{\delta-\delta_0} - \frac{b_1}{\delta^2}$,

$$K_0 = \frac{a_2}{\delta-\delta_0} - \frac{a_1}{\delta^2} + \beta_0^0.$$

Putting (35) in (33) we get

$$U_0(r,z) = \Phi_0(r)u_0(z) - \psi_0(r), \qquad (36)$$

where $\Phi_0(r) = (\delta - r)\left(\frac{b_0 a_1 - b_1}{\delta r} + \frac{b_2 - b_0 a_2}{\delta-\delta_0}\right) + b_0$

and $\psi_0(r) = (\delta - r)\left(\frac{(d_0 a_1 - d_1)}{\delta r} + \frac{d_2 - d_0 a_2}{\delta-\delta_0}\right) + d_0$.

Now integrating partial differential equation (5) and taking account equation (28) get

$$\frac{2}{\delta^2-\delta_0^2} r\frac{\partial U_0}{\partial r}\bigg|_{\delta_0}^{\delta} + \frac{d^2u_0}{dz^2} = 0. \qquad (37)$$

Using boundary conditions (2), (3) and (33)

$$\frac{d^2u_0}{dz^2} - k^2 u_0 = -Q_2, \qquad (38)$$

where

$$k^2 = \frac{2\left(\beta_0^0\Phi_0(\delta_0) + \beta_0\Phi_0(\delta)\right)}{\delta^2-\delta_0^2},$$

$$Q_2 = \frac{2\left(\beta_0^0\left(1+\psi_0(\delta_0)\right) + \psi_0(\delta)\beta_0\right)}{\delta^2-\delta_0^2}.$$

Solution of differential equation (38), using boundary condition (11) looks as follows

$$u_0(z) = C_2\cosh(k(z-1)) + \frac{Q_2}{k^2}. \qquad (39)$$

As we can see we have solved function $u_0(z)$ and now problem reduces to the problem of finding constant C_2.

3.3.2 Solution for lower Wall

Using equation (34) and (26) and putting them in to conjugation condition (9) at value $r = \delta$

$$g_0(z) = C_1\left(\mu_1 I_0(\mu\delta) + K_0(\mu\delta)\right)\Phi(z) \qquad (40)$$

Equation (19) we can rewrite as

$$\Phi(z) = F_0 - F_1\cosh(\rho z), \qquad (41)$$

where $F_1 = \dfrac{1}{\dfrac{\sinh(1)}{\beta b} + \cosh(1) - \sinh(1)}$ and

$$F_0 = \left[\frac{\sinh(1)}{\beta b} + \cosh(1)\right]F_1.$$

We can continue with equation (40) and (41) and get $g_0(z) = C_1\left(\tilde{F}_0 - \tilde{F}_1\cosh(\rho z)\right)$, where (42)

$$\tilde{F}_i = \left(\mu_1 I_0(\mu\delta) + K_0(\mu\delta)\right)F_i.$$

Using (26) and (40) we get following derivation at value $r = \delta$

$$\frac{\partial U}{\partial r}\bigg|_{r=\delta} = C_1\mu\left(\mu_1 I_1(\mu\delta) - K_1(\mu\delta)\right)\Phi(z). \qquad (43)$$

But from (9), (41) and (43) we get

$$\delta\frac{\partial U_0}{\partial r}\bigg|_{r=\delta} = \delta\frac{\beta_0}{\beta}C_1\mu\big(\mu_1 I_1(\mu\delta) - K_1(\mu\delta)\big)\Phi(z) \text{ or}$$

$$\delta\frac{\partial U_0}{\partial r}\bigg|_{r=\delta} = C_1\left(\tilde{\tilde{F}}_0 - \tilde{\tilde{F}}_1\cosh(\rho z)\right), \text{ where} \qquad (44)$$

$$\tilde{\tilde{F}}_i = \delta\frac{\beta_0}{\beta}\mu\big(\mu_1 I_1(\mu\delta) - K_1(\mu\delta)\big)F_i\,,\ i = 0,1.$$

Using equation (42) and derivation of equation (34) at value $r = \delta_0$ we get

$$\frac{\partial U_0}{\partial r}\bigg|_{r=\delta_0} = C_1\left(\tilde{F}_0 - \tilde{F}_1\cosh(\rho z)\right)\left(\frac{a_2}{\delta-\delta_0} - \frac{a_1}{\delta_0^2}\right) +$$

$$+\left(\frac{b_1}{\delta_0^2} - \frac{b_2}{\delta-\delta_0}\right)u_0(z) + \frac{d_2}{\delta-\delta_0} - \frac{d_1}{\delta_0^2} \qquad (45)$$

Now submitting equations (41) and (42) in equation (37) we get differential equation

$$\frac{d^2u_0}{dz^2} - u_0 G = FC_1 + H + EC_1\cosh(\rho z)\,, \qquad (46)$$

where

$$G = \frac{2\delta_0}{\delta^2 - \delta_0{}^2}\left(\frac{b_1}{\delta_0^2} - \frac{b_2}{\delta-\delta_0}\right),$$

$$E = \frac{2C_1}{\delta^2 - \delta_0{}^2}\left(\tilde{\tilde{F}}_1 - \delta_0\tilde{F}_1\left(\frac{a_2}{\delta-\delta_0} - \frac{a_1}{\delta_0^2}\right)\right),$$

$$F = \frac{2}{\delta^2 - \delta_0{}^2}\left(\delta_0\tilde{F}_0\left(\frac{a_2}{\delta-\delta_0} - \frac{a_1}{\delta_0^2}\right) - \tilde{\tilde{F}}_0\right),$$

$$H = \frac{2}{\delta^2 - \delta_0{}^2}\left(\frac{d_2}{\delta-\delta_0} - \frac{d_1}{\delta_0^2}\right).$$

Solution of differential equation looks following

$$u_0(z) = C_3 e^{\sqrt{G}z} + C_4 e^{-\sqrt{G}z} +$$

$$+\frac{EC_1}{\rho^2 - G}\cosh(\rho z) - \frac{FC_1 + H}{G}\cdot \qquad (47)$$

From condition (8) and (47) follows that constants $C_3 = C_4$, therefore $u_0(z)$ can rewrite in the following way

$$u_0(z) = C_3\cosh(\sqrt{G}z) +$$

$$+\frac{EC_1}{\rho^2 - G}\cosh(\rho z) - \frac{FC_1 + H}{G} \qquad (48)$$

putting together (34), (42) and (48) we get

$$U_0(r,z) = C_1 u(\delta)\Phi(z)\left((\delta - r)\left(\frac{a_1}{\delta r} - \frac{a_2}{\delta-\delta_0}\right) + 1\right)$$

$$+(\delta - r)\left(\frac{b_2}{\delta-\delta_0} - \frac{b_1}{\delta r}\right)\cdot \qquad (49)$$

$$\left(C_3\cosh(\sqrt{G}z) + \frac{EC_1}{\rho^2 - G}\cosh(\rho z) - \frac{FC_1 + H}{G}\right) +$$

$$+(\delta - r)\left(\frac{d_1}{\delta r} - \frac{d_2}{\delta-\delta_0}\right)$$

Now solution for wall contains only two unknown constants C_1 and C_3.

3.4 Solution

We have two additional conditions on function $u_0(z)$, respectively

$$u_0(b)_{b-0} = u_0(b)_{b+0} \qquad (50)$$

and

$$\frac{\partial u_0}{\partial z}\bigg|_{b-0} = \frac{\partial u_0}{\partial z}\bigg|_{b+0}. \qquad (51)$$

Also in point (δ, b) values of functions $U_0(r,z)$ and $U(r,z)$ must be equivalent

$$U_0(\delta, b) = U(\delta, b) \qquad (52)$$

To satisfy equation (50) we need to use (39) and (48)

$$C_2\cosh(k(b-1)) + \frac{Q_2}{k^2} + \frac{H}{G} =$$

$$= C_3\cosh(\sqrt{G}b) + C_1\left(\frac{E}{\rho^2 - G}\cosh(1) - \frac{F}{G}\right), \qquad (53)$$

but to satisfy equation (51) - derivative of (39) and (48)

$$\sqrt{G}C_3\sinh(\sqrt{G}b) + \frac{E\rho}{\rho^2 - G}C_1\sinh(1) =$$

$$= C_2 k\sinh(k(b-1)). \qquad (54)$$

And to satisfy equation (52) we will use equations (26) and (36)

$$C_1 u(\delta)\Phi(b) = -d_0 +$$

$$= +b_0\left(C_3\cosh(\sqrt{G}b) + \frac{EC_1\cosh(1)}{\rho^2 - G} - \frac{FC_1 + H}{G}\right). \qquad (55)$$

And now problem reduces to solution of three linear equations (53), (54) and (55) for three unknown constants $C_i, i = 1,2,3$.

$$\begin{cases} C_1 u(\delta)\Phi(b) = -d_0 + \\ = +b_0\left(C_3\cosh(\sqrt{G}b) + \dfrac{EC_1\cosh(1)}{\rho^2 - G} - \dfrac{FC_1 + H}{G}\right) \\ \sqrt{G}C_3\sinh(\sqrt{G}b) + \dfrac{E\rho}{\rho^2 - G}C_1\sinh(1) = \\ = C_2 k\sinh(k(b-1)) \\ C_2\cosh(k(b-1)) + \dfrac{Q_2}{k^2} + \dfrac{H}{G} = \\ = C_3\cosh(\sqrt{G}b) + C_1\left(\dfrac{E}{\rho^2 - G}\cosh(1) - \dfrac{F}{G}\right). \end{cases} \tag{56}$$

After solving system (56) we can put constants $C_i, i = 1,2,3$ to equation (27) or (49) and calculate value of temperature at any point of our 2D domain.

4 1D Solution as the Simple Case of the 3D Solution

Using following integral values for equations (5) and (8)

$$v_0(r) = \int_0^1 U_0(r,z)dz \tag{57}$$

and for equation (10)

$$v(r) = \frac{1}{b}\int_0^b U(r,z)dz \tag{58}$$

we get new 1D problem

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_0}{dr}\right) = 0,\ r \in (\delta_0, \delta) \tag{59}$$

$$\frac{b}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \beta v,\ r \in (\delta, \delta + l) \tag{60}$$

with following boundary conditions

$$r\frac{dv_0}{dr} + \beta_0^0(1 - v_0) = 0,\ r = \delta_0 \tag{61}$$

$$r\frac{dv}{dr} + \beta v = 0,\ r = \delta + l \tag{62}$$

$$v_0\big|_{r=\delta} = v\big|_{r=\delta}, \tag{63}$$

$$\beta\frac{dv_0}{dr}\bigg|_{r=\delta-0} = \beta_0\frac{dv}{dr}\bigg|_{r=\delta+0} \tag{64}$$

$$\beta\left(r\frac{dv_0}{dr} + \beta_0 v_0(1-b)\right)\bigg|_{r=\delta} = rb\beta_0\frac{dv}{dr}\bigg|_{r=\delta} \tag{65}$$

The solution of problem (59)-(65) can be written in following form:

$$\begin{cases} v_0(r) = C_1\ln r + C_2 \\ v(r) = C_3 I_0\left(\sqrt{\dfrac{\beta}{b}}r\right) + C_4 K_0\left(\sqrt{\dfrac{\beta}{b}}r\right) \end{cases}, \tag{66}$$

where I_0, K_0 is Bessels' modified functions.

Here the four unknown constants can be easy determined from the four boundary and conjugations conditions (61)-(64):

$$\begin{cases} C_1 = \dfrac{\beta_0^0(C_2 - 1)}{1 - \beta_0^0\ln\delta_0} \\ C_3 = C_4\dfrac{(\delta+l)\sqrt{\dfrac{\beta}{b}}K_1\left(\sqrt{\dfrac{\beta}{b}}(\delta+l)\right) - \beta K_0\left(\sqrt{\dfrac{\beta}{b}}(\delta+l)\right)}{(\delta+l)\sqrt{\dfrac{\beta}{b}}I_1\left(\sqrt{\dfrac{\beta}{b}}(\delta+l)\right) + \beta I_0\left(\sqrt{\dfrac{\beta}{b}}(\delta+l)\right)} \\ C_1\beta\left(1 + \beta_0(1-b)\ln\delta\right) + C_2\beta\beta_0(1-b) = \\ = C_3\delta\beta_0\sqrt{\beta b}I_1\left(\sqrt{\dfrac{\beta}{b}}\delta\right) - C_4\delta\beta_0\sqrt{\beta b}K_1\left(\sqrt{\dfrac{\beta}{b}}\delta\right) \\ C_1\ln\delta + C_2 = C_3 I_0\left(\sqrt{\dfrac{\beta}{b}}\delta\right) + C_4 K_0\left(\sqrt{\dfrac{\beta}{b}}\delta\right) \end{cases} \tag{67}$$

5 Conclusion

We have constructed some approximate three dimensional analytical solutions for a periodical system with cylindrical fin when the wall and the fin consist of materials which have different thermal properties.

Acknowledgements:
Research was supported by European Social Fund and Council of Sciences of Latvia (grant 05.1525).

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