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Contribution to numerical study of 2D free-surface waveless flow

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Abstract: - The numerical two-dimensional free-surface steady flow of an incompressible inviscid fluid, over an obstacle lying on the bottom of a channel, is examined. The first numerical model presented uses quadrangular finite elements P1. For a supercritical regime, the Finite Volume Method is also used. The obtained results agree quite well, even for highly nonlinear configurations, with those of other authors.

Key-Words: - free surface – finite element method – finite volume method –hydraulic channel – hydraulic jump – supercritical regime.

1 Introduction

The numerical solution of flows that are partially bounded by a freely moving boundary is of great importance in fluid dynamics. Indeed, from the mathematical point of view, the free-surface is an unknown and two boundary conditions, one being nonlinear, must be satisfied on it. The problem of flow over an obstacle has applications in many physical situations, from the flow of water over rocks [1], [2], [3], [4] for example, to atmospheric and oceanic flows over topographic obstacles [5], [6], [7]. The parameter which plays an important role is the Froude number.

2 Mathematical formulation

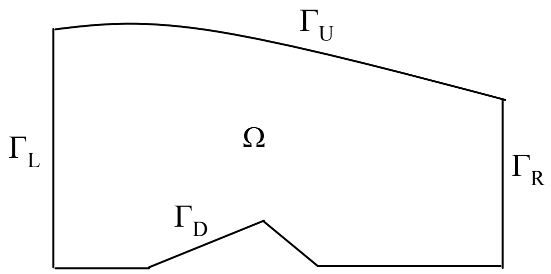
2. 1 Governing equations

Let us consider the two dimensional free-surface supercritical flow of an inviscid incompressible fluid over an obstacle lying on the bottom of a hydraulic channel.

Experiments carried out in channels show that, according to the value of the Froude number, the free-surface looks like:

- a unique elevation over the obstacle then gradual going back to the undisturbed level ($y = 1$) for a supercritical flow upstream ($F > 1$).
- a depression in the vicinity of the obstacle followed or not by surface waves for a subcritical flow ($F<1$).

The two-dimensional fluid domain Ω is limited upstream by the vertical boundary Γ_L, downstream by the vertical boundary Γ_R, the non uniform bottom Γ_D and the unknown free-surface Γ_U. Taking into account the previous remark, one tries to solve the problem by assuming a downstream horizontal velocity. By using the streamfunction ψ, one leads to a non linear boundary problem.



In non dimensional variables, the problem is therefore described by the following equation and boundary conditions which can be written [8]:

$\Delta\psi = 0$	in Ω	(1a)
$\psi = y$	on Γ_L	(1b)
$\partial\psi / \partial x = 0$	on Γ_L	(1c)
$\psi = 0$	on Γ_D	(1d)
$\partial\psi / \partial x = 0$	on Γ_D	(1e)
$\dfrac{\partial\psi}{\partial n} = \sqrt{1+\dfrac{2}{F^2}(1-y_0)}$	on Γ_U	(1f)
$\psi = 1$	on Γ_U	(2)

where y_0 is the free-surface position measured from the horizontal bottom, $F = U/\sqrt{gh}$ the upstream Froude number, assumed greater than one, and $\vec{n}$ the unit outward normal vector.

The relations (1f) and (2) are known as, respectively, dynamic and kinematic condition. The existence of these two free-surface boundary conditions allows us to consider the two iterative processes:

i) Associate the kinematic condition to the problem and then determine the free-surface position verifying the dynamic condition and so on until convergence.

ii) Associate the dynamic condition to the problem and then determine the new geometry verifying the kinematic condition.

We can show, from a simple example ([9]), that the dynamical approach (ii) is stable for F>1. If M is a point on the free-surface, N the point just beneath it and M' the new location verifying the condition (2), we have chosen the linear extrapolation:

$[\psi(N)-\psi(M)]\, y(M') = [1-\psi(M)]\, y(N)-[1-\psi(N)]\, y(M)$ (3)

Let us consider the flow in an horizontal channel with an also horizontal free-surface $y(x) = 1 + \delta$. The analytical solution of the system (1) is:

$$\psi(y) = y\sqrt{1-\frac{2\delta}{F^2}} \Rightarrow \psi(M) = (1+\delta)\sqrt{1-\frac{2\delta}{F^2}} = (1+\delta)\left(1-\frac{1}{F^2}\right) + o(\delta^2)$$

The equation (3) gives then: $y(M') = 1+\frac{1}{F^2} + o(\delta^2)$.

The algorithm is then a priori stable only if F>1.

Finally, the stream function $\psi(x,y)$ and the free-surface $y_0(x)$ are the solution of two problems P_1 and P_2.

The first one consists on determining $\psi(x,y)$, which is solution of the system (1) of equations, for a known free-surface $y_0(x)$. The problem P_2 allows the determination, for the previous distribution $\psi(x,y)$, of a new free-surface location, i.e. $y_0(x)$, solution of the equation (3).

2. 2 Finite Element Method approach

The domain Ω is subdivided into triangles having all a vertical side by taking n points on Γ_L and Γ_R and m points on Γ_D and Γ_U, i.e. NS = nm grid points:

Making use of finite elements of P_1 type, we write $\Psi(x,y)$ in the form:

$$\psi(x,y) = \sum_{I=1}^{NS} \Psi_I\, \eta_I(x,y) \qquad (4)$$

where ψ_I denotes the nodal values of $\psi(x,y)$ and $\eta(x,y)$ a trial linear function which yields 1 at the considered grid point and zero in all others. With the weak formulation of equation (1.a), using Green-Riemann identity, we get:

$$\int_\Omega \vec{\nabla}\psi . \vec{\nabla}\eta_I\, ds - \int_\Gamma \eta_I \frac{\partial\Psi}{\partial n} dl = 0 \qquad (5)$$

where $\vec{n}$ is the outward normal vector on the boundary Γ of the domain Ω. The test functions $\eta_I(x,y)$, chosen in the integral formulation, are those defined at the nodal points of the grid where the streamfunction $\psi(x,y)$ is unknown.

Taking into account the boundary conditions (1.b) to (1.e), the relation (5) becomes:

$$\int_\Omega \vec{\nabla}\psi . \vec{\nabla}\eta_I\, ds - \int_{\Gamma_U} \eta_I \sqrt{1+2(1-y_0)F^{-2}}\, dl = 0$$

where the border Γ_U, of equation $y_0(x)$, is assumed known.

Rewrite the relation (4) in the form:

$$\psi(x,y) = \sum_{I=1}^{NI} \psi_I\, \eta_I(x,y) + F_1(x,y) \qquad (6)$$

where NI is the number of unknowns.

$F_1(x,y)$ being therefore a known function, the determination of the nodal values ψ_I of the streamfunction $\psi(x,y)$ is done by the resolution of a system of linear algebraic equations which matrix is symmetric positive defined.

We first note that the location of the free-surface depends weakly on the space discretization when the step is less than 0.25. Finally, we have adopted the following steps: **dy** = 0.2 and **dx** =L/10 to L/20, where L is the length of the obstacle.

Four obstacles, of maximum height is **b**, were tested: a non symmetrical one of equation:

$y = 27\, b\, x(x-L)^2/4L^3$,

an arch of sinusoid of equation:

$y = b(1 - \cos 2\pi x/L)/2$,

a triangle, a semicircle and a step.

We give, at the figure 1, some results obtained for the non symmetrical obstacle with a configuration highly non linear and a low Froude number.

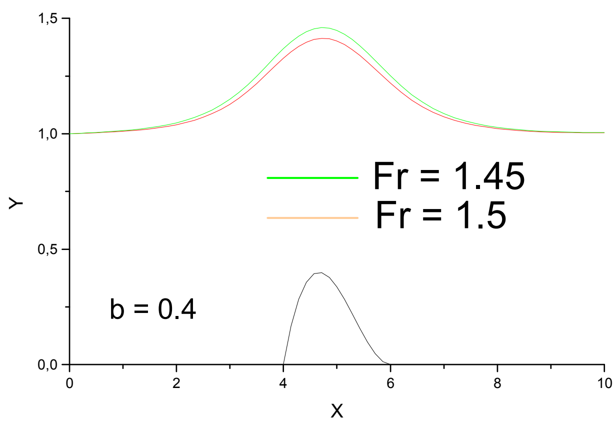


Figure 1

The figure 2 shows similar results, for a symmetrical obstacle which is an arch of sinusoid.

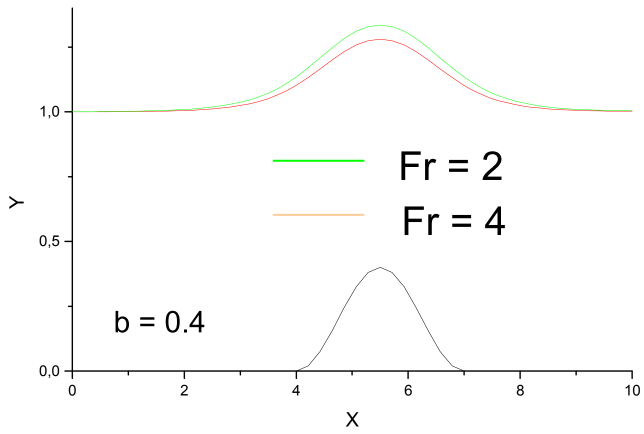


Figure 2

We show, in the figure 3, the results obtained with our numerical model in the case of a semi infinite step. We have chosen the same parameters as those of [11] and the agreement is very good.

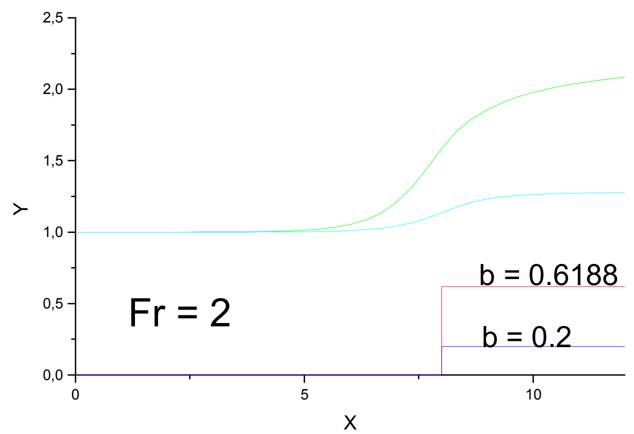


Figure 3

In the figure 4, we represent the free-surface elevation for two semicircles of radius 0.7 and 1.1. For the latter, we note that the obstacle's height exceeds the initially undisturbed free-surface.

It is necessary to notice that, in the case of a high obstacle exceeding the initially undisturbed free-surface, a fast convergence of the iterative process can de obtained if we start the process with a free-surface of sinusoidal form. Let us recall that the convergence is obtained, in almost cases, for an error parameter of 10^{-4}, in less than 10 iterations.

The good agreement of our results, with those of other authors ([9], [10]) proves, if need be, the validity of the numerical model.

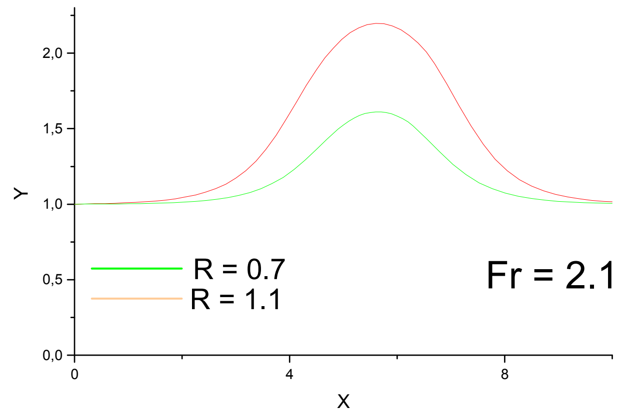


Figure 4

The waveless phenomena can also exist with an upstream subcritical flow and a supercritical flow downstream of the obstacle. In this case, the numerical model is different, as that developed in [12]. This last mode can be preceded by the regime in which a turbulent hydraulic jump occurs (photo 1).

We show, in the figure 5, extracted from [12], showing an example of the obtained result.

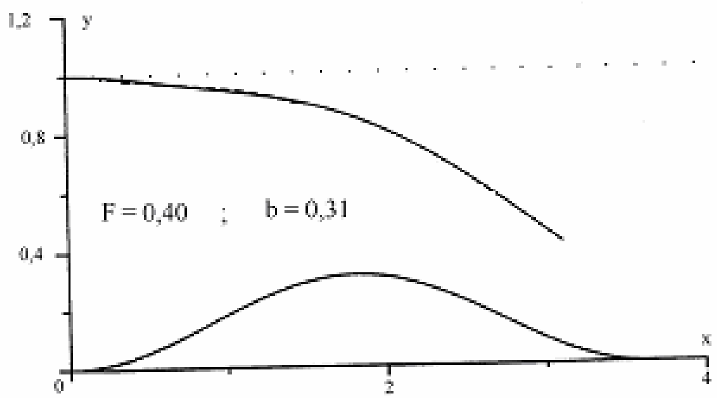


Figure 5

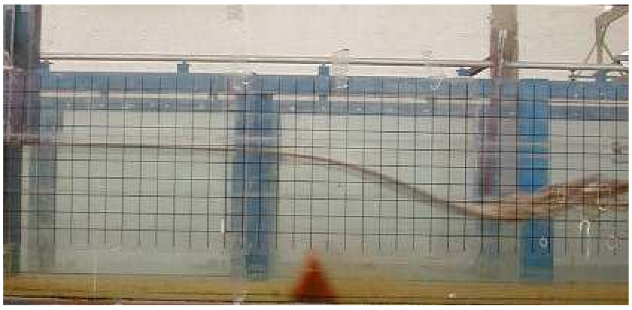


Photo 1

2. 3 Finite Volume Method approach

The finite volume method is more adapted for conservative problems which can be put in the form:

$$\varepsilon \frac{\partial \psi}{\partial t} + \mathrm{div}\,[\vec{F}(\psi)] = S_\psi \qquad (7)$$

$\varepsilon = 1$ for unsteady problems

$\vec{F}(\psi)$ is the total flux vector

S_ψ is a source term

For our steady problem, (7) is simplified because of the absence of a source term:

$$\mathrm{div}\,[\vec{F}(\psi)] = 0 \qquad (8)$$

The function ψ being harmonic ($\Delta\psi = 0$), the vector $\vec{F}$ is thus related to ψ by:

$$\vec{F} = \overrightarrow{\mathrm{grad}}\,\psi \qquad (9)$$

For a domain Ω_K of boundary occupied by the cell K, and using the Green's theorem, (8) can be written:

$$\int_{\Gamma_K} \vec{F}\,.\,\vec{n}\,ds = 0 \qquad (10)$$

A given cell K(i,j) has two borders (i-1), (i+1) along **x** and two others (j-1)), (j+1) along **y**. The boundary Γ_K in the equation (10) is then the union of these four boundaries which are necessary to the evaluation of the components of the vector $\vec{F}$, related to ψ by the equation (9), at the centre of the cell.

As for the finite element method, the dynamical approach is used to ensure the convergence of the iterative process issued from the equation (10) associated to the boundary conditions (1b), (1d), (1e), (1f) with the relation (3).

We show, in the figures below, some results obtained with the preceding described finite volume method. The figure 6 is related to a triangular obstacle.

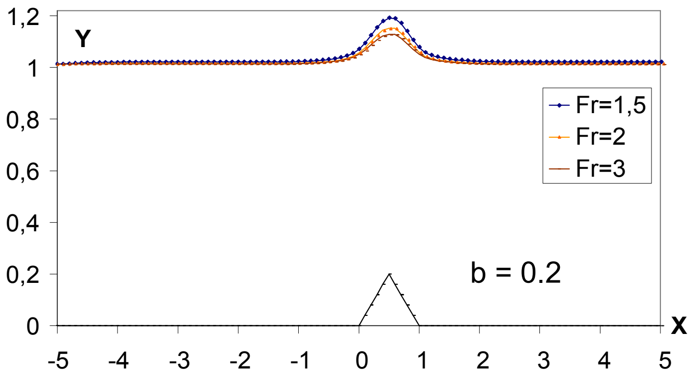


Figure 6

In the figure 7, for the same Froude numbers and obstacle's height, the elevation for the semi infinite step is greater than the corresponding free-surface for the triangular obstacle.

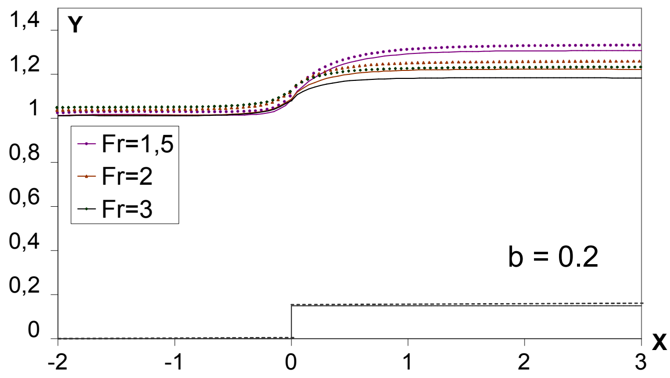


Figure 7

4 Conclusion

The numerical methods used to solve the problem of the supercritical flow, above an obstacle lying on the bottom of a hydraulic channel, gave very satisfactory results since they corroborate those already given by other authors. In the first numerical method, i.e. the Finite Element Method, the problem consists on solving the Laplace equation using quadrangular finite elements. The second one deals with the Finite Volume Method in which we make use of the gradient of the streamfunction ψ. The used iterative process is rather simple and is based on the treatment of the dynamic and kinematic conditions on the free-surface. We showed, on a simple case, that it is more judicious to solve the problem including the Bernoulli's condition and then to seek a new free-surface satisfying the kinematic condition. For this, an extrapolation by a linear variation of the streamfunction ψ, in the vicinity of the free-surface, is sufficient.

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