# New Aspect to Random Vortex Method Description, Base on Multiple Scale Method

N.M. NOURI, A. SARRESHTEHDARI Mechanical Engineering Faculty Iran University of Science and Technology (IUST) Narmak, Tehran, Iran

*Abstract:* - improvement of the random vortex method (RVM), a numerical scheme that combines the representation of the vorticity field by a number of Lagrangian vortex elements of finite cores with a stochastic simulation of diffusion using random walk, is investigated. Multiple scales method Base on Rankin vortex model, used to derive different orders of vorticity transport equation, and physical description of terms presented. This new aspect presents the reason of limitation of vortex method and explains how it is possible to develop this method, base of understanding of physical behavior of flow in various conditions. Base on this view point it is possible to explain why vortex method in some of applications don't have satisfying Results.

Key-Words: - Random Vortex Method, Multiple Scale, Model Improvement

# 1 Introduction

Many of researchers have used various vortex methods for studying different types of fluid flow problems. A.J. Chorin [1] has presented the random vortex method for solving the time-dependent Navier-Stokes equation in two dimensions at high Reynolds number and an application to flow past a circular cylinder. He used distribution of vortex blobs as a model of vortex generation to satisfy noslip condition on the wall. This model has been expanded for different applications by other researchers. In other works, A.J. Chorin [2] carried out the boundary layer simulation. He used distribution of vortex sheets as a model of vortex generation near the walls. The accuracy of the results far from the wall is acceptable, but the near wall solutions are not reliable. The main reason of this deficiency can be the mechanism of vorticity generation. Ghoniem, Chorin and Oppenheim [3] considered a correction to the zone of influence of a vortex sheet for improving the near wall solution. Ghoniem and Cagnon [4] discussed the effect of some numerical parameters such as initial strength of vortex sheets and number of vortices in the rate of convergence of laminar recalculating flow. They have shown that the rate of convergence is a function of initial strength of vortex sheets and the number of vortices.

The general aspects of the results have a good agreement with the experimental data, but the local velocity profiles near the walls are inaccurate. In other investigation, A.J. Chorin [5] considered that vortex sheets transform to blob when sheets find themselves out of a specified position and time. The simulations indicate that in special conditions, the results were insensitive to the computational parameters. Hou [6] and Henri Cottet et al. [7] have studied the accuracy of vortex method by using variable size vortex particles.

In summary, the model of vortex generation in RVM is based on no-slip condition along a line segment on the boundary by producing the vortex sheet. If the vortex sheets jumped out from a specified distance near the wall, they will transform to vortex blob. Otherwise, they will stay in new random position and remain in vortex sheet pattern [8]. Generation, distribution of vortex sheets, their amount of strength and another important parameter in this method have significant effect on the flow field properties like as velocity profile or local shear stress in vicinity of the wall. In general, this method does not detect accurate velocity profile specially in vicinity of the wall, and various flow regimes and the average velocity profiles in this region are inaccurate. This method of vortex generation in some applications that local velocity and skin friction near the wall is important is not applicable.

Recently new approaches in Vortex method introduced and used to more precision results and using in more complex flow fields, e.g. Nouri and Ebrahimi used new method to vortex generation [9] and Nouri and Eslami developed a method to add free stream turbulence base on vortex method [10], also Ramachandran study and developed High resolution two dimensional Random vortex [11]. In present paper, new aspect to random vortex method base on Rankin vortex model and multiple scale method developed. This aspect describes when separation of diffusion and convection terms in N-S equation, which is a usual method used in RVM, is allowable and how it is possible to do. This description is against normal usage of RVM which in all of cases separate part of diffusion and convection phenomena terms without care to flow properties.

# **2 Problem Formulation**

In present research, trajectory of a spinning disk in a uniform stream flow, studied to get a clear physical understanding of the highly oscillatory motion of a Rankin vortex about its mean trajectory, later the same analogy employed to description of the motion of a vortex with a vortical core to get important time scales in Rankin vortex model. Base on these time scales the N-S equation expanded, diffusion and convection parts separated with mention to their physical concepts.

#### 2.1 Rankin's vortex time scales

The equation of motion of the center of disk with mass M, in a uniform stream flow, Figure (1), with  $U_{\infty}$  velocity and  $\rho$  density is,



Fig.1, Spinning disk in an uniform flow

Where M' is virtual mass and accounts for the inertia of the surrounding fluid having an equivalent mass, Z is complex plane,  $\Gamma$  is the circulation around the disk and Q is relative velocity between background uniform stream flow and spinning disk,

$$Q(t) = U_{\infty} - \mathbf{Z}^{\mathbf{X}} \tag{2}$$

The solution of (1) and (2), which is the trajectory of the center of the disk, is

$$Z = U_{\infty}t + Ae^{i2\pi t/T} + c \qquad (3)$$
  
Where  
$$A = iTQ(0)/2\pi,$$
  
$$T = 2\pi/\omega = 2\pi^2 a^2 (\rho + \rho_0)/(\rho\Gamma) \qquad (4)$$

In the above *a*,  $\rho_0$  are radius of circulating disk and its density respectively, and c is a constant value.

The second term in RHS of equation (3) represents the oscillatory part of trajectory relative to the mean trajectory defined by  $U_{\infty}t + c$ . Here  $\omega$ , T and A are the frequency, period and complex amplitude of the oscillations, respectively, figure (2). The mean trajectory in (3) shows that the disk drifts with the background velocity  $U_{\infty}$ . The term A represents the diviation of the initial position from the mean trajectory and the amplitude of an oscillatory motion while the first term in equation (3) is a uniform motion.



Fig.2, Trajectory of spinning disk in uniform flow

Figure (2) Shows the oscillatory and mean trajectory of a disk that starts with zero velocity from the origin, i.e.,  $Z_0=0$  and  $Z_0'=0$ . At t=0, the Joukowski force is  $\rho U_{\infty} \Gamma$  acting in the downward direction and the disk is gaining a downward velocity. This in turn induces a horizontal; component of the Joukowski force, increases xdirection velocity from zero and thus reduces the horizontal component of relative velocity. The wave length, which is the horizontal distance traveled during one period, is  $U_{\infty}T$ . The vertical shift  $U_{\infty}T/2\pi$  of the mean trajectory given in equation (4) is equal to the average of the vertical displacement over one period. This completes the physical description of the oscillatory and mean trajectory of a spinning disk in a uniform stream.

Now consider the limiting case of "vanishing mass" or rather "vanishing radius" with finite  $\Gamma$ ,  $U_{\infty}$ . In order to setup the analogy for the motion of a Rankin vortex, let the density of the disk equal that of the fluid,

$$\rho_0 = \rho = 1 \tag{5}$$

And with introduce the normal length and time scales:

 $l = \Gamma / U_{\infty}$ , and  $l / U_{\infty} = \Gamma / U_{\infty}^{2}$  (6)

For the limiting case of "vanishing radios", consider the radios "a" to be the very small relative to "l", i.e.  $a = \varepsilon l = \varepsilon \Gamma / U_{\infty}, \text{ with } 0 < \varepsilon = 1$  (7)

In order to avoid the introduction of new symbols for dimensionless quantities,  $l, l/U_{\infty}$  used as unit length and time with  $U_{\infty} = 1$  and  $\Gamma = 1$  and thus render the quantities in (1) to (4) dimensionless. In particular z and t become the normal space and time variables. Also, noted,

$$T = 4\pi^2 \varepsilon^2$$
 and  $A = 2\pi i \varepsilon^2 Q(0)$  (8)

In the limit as  $\varepsilon = a \rightarrow 0$ , the period T, Amplitude A and the initial shift of the mean trajectory vanish as  $O(\varepsilon^2)$ . Consequently, in the normal time scale, the inertia of the disk (like as Rankin vortex similarly) is  $O(\varepsilon^2)$  and it appears to be moving with the velocity of the center of the disk (or vortex) fluctuates around  $U_{\infty}$  by order O(1).

This means that the motion of a Rankin vortex in a flow field has two time scales at least.

#### 2.2 Expansion of N-S equation

In an incompressible two-dimensional Newtonian fluid flow, the governing equations are the continuity and Navier Stokes. In terms of velocity, pressure gradient, the differential equations are,  $\nabla^2 U = 0$  (0)

In a Cartesian coordinate system. In twodimensional flow, vorticity as the curl of velocity vector will be a scalar,

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{11}$$

The vorticity transport equation will obtain by taking the curl of momentum equation (10) and using mass conservation (9) as following,

$$\rho \frac{D\omega}{Dt} = \rho \left( \frac{\partial \omega}{\partial t} + (U \cdot \nabla) \omega \right) = \mu \nabla^2 \omega$$

$$or \qquad \left( \frac{\partial \omega}{\partial t} + (U \cdot \nabla) \omega \right) = \nu \nabla^2 \omega$$
(12)

Base on introduced time scales,  $t = t(T \ \tau) \implies$ 

$$t = t(T, \tau) \implies (13)$$
  
$$(T = \varepsilon^{0} \cdot t \,\& \tau = \varepsilon^{2} \cdot t) \Longrightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial T} + \varepsilon^{2} \frac{\partial}{\partial \tau}$$

In other words,

$$\left. \begin{array}{l} T = T_0 = \varepsilon^0 . t = t \\ \tau = \varepsilon^2 . t \end{array} \right\} \implies \tau = \varepsilon^2 . T \qquad (14)$$

These two time scale shown schematically in figure (3).

An expansion of two various order of viscosity introduced as,

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\varepsilon}^2 \boldsymbol{v}_\alpha \tag{15}$$

Where  $v_0$  is kinematic viscosity of fluid in flow while  $v_{\alpha}$  is a property of flow that effect as a new viscosity term, but it different from properties of fluid.



Fig.3, schematic of two time scales

Substitution equations (13) and (15) in the (12) it yields,

$$\frac{\partial \omega}{\partial T} + \varepsilon^2 \frac{\partial \omega}{\partial \tau} + (U.\nabla)\omega = v_0 \nabla^2 \omega + \varepsilon^2 v_\alpha \nabla^2 \omega$$
(16)

Separation of various orders, lead to found follow equation,

$$\frac{O(\varepsilon^{0}) = O(1)}{\frac{\partial \omega}{\partial T} + (U.\nabla)\omega} = v\nabla^{2}\omega$$
(17)
And,
$$\frac{O(\varepsilon^{2})}{\frac{\partial \omega}{\partial \tau}} = v_{\alpha}\nabla^{2}(\omega)$$
(18)

As it shown in above, equation (17) describes the vorticity field in normal time step, it is the time step in which convection phenomena appeared. In this time scale, diffusion related to kinematic viscosity of fluid and this equation is as similar as initial figure of vorticity transform equation but it has a different note, because it is confined to normal time scale and small time scale separated from the first configuration of initial form of vorticity transform equation. Equation (18) just shows a diffusion term in the small time scale base on different viscosity, which called flow or vortex viscosity. This viscosity term related to flow properties and describes the amount of new diffusion phenomena base on flow regimes.

Therefore, it is need, to solve below system of equations simultaneously, to finding vorticity field in a flow problem

$$\begin{cases} \frac{\partial \omega}{\partial T} + (U \cdot \nabla) \omega = v \nabla^2 \omega \\ \frac{\partial \omega}{\partial \tau} = v_{\alpha} \nabla^2 (\omega) \end{cases}$$
(19)

### **3** Usage in Problem Solution

In this part, proper usage of separated equations, and their simplifying will be described, and boundary conditions will be introduced.

#### 3.1 Flow Regimes Notifications

Using equation (15), it is possible to write

$$v = v_0 + v_T = v_0 + \varepsilon^2 v_\alpha \tag{20}$$

Where,  $v_T$  is turbulence viscosity.

First let to focus on laminar flow, in this regime there is not vortex viscosity in the flow and the important viscosity related to fluid properties, therefore

$$v_0 \gg v_T \implies v_0 \gg \varepsilon^2 v_\alpha \tag{21}$$

And it is possible to ignore second equation, (18), and just use of solution of equation (17), to find the vorticity field of flow. Therefore equation (19) change to

$$\frac{\partial \omega}{\partial T} + (U.\nabla)\omega = v\nabla^2 \omega \tag{22}$$

This equation can be solved as usual vortex method, that equation (22) is solved in two fractional steps by implementing the two mechanisms of transport in each time step of the computations individually

$$\begin{cases} \frac{\partial \omega}{\partial t} + (U.\nabla)\omega = 0\\ \frac{\partial \omega}{\partial t} = v\nabla^2(\omega) \end{cases}$$
(23)

In the first fraction step, the transport of vorticity due to convection obtained from the solution of first expression in above equation, in terms of the Lagrangian displacement of a set of finite vortex elements. In the second step, the solution of the second expression simulated stochastically by the random walk displacement of the same vortex elements. The boundary conditions, satisfied by adding a potential velocity field in the convection step, and by creating extra vortex elements to satisfy the no-slip condition in the diffusion step. In the following boundary conditions described in more details.

In the turbulent flow vortex viscosity or viscosity that related to flow properties becomes more important, and its amount is higher than fluid viscosity, therefore

$$V_0 \ll V_T \implies V_0 \ll \mathcal{E}^2 V_\alpha \tag{24}$$

Hence, it is possible to neglecting fluid viscosity with compare to flow viscosity, and rewriting equation lead to

$$\begin{cases} \frac{\partial \omega}{\partial T} + (U \cdot \nabla) \omega = 0 \\ \frac{\partial \omega}{\partial \tau} = v_{\alpha} \nabla^{2}(\omega) \end{cases}$$
(25)

This is similar to RVM essential expressions, which used to finding vorticity field in problems.

It is important to note that this presentation of two separated equations, to finding vorticity field, is similar to RVM, but it essentially different, because in usual RVM, equation (12) separated with another view point, that describes, this operation is allowable because the time steps in solution process kept very small, although in present paper base on Rankin vortex model described that this separation is allowed because it is possible to use to different time scales, and specially two viscosity, instead of unique viscosity which used in a normal usage of RVM.

#### **3.2 Boundary Condition**

To solution of above equation, boundary conditions are needed.

For near wall location, base on no-slip condition it can be wrote,

$$\frac{y=0}{U=U_0} = 0 \implies U + \varepsilon^2 U_\alpha = 0$$
$$\implies (U=0, \ U_\alpha = 0)$$
(26)

By taking the curl of momentum equation

$$\omega = \omega_{w} \implies \omega + \varepsilon^{2} \omega_{\alpha} = \omega_{w}$$
$$\implies (\omega = \frac{u}{y}, \varepsilon^{2} \omega = \omega_{w} - \frac{u}{y})^{(27)}$$

For locations far from the wall, boundary condition change to

$$\frac{y = \infty}{U = U_{\infty}} \implies U + \varepsilon^2 U_{\alpha} = U_{\infty}$$
$$\implies (U = U_{\infty}, \ \varepsilon^2 U_{\alpha} = 0)$$
(28)

Therefore

$$\omega = 0 \qquad \Rightarrow \qquad \omega + \varepsilon^2 \omega_{\alpha} = 0$$
  
$$\Rightarrow (\omega = 0, \ \varepsilon^2 \omega = 0) \qquad (29)$$

It means that, no new vorticity add to flow, far from the wall without some things to change the shear stresses.

# 4 Conclusion

In present paper vorticity transport equation was separated to two parts base on two time scale of dynamic behavior of Rankin vortex model. The two main features of this work are: (i) usual usage of RVM is not valid for high Reynolds flows, because in turbulent regime time step, that related to fluctuations of flow, differ from normal time step, and the dominant viscosity is flow or vortex viscosity instead of fluid viscosity in the laminar flow, (ii) it is important to solve two separated convection equations of and diffusion simultaneously base on small time scale and flow (vortex) viscosity, may be found from another research work, for example base on Boussinesq theorem, and not with the fluid viscosity as usual in RVM, this work will be present in later publications, (iii) usage of usual RVM is acceptable for laminar flow with description of separation of convection and diffusion terms from initial equation, because in this regime there is one time step, that it is possible to separate two phenomena with small error in results, in this case the reason is very small time steps, that let to solve two equations of convection and diffusion separately and summarize their results after very small time step.

References:

- Chorin, A. J, The Evolution of a Turbulent Vortex, *Computational Fluid Mechanics*, Vol.83, 1983, pp. 205-213.
- [2] Chorin, A. J, Vortex Sheet Approximation of Boundary Layers, *Journal of Computational physics*, Vol.27, No.3, 1978, pp. 123-137.
- [3] Ghoniem, A. F., Chorin, A. J. and Oppenheim, A. K., Numerical Modeling of Turbulent Flow in a Combustion Tunnel, *Philosophical Transactions of Royal Society of London*, Series A, Vol.304, 1982.
- [4] Ghoniem, A. F. and Cagnon, Y., Journal of Computational Physics, Vol.68, 1987, pp. 346-377.
- [5] Chorin, A. J., Numerical study of slightly viscous flow, *Journal of Fluid mechanics*, Vol. 57, 1973, pp. 785-796.

- [6] Hou, T. Y., Convergence of variable blob vortex method for the Euler and Navier Stokes equations, SIAM Journal Numerical Analysis, Vol.27, 1990,pp.1387-1390
- [7] Henri Cottet, G. Koumoutsakos, P. and Salihi, M. L. O., Vortex Methods with Spatially Varying Cores, *Journal Of Computational Physics*, Vol.162, 2000, pp.164–185.
- [8] Chorin, A. J., Vortex models and boundary layer instability, SIAM Journal on Scientific and Statistical Computing, Vol.1, No.1, 1980.
- [9] Nouri, N. M., Ebrahimi, M., Bubble Interaction in an Incompressible, Two Phase Flow Around a 2D Cylinder by Random Vortex Method and Lagrangian Particle Dynamics, 13th Annual (International) Mechanical Engineering Conference, Isfahan University of Technology, Isfahan, Iran, May 2005.
- [10] Nouri, N. M., Eslami, S., Modeling of freestream turbulence with random vortex method (RVM) in incompressible flow around circular cylinder, *Flow Dynamics Conferences, Shiraz, Iran*, 2005.
- [11] P. Ramachandran, Development and study of a High- Resolution 2-D Random Vortex Method, *Ph.D. Thesis Department of Aerospace Engineering Indian Institute of Tech., Madras,* June 2004.
- [12] Ghoniem, A. F., Vortex Simulation of Laminar Recirculating, *Journal of Computational physics*, No.68, 1987, pp. 346-377.
- [13] Majda A., Bertozzi A. L., *Vorticity and Incompressible flow*, Cambridge Univ. Press, 2001.
- [14] L. A. Barba1, A. Leonard and C. B. Allen, Advances in viscous vortex methods—meshless spatial adoption based on radial basis function interpolation, *INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS*, Int. J. Numer. Meth. Fluids 2005, No. 47, pp. 387– 421
- [15] Henri Cottet, G. Koumoutsakos, P. and Salihi, M. L. O., Vortex Methods with Spatially Varying Cores, *Journal Of Computational Physics*, Vol.162, 2000, pp.164–185.
- [16] Chan WM, Gomez RJ (III), Rogers SE, Buning PG. Best practices in overset grid generation, *AIAA*, #2002-3191; 32nd Fluid Dynamics Conference, St. Louis MI, 2002.