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Combination of Extended Random Vortex and Boundary Element Methods for Flat Plate Flow

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Abstract: - Flow over flat plate was investigated using combination of random vortex and boundary element methods. Use was made of Helmholtz principle in order to decompose the flow field into potential and rotational parts. Boundary Element Method (BEM) was used to solve Laplace equation for potential flow field in which the boundaries discritize to some elements. Contact layer model was utilized to generate vorticity over boundaries in order to compute the effects of rotational field. Generated vortexes simulate the rotational field via convection and diffusion. Comparison was made in two cases, Re=1000, Re=10000, and the obtained results confirms the applicability of the proposed method.

Key-Words: Boundary Element Method, Random Vortex Method, Contact Layer

1 Introduction

Using random vortex method (RVM) has some advantages for solving a flow field because pressure terms are omitted in the vorticity transport equations and the continuity equation is satisfied inherently. This method was first proposed by Chorin [1] for high-Reynolds flow.

The effect of parameters such as the strength of the vortexes and the number of vortexes over the rate of convergence in laminar flow were studied by Ghoniem and Cagnon [2]. Their study shows that the rate of convergence is a function of the number and strength of vortexes in the field. The precision of the obtained results was investigated by Henri Cottet [3]. Marshall and Grant [4] construct a diffusion velocity method for axisymmetric flows. Milane [5] uses a diffusion velocity method to compute LES solutions in a 2D mixing layer.

M. Gallati and G. Braschi [6] used random vortex to analysis flow past a cylinder.

In the above mentioned works, sheet vortex generation was utilized as a model of vortex generation satisfying no-slip condition on walls. Superposition of Basic flows and image methods were used to study normal boundary condition in rotational part of the flow field solution.

Clarke and Tutty [7] used 2-D boundary element method with 2-D elements for potential field and random vortex method for rotational field of the whole flow field. Marshall and Grant [8] used a combination of source and vortex panels for studying a blade in a vortex core. In recent works, potential field will be obtained by applying the integral form of the Laplace equation Ploumhans [9] and Khatir [10]. Elliptical boundary integral was investigated by Beal [11] by calculating the strength of a double layer potential over boundaries. Combination of RVM and BEM was used by Gharakhani and Ghoniem [12] to study internal flows. In this study, we used BEM and RVM with contact layer model to investigate the flow field over a flat plate. Boundary layer thickness, shear stress and velocity profiles were calculated. Finally, the analytical result compared with the obtained results which confirms the applicability of the proposed method.

2 Problem Formulation

We considered an unsteady 2-D flow field over a flat plate with zero-angle of attack. The fluid was assumed to be Newtonian, viscous and incompressible. The governing equations are as follows,

$$\nabla \bullet u = 0 \tag{1}$$

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}.\nabla)\mathbf{U} = -\nabla P + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{U} + \mathbf{f} \tag{2}$$

$$\mathbf{U} = 0 \qquad \textit{on the walls} \tag{3}$$

$$\mathbf{U} = (u, v) = (1, 0) \qquad \textit{far from the walls} \tag{4}$$

3 BEM and RVM combination

Considering Helmholtz principle, one can decompose a field as a summation of a curl-free and a divergence-free parts.

$$u = \nabla\varphi + \nabla \times A \tag{5}$$

Applying divergence operator in both sides of Eq. (5) and considering the incompressibility relation $(\nabla \bullet u = 0)$, Laplace equation can be obtained for the scalar potential field. The boundary conditions are as follows,

$$\nabla\phi.\hat{n} = \frac{\partial\phi}{\partial n} = 0 \tag{6}$$

$$\nabla\phi|_{\infty} = U_{\infty} \tag{7}$$

The potential field was calculated by boundary element method using boundary integral equation.

3.1 Potential field via BEM

The basic of the BEM is derived from the Green's identity over the boundaries of a body [14].

$$\int_V f\nabla^2 w \, dv_p + \int_{\partial V} f \frac{\partial w}{\partial \mathbf{n}} dS = \int_{\partial V} w \frac{\partial f}{\partial \mathbf{n}} dS \tag{8}$$

$\nabla^2 f = 0$ and $\nabla^2 w = \delta(\xi,\eta)$ are the basic equations in which $p = (\xi,\eta)$ is a singularity. Equation (8) can be rewritten as follows,

$$c_p f_p + \int_\Gamma f \frac{\partial w}{\partial \mathbf{n}} dS = \int_\Gamma w \frac{\partial f}{\partial \mathbf{n}} dS \tag{9}$$

where the value of c_p depends on the location of $p = (\xi,\eta)$.(inside or outside of the integral boundary)

$$c_p = \begin{cases} 1 & \mathrm{p} \in \mathrm{D} \\ \frac{1}{2} & \mathrm{p} \in \Gamma \ \text{ AND } \ \Gamma \text{ smooth at p} \\ \frac{\text{internal angle}}{2\pi} & \mathrm{p} \in \Gamma \ \text{ AND } \ \Gamma \text{ not smooth at p} \end{cases} \tag{10}$$

In this method, the boundaries are discretized, so the integrals of equations (8) and (9) can be evaluated numerically. The values of $\frac{\partial f}{\partial \mathbf{n}}$ in each elements are specific, thus the equation will be just a function of values of f over the boundaries.

Therefore, by substituting the basic solution of the singular point in each boundary node and calculating the integrals, we arrive at a system of algebraic equations.

The solution of the system will be the value of f in each node. For calculating f in each point of the field, the singularity is located in that point and then the integrals will be calculated over the boundaries. Following this procedure, the potential field of the flow field will be obtained. This part of the solution is used as a background flow in each iteration in RVM. The value of $\frac{\partial f}{\partial \mathbf{n}}$ in each node can be calculated via the induced velocity of the vortexes and will be the boundary condition in the next step for the potential field.

3.2 Rotational field via RVM with contact layer

The vorticity transport equation is,

$$\frac{\partial\omega}{\partial t} + (\mathbf{U}.\nabla)\omega = \frac{1}{\mathrm{Re}}\nabla^2\omega + \nabla \times \mathbf{f} \tag{11}$$

where $(\vec{U}.\nabla)\omega$ and $(1/\mathrm{Re})\nabla^2\omega$ are the convection and diffusion of vorticity terms, respectively. $\nabla \times f$ is the source term where $\vec{f}$ is the body force which play the role of generation of vorticities in a contact layer element [13,15]. If the time step is considered to be enough small, the process of transportation of vorticity can be splitted into two mechanisms. Each mechanism can be solved in a small time step.

$$\frac{\partial\omega}{\partial t} + (\mathbf{U}.\nabla)\omega = 0 \tag{12}$$

$$\frac{\partial\omega}{\partial t} = \frac{1}{\mathrm{Re}}\nabla^2\omega + \nabla \times \mathbf{f} \tag{13}$$

In the first time step, transportation of vorticity is affected by convection, Eq. (12). In the second step, Eq. (13) is solved by generation of vorticity and diffusion of the vorticities. In this study, use was made of contact layer element model near the boundaries. The diffusion effect is more powerful than convection effect near the boundaries. Moreover, in this region, $\partial v/\partial x$ is less than $\partial u/\partial y$. So, the vorticities in this region can be calculated as,

$$\omega \approx -\partial u/\partial y \tag{14}$$

Assume that the vorticity is distributed near the wall by the thickness of α, smoothly (Fig. 13). The vorticity in this region can be evaluated via Eq. (14). Therefore, the asymptotic solution in the contact layer element is,

$$\omega = \omega_0 + \omega_1 y + \omega_2 y^2 + \cdots + \omega_n \tag{15}$$

where $y = O(\alpha)$ and ω_i is the order of different vorticities in the contact layer element. In the first time step, the vorticity is generated because of external forces and in the second step, these vorticities diffuse in the flow field. The thickness of α is divided into N element. Each element has ω_i and its surface is dA_i, thus the circulation is,

$$\Gamma_i = \omega_i.dA_i \quad (i = 1,2,\ldots N) \tag{16}$$

Γ_i can be divided into some bubble vortexes in each time step. The distribution of vortexes in each element should result in linear profile of induced velocity, Fig. 12, and the shear stress in the element will be satisfied, Fig. 15. These vorticities diffuse into the flow field according to,

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \tag{17}$$

where ν is the kinematic viscosity. The Green's function in one-dimension is,

$$Gr(y,t) = \sqrt{1/4\pi \nu t}.\exp\left(\frac{-1}{4\nu t} y^2\right) \tag{18}$$

Equation (18) is the same as a probability density function.

$$P(\eta;t) = \sqrt{1/2\pi\sigma^2}.\exp\left(-\frac{1}{2\sigma^2}\eta^2\right) \tag{19}$$

where η is a Gaussian parameter and $\sigma = \sqrt{2\nu t}$ is a standard deviation. Green's function in two-dimension is,

$$Gr'(x,y,t) = \frac{1}{4\pi \nu t}.\exp\left(\frac{-1}{4\nu t}\left(x^2 + y^2\right)\right) \tag{20}$$

Random motion of bubble vortex in contact layer element is alike to their motion in the flow field. $y_i < \alpha$ means that the vortex is still in the contact layer and it will be eliminated. In the next time step, they will be appeared in a new position regard to boundary conditions. If y_i is greater than α, the vortex was released to the flow field. So, we can write,

$$\mathbf{x}_j(t + \Delta t) = \mathbf{x}_j(t) + \sum_k \mathbf{U}(\mathbf{x}_{jk}) \tag{21}$$

where $\boldsymbol{\eta}_j = (\eta_{xj}, \eta_{yj})$ is a two dimensional Gaussian number and $\mathbf{U}(\mathbf{x}_{jk})$ is the velocity included both potential background flow and vortex effects. In contact layer model, the velocity is assumed to be linear in the contact layer element. Thus, α should be chosen enough small to satisfy the linear velocity profile criteria. If we consider σ equals α and as $\sigma = \sqrt{2\nu\Delta t}$, Δt can be calculated and so on the other parameters will be specified.

5 Combination of BEM and RVM with contact layer element model

In each iteration, flow field without consideration of boundaries will be found by summation of free velocity field and induced vortex velocity. This velocity field is considered as a initial condition for the next iteration. The velocity in each point in the field can be obtained by using integral form of Laplace equation over boundaries. It leads to a system of n equations and the velocity is achieved for each point in the domain of the solution. Therefore, the vorticities in the contact layer element produced according to the obtained velocity. These vorticities simulate the convection term in the vorticity transport equation. Simulation of diffusion term is described in sec. 4.

6 Results and Discussion

In this study, combination of BEM and RVM with contact layer element model is used to study the velocity field over a flat plate. Two kinematic viscosities are considered with $U_\infty = 1$ and $L = 1$. The plate is divided into 250 panels with thickness of α. 900 nodes are considered for the boundaries. The parameters are listed in table 1. Γ_{max} stands for the maximum of circulation for each bubble vortex. Considering the turbulence intensity less than 0.2%, the value of Γ_{max} is calculated. δ is the radius of bubble vortex core. We do not generate vortex over 1 meter (the length of the plate). Figures 1 and 2 show the boundary layer thickness. Velocity profiles for different cases illustrated in Figs. 3-10.

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Table 1: Parameters in Random Vortex Method (RVM)

Computational Parameter	$\mathrm{Re}_L = 1000$	$\mathrm{Re}_L = 10000$
$\Gamma_{\max}\left(m^2/S\mathrm{ec}\right)$	7.59×10^{-5}	7.59×10^{-5}
$\alpha(m)$	2×10^{-3}	5×10^{-4}
$\sigma(m)$	2×10^{-3}	5×10^{-4}
$\delta(m)$	2×10^{-3}	5×10^{-4}
$\Delta t\left(Sec\right)$	2×10^{-3}	1.25×10^{-3}

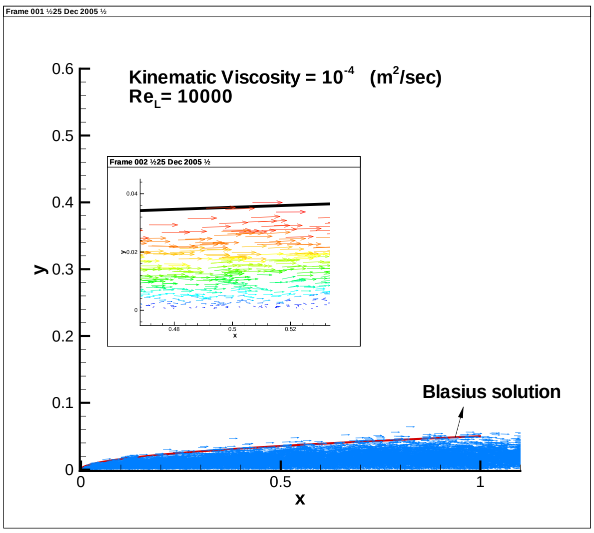

Fig 1. Boundary layer thickness with υ=0.0001 m^2/s

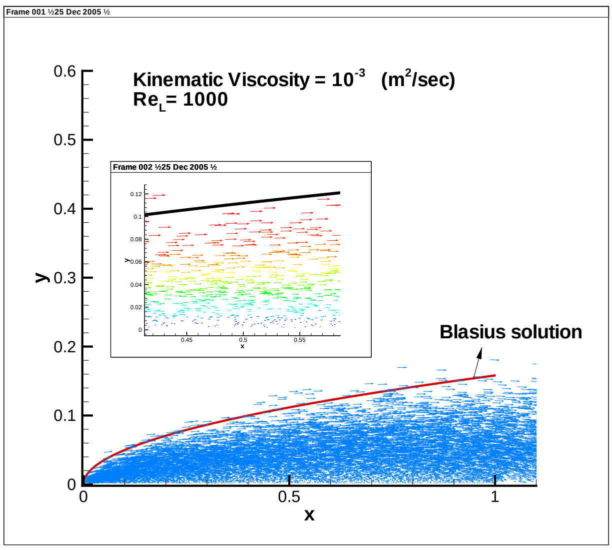

Fig 2. Boundary layer thickness with m^2/s υ=0.001

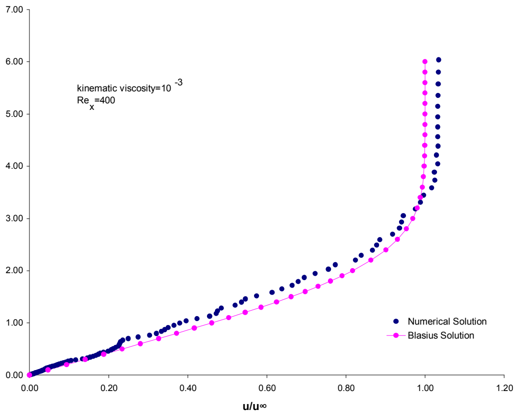

Fig 3. Velocity profile, x=0.4, υ=0.001 m^2/s

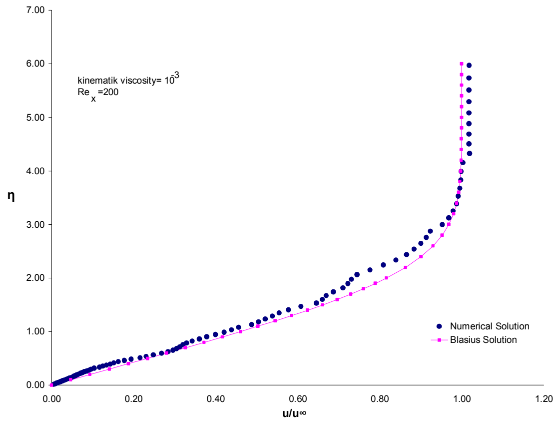

Fig 4. Velocity profile, x=0.2, υ=0.001 m^2/s

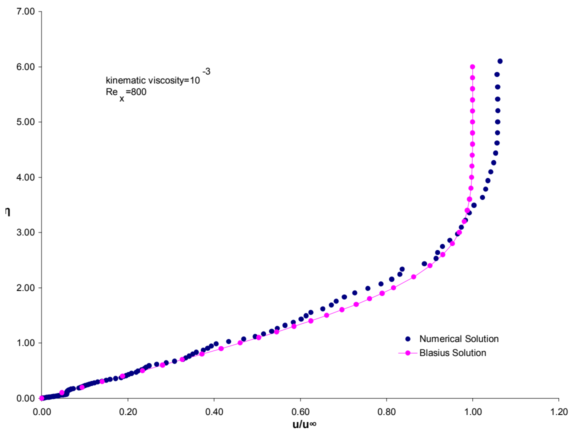

Fig 5. Velocity profile, x=0.8, υ=0.001 m^2/s

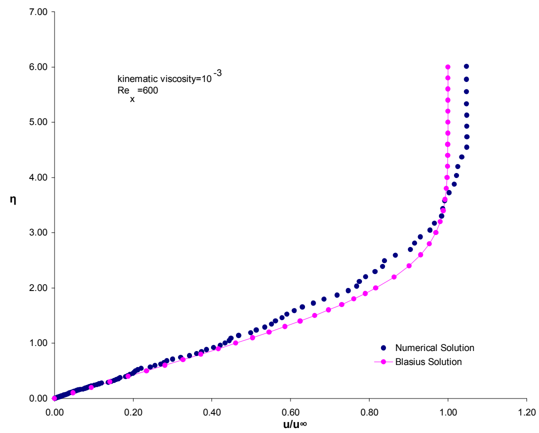

Fig 6. Velocity profile, x=0.6, υ=0.001 m^2/s

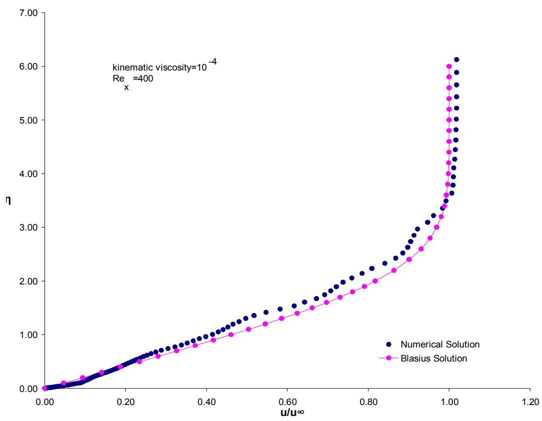

Fig 7. Velocity profile, x=0.4, υ=0.0001 m^2/s

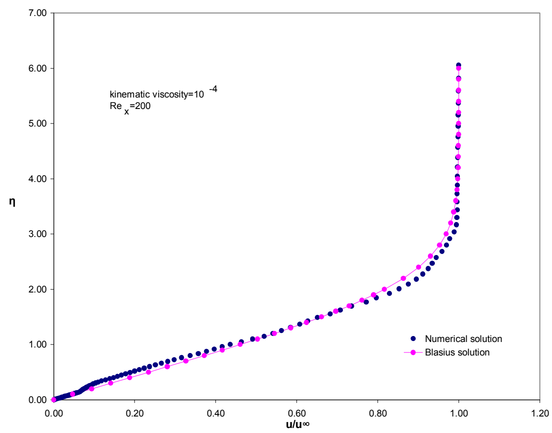

Fig 8. Velocity profile, x=0.2, υ=0.0001 m^2/s

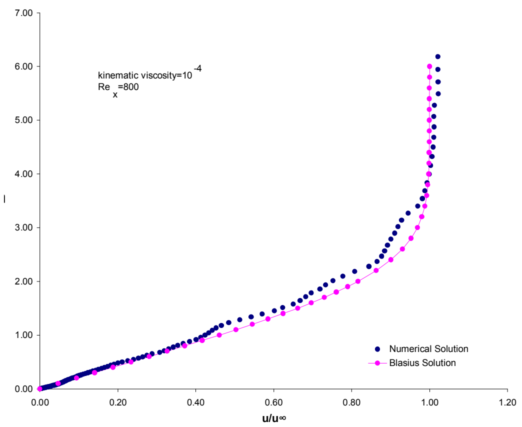

Fig 9. Velocity profile, x=0.8, υ=0.0001 m^2/s

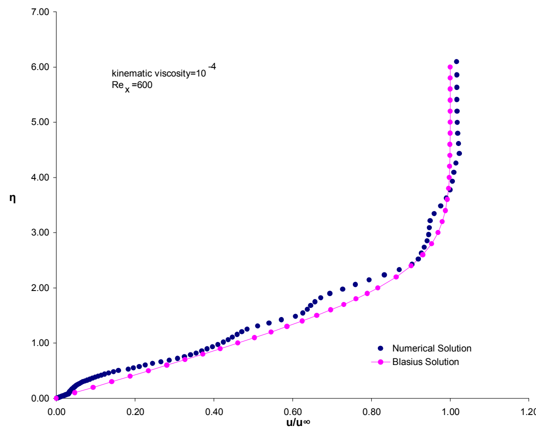

Fig 10. Velocity profile, x=0.6, υ=0.0001 m^2/s

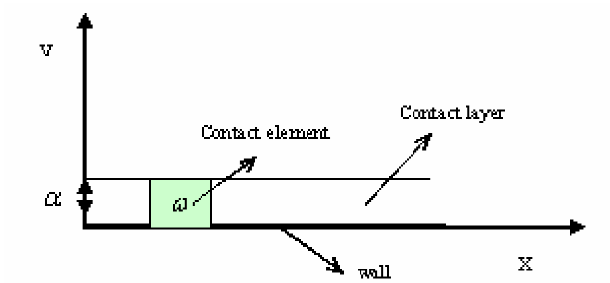

Fig 11. Contact layer element

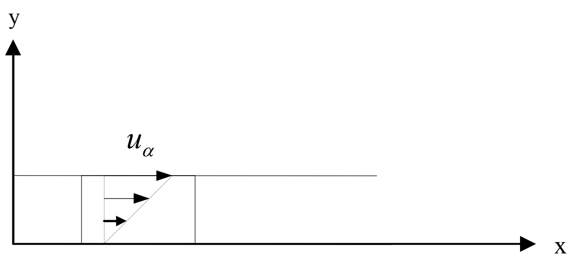

Fig 12. Velocity profile in contact layer element

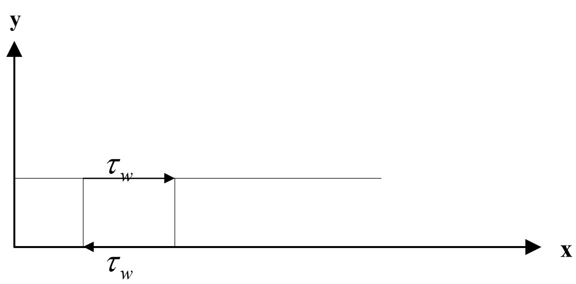

Fig 13. Shear stress over wall and contact layer element

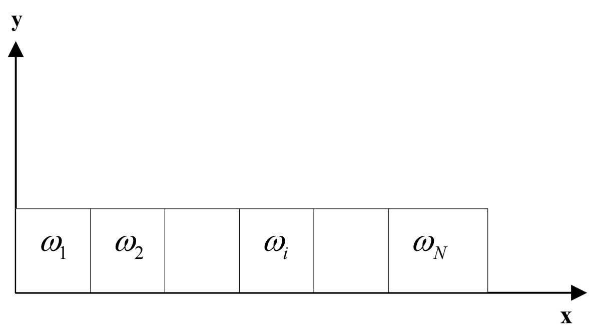

Fig 14. discritization of vorticities in a contact layer element