Proceedings of the 4th WSEAS Int. Conf. on HEAT TRANSFER, THERMAL ENGINEERING and ENVIRONMENT, Elounda, Greece, August 21-23, 2006 (pp380-384)

Comparison Between The Analytical And Numerical Methods Of Solving One-Dimensional Transient Heat Conduction Problems

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ABSTRACT:

The use of numerical methods for solving heat transfer problems is a result of the complexity of the analytical solution associated with practical engineering problems.

The present work represents a numerical solution using explicit and implicit techniques and the comparison with the analytical solution; of a heat transfer through a large slab of 0.3 mt thick steel armour plate initially at a uniform temperature of 710 ^oC. One surface is maintained at 710 ^oC while air is blown over the other surface which gives rise to an average heat-transfer coefficient of 113.4 $\frac{w}{m^2C}$. The temperature of air is T_∞ = 318 ^oC. The surface temperature and the distribution after one hour had elapsed were measured.

The results obtained of both implicit and explicit methods gave the same degree of accuracy for the problem solved and also the result obtained agree with the analytical solution which was developed by the author in previous paper[1]; and only an error of about 1% is measured.

This work shows that the implicit method has the disadvantage of requiring complete set of calculations; so that the implicit method used when the time increments of physical or boundary conditions impose excessively small time increments for the convergence of the solution by the explicit formulation.

Key words: Heat transfer, large slab, analytical solution, explicit method, implicit method.

Introduction

Heat transfer and temperature distribution in the slabs made of steel or other material play an important role in thermal applications.

Slabs in casting mould, or slabs in buildings structures or slabs in engines (Internal Combustion) have some times complex geometry; which make the use of numerical method easier for solving heat transfer problems rather than the complexity of the analytical solution associated with there practical engineering applications also non uniform boundary conditions, time dependent boundary conditions and temperature dependent properties.

Heat transfer in slabs has been the subject of investigations for many researchers [1-6]. In these studies i.e M.Riyad H.Abdelkader[1] developed and verified an analytical solution for this problem. The final formula obtained was

$$T(x,t) - T_0 = [T_0 - f(t)]\frac{\frac{hL}{K}}{1+\frac{hL}{K}}\frac{x}{L} +$$

$$2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 \alpha t}}{(\lambda_n L)^2 + \frac{hL}{K}\sin^2\lambda_n L}[T_0 - f(0)]\frac{hL}{K}\sin(\lambda_n L)\sin(\lambda_n x)$$

I.Kreja and et al[2] deals with the elaborated numerical analysis of some examples which elucidate most important features of the computer program prepared for the proposal formulation of heat transfer and temperature gradient at the phase change interface results for freezing slab are obtained in simulation of the 2-Dimensional infinite media.

Micheal B. and et al[3] used a lines method technique for solving partial differential equations (PDEs) by typically using finite difference relationships for the spatial derivatives and ordinary differential equations for the time derivatives. A problem in unsteady-state heat transfer in a slab is numerically solved by the lines method. And it was found that the results which are obtained by the lines method indicated that there is a general agreement between the lines method and the results which are obtained by the hand calculation of a finite difference solution.

Lars and et al[4] consider an inverse heat conduction problem, the sideways heat equation, which is the model of the problem where temperature distribution on both sides of a thick wall should be determined. Numerical measurements are executed on one side of the thick wall. The numerical implementation of Fourier and wavelet methods for solving the sideway heat equations was discussed. It was found comparable accuracy between the numerical experiments and Daubechies wavelet.

Blet and et al[5] used the finite element and boundary elemnt techniques to describe the temperature distribution and heat flux through slabs of continous casting process.

Richard and et al[6] developed and verified a fundamental based model for low temperatures radiant system and used a computer program to evaluate temperature distribution and heat flux variation through one dimensional transient conduction heat problem.

Lars Eldl'en[7] consider a Cuchy problem for heat equation in quarter plane where data are given and a solution is sought in the interval 0<x<1, where temperature distribution on a thick wall should be determined for manufacturing purposes, the heat equation is discretuzed by a differential – difference equation, where the time derivative has been replaced by finite difference. An error estimate is obtained and gives information about how to choose the step length in the time discretization.

In this work the one dimensional heat problem which is time dependent is solved numerically by Explicit and implicit methods which are introduced, and their degree of agreement with the analytical method[1] is investigated.

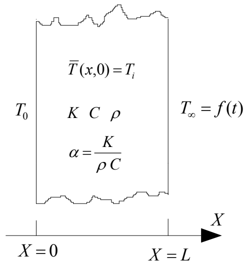

Figure 1: schematic representation of steel slab

Nomenclature

L	Thickness of wall m	T_{i+1}^{t+1}	Nodal temperature C
α	Thermal diffusivity = k/ρc m^2/hr	θ	Time (= t. Δθ) hr
k	Conduction coefficient w/m°C	A	Coefficient matrix
h	Convection coefficient w/m^2s	B	Right hand side-vector
N	Number of nodes	P	$(\Delta x^2/\propto\Delta\theta)+2$
Δx	Axial increment m.	Q	$\Delta x^2/\propto\Delta\theta$
Δθ	Time increment hr	t	Time hr
T_∞	Environment temperature ˚C	T_i^1	Initial temperature ˚C
i+1	Distance index	T_i^{t+1}	Unknown temperature vector ˚C
t+1	Time index		

Numerical solution

In the numerical solution, two methods of solution were applied, the explicit and implicit methods. Steps involved in the numerical solution are as follows:

1. All relevant information were assembled about the problem including geometry, boundary conditions and physical properties.

2. The slab was divided into (N-1) equal parts resulting in N nodal planes as shown in fig 2.

3. We assumed that
(a) The temperature of an element is represented by that at the node.
(b) The thermal conductivity to be used for the heat flow is constant.

4. The energy balance was preformed on each element leading to an algebraic equation for the node representing the element.

5. All equations were arranged in a suitable form so that they can be solved by explicit and implicit methods and the solution was obtained by using the computer.

1 <u>Explicit method</u>

The slab of width, L, in the x direction having no temperature gradients in the y and the z directions. The slab is initially at a uniform temperature Ti. From time θ=0 onwards, the left –face is maintained at constant temperature T_0 and the right face is exposed to convective heat loss. We are interested in obtaining a numerical solution to this problem. The slab width, L, is divided into (N-1) equal parts resulting in N nodal planes as shown in figure (2). All the internal nodes have material of width ($\frac{\Delta x}{2}$) associated them on either side of their center plane, while the boundary nodes have material of width ($\frac{\Delta x}{2}$) on one side only.

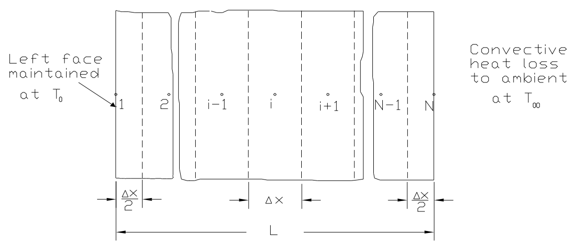

Figure 2: Grid for a slab of width L initially at temperature of Ti

The energy conducted into node i across-sectional area, A, during a unit time is given by

$$Q_{(i-1)\to i} = KA\,\frac{T_{i-1}^t - T_i^t}{\Delta x}$$

$$Q_{(i+1)\to i} = KA\,\frac{T_{i+1}^t - T_i^t}{\Delta x}$$

where the superscript t denotes temperatures at time $\theta = t\Delta\theta$
We use a superscript to represent time to emphasize the unsteady nature of the problem and maintain a distinction between the space coordinates and the time coordinate.

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In view of the unsteady nature of the problem, there is change of internal energy. ΔUi in time Δθ which is given by

$$\Delta U_i = \rho(A.\Delta x)C(T_i^{t+1} - T_i^t)$$

Where ρ is the density of the slab, c is the specific heat of the slab, and T_i^{t+1} denotes the temperature of the node i at time (θ + Δθ). To satisfy conservation of energy

$$\Delta U_i = (Q_{(i-1)\to i} + Q_{(i+1)\to i})\Delta\theta$$

Or

$$\rho A\Delta XC(T_i^{t+1} - T_i^t) = KA\frac{T_{i-1}^t - T_i^t}{\Delta x}\Delta\Theta + KA\frac{T_{i+1}^t - T_i^t}{\Delta x}\Delta\Theta \quad \ldots(1)$$

Where the superscript (t+1) denotes the temperature at time (θ+Δθ).

Solving for T_i^{t+1}, we obtain

$$T_i^{t+1} = \frac{k\Delta\Theta}{\rho c(\Delta x)^2}(T_{i-1}^t - 2t_i^t + T_{i+1}^t) + T_i^t \quad \ldots(1a)$$

Or

$$T_i^{t+1} = \frac{1}{\beta}[(T_{i-1}^t + T_{i+1}^t) + (\beta - 2)T_i^t]$$

Where

$$\frac{1}{\beta} = \frac{K}{\rho C}\frac{\Delta\Theta}{(\Delta x)^2} or \beta = \frac{(\Delta x)^2}{\alpha\Delta\Theta} \quad \ldots(1b)$$

Equation (1-a) expresses the temperature at node i at time (θ +Δθ) in terms of the temperature at time θ. In this case since the temperature for θ = 0 are all equal to T_i and are known, future temperatures at time Δθ at all the internal nodes can be computed with a pre-selected value of β. Once the temperature at all the nodes for time Δθ are calculated, their values are used as input when computing the temperature for time 2Δθ, etc.

The equation for boundary nodal is

$T_i^t = T_0$ For all θ > 0 or for t =1, 2, 3, ----

The equation for the boundary node N can be obtained by performing an energy balance on the Nth node. It yields

$$\Delta\Theta(Q_{(N-1)} \to +Q_{conv\to N}) = \Delta U_N$$

When appropriate expressions are substituted for each term in the above equation, we obtain

$$KA\frac{T_{N-1}^t - T_N^t}{\Delta X}\Delta\Theta + hA(T_\infty - T_N^t)\Delta\Theta = \rho(A\frac{1}{2}\Delta X)C(T_N^{t+1} - T_N^t)$$

Rearrangement gives

$$T_N^{t+1} = \frac{2}{\beta}(T_{N-1}^t - T_N^t) + \frac{2h\Delta\Theta}{\rho C\Delta X}(T_\infty - T_N^t) + T_N^t$$

Or

$$T_N^{t+1} = \frac{2}{\beta}T_{N-1}^t + \frac{2h\Delta\Theta}{\rho C\Delta X}T_\infty + (1 - \frac{2}{\beta} - \frac{2h\Delta\Theta}{\rho C\Delta X})T_N^t$$
$$= \frac{2\alpha\Delta\Theta}{\Delta X^2}T_{N-1}^t + \frac{2h\Delta\Theta}{\rho C\Delta X}T_\infty + 1 - \Delta\Theta(\frac{2\alpha}{\Delta X^2} + \frac{2h}{\rho C\Delta X})T_N^t \quad (2)$$

The temp at node 1 will be fixed, whereas that at node N will continually change until steady state is reached, and equation 1-2 gives the nodal temperature.

This form of the difference equations is known as the explicit form, since the temperature T_i^{t+1}, corresponding to time (t +Δθ) can be solved for explicitly, only the temperatures, T_i^t, corresponding to time, t, appear in the right-hand sides of the equations 1-a and (2). This is due to the use T_i^t in the energy balance for all the nodes and the associated stability criterion is

$$\Delta\Theta \le \frac{1}{\frac{2K}{\rho C\Delta X^2} + \frac{2h}{\rho C\Delta X}} = \frac{1}{2}\frac{\Delta X^2}{\alpha}(\frac{1}{1 + (h\Delta X / K)}) \quad \ldots(3)$$

With ΔX =.05 m Δθ ≤ 0.0898 hr we assume Δθ = .05 hr since the term in brackets in equation 3 is always less than one, it is apparent that to satisfy the stability criterion at a surface node with convection requires a smaller time increment than at an interior node. Thus, the surface node becomes the controlling factor for the maximum permissible value for Δθ. When the unit surface conductance is large, the permissible value of the time increment may become so small that the computations by the explicit methods will require an exorbitant amount of time. In such cases the implicit method described later should be used.

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A computer program which can perform the computation for the explicit method was developed.

2 Implicit Method

The requirement that Δθ should be restricted in size to insure stability sometimes results in an extremely small time step, of the order of a fraction of a second, especially when transient conduction a multiple layers is involved.

The implicit formulation eliminates this restriction, but it involves solving simultaneous equations at each time step.

Consider equation (4), which is reproduced below

$$\rho A\Delta XC(T_i^{t+1} - T_i^t) = KA\frac{T_{i-1}^t - T_i^t}{\Delta X}\Delta\theta + KA\frac{T_{i+1}^t - T_i^t}{\Delta X}\Delta\theta \quad (4)$$

The left-hand side represents the change of the internal energy due to the flow of heat associated with the "present" temperature gradients at time, θ, as written on the right-hand side. It is equally plausible that the change in internal energy is due to the flow of heat associated with temperature gradients at time (θ+Δθ) leading to the following equation.

$$\rho \Delta X C (T_i^{t+1} - T_i^t) = K \frac{T_{i-1}^{t+1} - T_i^{t+1}}{\Delta X} \Delta\theta + K \frac{T_{i+1}^{t+1} - T_i^{t+1}}{\Delta X} \Delta\theta \qquad (5)$$

The above equation contains only one temp at time θ, that is, T_i^t whereas all the rest of the terms contain temperatures at time (θ+Δθ). Therefore, temperatures at (θ+Δθ) cannot be solved for explicity; instead, one has to solve the equations for all the nodes simultaneously after setting up equations for the internal and the boundary nodes. Such as implicit formulation is stable regardless of the value of the time increment, Δθ that is chosen. An excessively large value of Δθ will cause large errors inherently associated with the finite difference method.

From equation (5)

$$\frac{\rho \Delta X C \, (T_i^{t+1} - T_i^t)}{\Delta \theta} = \frac{K}{\Delta X}[(T_{i-1}^{t+1} - T_i^{t+1}) + (T_{i+1}^{t+1} - T_i^{t+1})]$$

Solving for the nodal temperature T_i^{t+1}:

$$T_i^{t+1}\left(\frac{1}{\alpha}\frac{\Delta X^2}{\Delta \theta} + 2\right) = \frac{1}{\alpha}\frac{\Delta X^2}{\Delta \theta} T_i^t + T_{i-1}^{t+1} + T_{i+1}^{t+1}$$

The expression for T_i^{t+1} involves other unknowns at the time level; hence the value of T_i^{t+1} cannot be determined from one equation as in the explicit method. Instead, all nodal temperatures are determined at once by solving all heat balances simultaneously by matrix inversion.

The heat balance at a surface point (node N) gives:

$$T_N^{t+1}\left(\frac{\Delta X^2}{2\alpha\Delta\theta} + \frac{h\Delta x}{K} + 1\right) = \frac{h\Delta X}{K} T_\infty + T_{N-1}^{t+1} + \frac{X^2}{2\alpha\Delta\theta} T_N^t$$

The heat balances for all 7 nodal points may be written in concise matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & P & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & P & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & P & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & P & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & P & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & \frac{P}{2} + \frac{h\Delta X}{K} \end{bmatrix} \begin{bmatrix} T_1^{t+1} \\ T_2^{t+1} \\ T_3^{t+1} \\ T_4^{t+1} \\ T_5^{t+1} \\ T_6^{t+1} \\ T_7^{t+1} \end{bmatrix} = \begin{bmatrix} 1300 \\ QT_2^t \\ QT_3^t \\ QT_4^t \\ QT_5^t \\ QT_6^t \\ \frac{Q}{2}T_7^t + \frac{h\Delta X}{K}T_\infty \end{bmatrix}$$

Where $P = \left(\frac{\Delta X^2}{\alpha\Delta\theta} + 2\right)$, $Q = \frac{\Delta X^2}{\alpha\Delta\theta}$

This is a matrix equation of the form $A \cdot U = B$ which may be solved for U by finding the inverse matrix A, A^{-1} or U= $A^{-1} \cdot B$.

The transient temperature distribution is computed by executing such a matrix inversion at each time t. A computer program which can perform these computations was developed.

Results

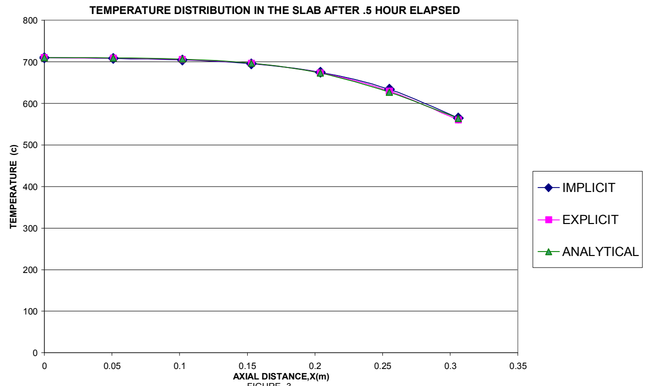

FIGURE -3-

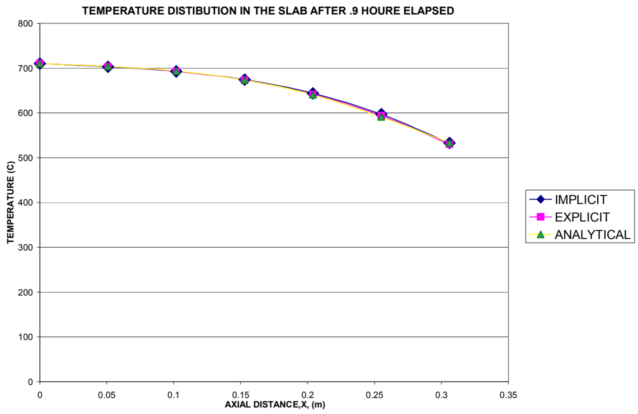

FIGURE--4-

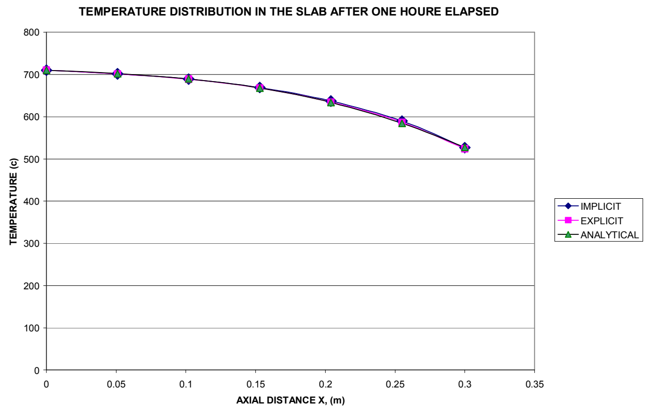

FIGURE -5-

Discussion and Conclusion

The plots in figures 3, 4 and 5 reveal that the explicit and implicit methods give essentially the same degree of accuracy for the problem solved. The little difference there can be attributed to round off error in the calculations. Also there is a small difference between the analytical solution and the numerical solutions. This error has a maximum of about 1%.

When the unstable case was run, the explicit solution contained some negative temperatures and asterisks were printed meaning the temperatures could not be calculated or printed. Over all, the results of this investigation turned out as expected. It was confirmed that the implicit method has the advantage that any time increment can be used. In fact, the time increment can be varied during the calculations. This was verified by running the unstable case. When this was done, the correct temperatures were obtained.

Also Confirmed was the fact that the implicit method has the disadvantage of requiring a complete set of calculations (i.e. iteration of matrix inversion) at each $\Delta\theta$ step. The implicit method is, therefore, used in practice when the physical or boundary conditions impose excessively small time increments for the convergence of the solution by the explicit formulation.

References

1- M.Riyad H.Abdelkader, "Analytical solution of one-dimentional transient heat conduction problem in thick metal slab", IASME Transations, Issue 9, vol.2, pp.1624-1629, 2005.

2- I,Kreja and B.B.Budkowaska "Fixed finite element model of heat transfer with phase change", International journal of offshore and polar engineering, vol.7.No.3, 1997.

3- Michael B. and Mordechai Shacham "The numerical methods of Lines for partial differential equations"

4- Lars Eld'en abd Fredrik Berntsson "Wavelet and Fourier methods for solving the sideways heat equation. SIAM J, scient.comput. vol.21 N.6 P.P 2187-2205,2000

5- Belet Kiflie and Dr.Ing Demiss – Almex [2000], Thermal analysis of continous casting process, ESME 5th Annual conference on manufacturing and process industry septemper2000.

6- Richard K. strand and Curtis O. Perdersen. Analytical verification of heat source Transfer functions. Blast support of fic, Dept of Mechanical and Industrial Engineering University of Illiniot at Urban. Champagion 1206 W.Green 6t. Urban 1261801 USA.

7- Lars Eld\'en "Numerical solution of the sideways heat equation by difference approximation in time report LITH – MAT – R – 1994 – 40.