Modelling of Pipelines in State-Space

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Abstract: – This paper deals with the-state space representation of a nonlinear model of a pipeline. First, the mathematical model of the pipeline is presented in its normalized form. Then the method of characteristics is used to transform the model into a form suitable for representing the model in state space. The resulting nonlinear state space-model is verified using measurements on a real pipeline.

Key Words: - Pipeline, Modelling and simulation, State-space representation.

1 Introduction

Observer-based-leak detection and localisation schemes, for example, require a pipeline model to compute the states of the pipeline without a leak [1], [2]. The first industrial applications demonstrated the performance of observerbased methods [3]. In these schemes the observer is derived using a mathematical model of the pipeline. The onedimensional compressible fluid flow through the pipelines is governed by nonlinear partial differential equations [4]. The pipelines are, therefore, distributed parameter systems. To date, there is no general closed-form solution for such systems Numerical approaches, like the method of characteristics, are used instead [5], yielding the computational background for the observer algorithms. These equations can also be written in the state-space representation form, which is suitable for the simulation and application of a broad variety of control algorithms. The aim of this paper is to present a nonlinear distributed parameter model of pipelines, to transform it into the state-space form and to verify it on data obtained by measurements on a real pipeline. The paper is organized as follows: In Section 2 the model of the pipeline is derived. It is a nonlinear distributed parameter model, so one of the possible solutions - the method of characteristics – is presented in Section 3. The equations of the method of characteristics are transformed into the state representation form in Section 4. The simulation of the obtained model and its verification on real pipeline data is given in the Section 5.

2 Mathematical model of the pipeline

The classical solution for unsteady-flow problems is obtained by using the equations for continuity, momentum and energy. These equations correspond to the physical principles of mass, momentum and energy conservation. Applying these equations leads to a coupled non-linear set of partial differential equations, which are very difficult to solve analytically. To date, there is no general closed-form solution. Further problems arise in the case of turbulent flow, which introduces stochastic flow behavior. Therefore, the mathematical derivation for the flow through a pipeline is a mixture of both theoretical and empirical approaches.

The following assumptions are made when deriving for the derivation of a mathematical model of the flow through pipelines:

- 1. **The fluid is compressible.** Compressibility of the fluid results in an unsteady flow.
- 2. **The flow is viscous.** Viscosity causes shear stresses in a moving fluid, and these stresses are taken into account.
- 3. The flow is adiabatic. No transfer of energy between the fluid and the pipeline is considered. Therefore, the temperature T along the pipeline is constant.
- 4. The flow is one-dimensional. All the characteristics of the pipeline, such as velocity v and pressure p, depend only on the position along the *x*-axis of the pipeline.



Figure 1: Illustration of characteristics

The application of the continuity and momentum equations [6] in conservative form for the one-dimensional case, the formula of Darcy and Weisbach [7] and the formula of Colebrook [7] lead to the following mathematical model of a pipeline [8]:

$$\frac{d\rho}{dt} + \rho \frac{\partial v}{\partial x} = 0 \tag{1}$$

$$\frac{dv}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\alpha + \frac{\lambda v|v|}{2D} = 0$$
(2)

$$\frac{dp}{dt} - a^2 \frac{d\rho}{dt} = 0 \tag{3}$$

with density $\rho(x)$, velocity v(x), pressure p(x), diameter D, dimensionless friction coefficient $\lambda(v)$, and (isentropic) speed of sound a of the fluid. $g \sin \alpha$ is the x-component of the standard gravity vector \mathbf{g} .

Using the method of characteristics [5] the following set of equations is obtained

$$dp + a \bar{\rho} dv + \tilde{p} ds = 0$$

$$a dt - ds = 0$$
(4)

is valid along C^+ , the second one

$$dp - a \bar{\rho} dv + \tilde{p} ds = 0$$

$$a dt + ds = 0$$
 (5)

is valid along C^- , and the third one

$$a^{2} \frac{d\rho}{dt} - \frac{dp}{dt} = 0$$

$$v \, dt - ds = 0$$
(6)

is valid along C^F . The specific pressure loss is given by

$$\tilde{p} \equiv \bar{\rho} \left(g \, \sin \alpha + \frac{1}{2} \, \frac{\lambda \, v \, |v|}{D} \right) \tag{7}$$

where C^+ , C^- and C^F are characteristics corresponding to the velocity of sound and the flow, respectively, as illustrated in Fig. 1 Introducing the normalized time t' and length s' as well as the *Mach-number* Ma

$$t' \equiv \frac{a t}{L} \qquad \qquad v' \equiv Fr^2 Ma$$
$$Ma \equiv \frac{ds'}{dt'} = \frac{\frac{ds}{L}}{\frac{a dt}{L}} = \frac{v}{a} \qquad \qquad s' \equiv \frac{s}{L}$$

where L is the length of the pipeline, the *normalized* piezometric head, the *normalized* specific pressure loss \tilde{p}' , the *normalized density*

$$h \equiv \frac{p}{g \,\bar{\rho} \,L} \qquad \tilde{p}' \equiv \frac{\bar{p}}{g \,\bar{\rho}} \qquad \rho' \equiv \frac{\rho}{\bar{\rho}} \qquad (8)$$

the normalized diameter $d \equiv \frac{D}{L}$, the normalized specific pressure loss \tilde{p}' is given by

$$\tilde{p}' \equiv \frac{\tilde{p}}{g\,\bar{\rho}} = \sin\alpha + R\,\lambda\,v'\,|v'| \tag{9}$$

where the resistance-number and the Froude-number

$$R \equiv \frac{1}{2} \frac{1}{Fr^2 d} \qquad \qquad Fr \equiv \frac{a}{\sqrt{g L}} \qquad (10)$$

we get the normalized model of a pipeline for (C^+)

$$dh + dv' + \tilde{p}' ds' = 0$$

$$dt' - ds' = 0$$
(11)

$$(C^{-})$$

$$dh - dv' + \tilde{p}' ds' = 0$$

$$dt' + ds' = 0$$
 (12)

and (C^F)

$$Fr^{2}d\rho' - dh = 0$$

$$Ma dt' - ds' = 0$$
(13)

3 Numerical Solution by the Method of Characteristics

We now want to find a numerical solution for (11) and (12) using (9). Applying the method of characteristics to the inner points $1 \le i \le N - 1$, gives

$$\begin{pmatrix} h_i(k+1) - h_{i-\xi}(k) \end{pmatrix} + \\ + \left(v'_i(k+1) - v'_{i-\xi}(k) \right) + \Delta t' \, \tilde{p}'_{i-\xi}(k) = 0 \\ \left(h_i(k+1) - h_{i+\psi}(k) \right) - \\ + \left(v'_i(k+1) - v'_{i+\psi}(k) \right) - \Delta t' \, \tilde{p}'_{i+\psi}(k) = 0$$

$$(14)$$

By adding and subtracting equations 14 we get

$$2 h_{i}(k+1) = \left(h_{i-\xi}(k) + h_{i+\psi}(k)\right) + \left(v_{i-\xi}'(k) - v_{i+\psi}'(k)\right) - \Delta t' \left(\tilde{p}_{i-\xi}'(k) - \tilde{p}_{i+\psi}'(k)\right)$$
$$2 v_{i}'(k+1) = \left(v_{i-\xi}'(k) + v_{i+\psi}'(k)\right) + \left(h_{i-\xi}(k) - h_{i+\psi}(k)\right) - \Delta t' \left(\tilde{p}_{i-\xi}'(k) + \tilde{p}_{i+\psi}'(k)\right)$$
(15)

For the (C^F) characteristic we get

$$\left(\rho_i'(k+1) - \rho_{i-\zeta}'(k) \right) -$$

$$- \frac{1}{Fr^2} \left(h_i(k+1) - h_{i-\zeta}(k) \right) = 0$$
(16)

For the inlet i = 0 and the outlet i = N we have to choose the appropriate boundary conditions. In this case we choose $h_0(k+1)$ for the inlet and $h_N(k+1)$ for the outlet. This leads to

$$v_{0}'(k+1) = v_{0+\psi}'(k) + \left(h_{0}(k+1) - h_{0+\psi}(k)\right) - \Delta t' \,\tilde{p}_{0+\psi}'(k)$$
$$v_{N}'(k+1) = v_{N-\xi}'(k) - \left(h_{N}(k+1) - h_{N-\xi}(k)\right) - \Delta t' \,\tilde{p}_{N-\xi}'(k)$$
(17)

With respect to the (C^F) characteristic the boundary condition for ρ at the inlet of the pipeline are required. The density ρ at the outlet is determined in the same way as for the inner points

$$\left(\rho'_N(k+1) - \rho'_{N-\zeta}(k) \right) -$$

$$- \frac{1}{Fr^2} \left(h_N(k+1) - h_{N-\zeta}(k) \right) = 0$$
(18)

The resulting solution algorithm is given by (15), (16) together with (17), (18).

4 The State-Space Representation

First of all, we rearrange (15);

$$2 h_{i}(k+1) = \left(h_{i-\xi}(k) + v'_{i-\xi}(k)\right) + \\ + \left(h_{i+\psi}(k) - v'_{i+\psi}(k)\right) - \Delta t' \left(\tilde{p}'_{i-\xi}(k) - \tilde{p}'_{i+\psi}(k)\right) \\ 2 v'_{i}(k+1) = \left(h_{i-\xi}(k) + v'_{i-\xi}(k)\right) +$$
(19)
$$+ \left(-h_{i+\psi}(k) + v'_{i+\psi}(k)\right) - \Delta t' \left(\tilde{p}'_{i-\xi}(k) + \tilde{p}'_{i+\psi}(k)\right)$$

and introduce (19) into (16):

$$2 \rho'_{i}(k+1) = 2 \rho'_{i-\zeta}(k) + \frac{1}{Fr^{2}} \left(\left(h_{i-\xi}(k) + v'_{i-\xi}(k) \right) + \left(h_{i+\psi}(k) - v'_{i+\psi}(k) \right) - (20) - \Delta t' \left(\tilde{p}'_{i-\xi}(k) - \tilde{p}'_{i+\psi}(k) \right) - 2 h_{i-\zeta}(k) \right)$$

The quantities $h_{i-\xi}(k)$, $v'_{i-\xi}$, $h_{i+\psi}(k)$, $v'_{i+\psi}(k)$, $\tilde{p}'_{i-\xi}(k)$ and $\tilde{p}'_{i+\psi}(k)$ are obtained by linear interpolation as follows

$$h_{i-\xi}(k) = \gamma h_{i-1}(k) + (1-\gamma)h_i(k)$$

$$v'_{i-\xi}(k) = \gamma v'_{i-1}(k) + (1-\gamma)v'_i(k)$$

$$\tilde{p}'_{i-\xi}(k) = \gamma \tilde{p}'_{i-1}(k) + (1-\gamma)\tilde{p}'_i(k)$$

$$h_{i+\psi}(k) = \gamma h_{i+1}(k) + (1-\gamma)h_i(k)$$

$$v'_{i+\psi}(k) = \gamma v'_{i+1}(k) + (1-\gamma)v'_i(k)$$

$$\tilde{p}'_{i+\psi}(k) = \gamma \tilde{p}'_{i+1}(k) + (1-\gamma)\tilde{p}'_i(k)$$

$$h_{i-\zeta}(k) = \gamma Ma^i h_{i-1}(k) + (1-\gamma Ma^i)h_i(k)$$

$$\rho'_{i-\zeta}(k) = \gamma Ma^i \rho'_{i-1}(k) + (1-\gamma Ma^i)\rho'_i(k)$$
(21)

where

$$\gamma = \frac{\Delta t'}{\Delta s'} \qquad \qquad Ma^i = \frac{v'_i}{Fr^2} \qquad (22)$$

Introducing (21) into (19) and (21) we obtain

$$2 h_{i}(k+1) = \gamma \left(h_{i-1}(k) + v_{i-1}'(k) \right) + 2 (1-\gamma)h_{i}(k) + \gamma \left(h_{i+1}(k) - v_{i+1}'(k) \right) - \Delta t' \gamma \left(\tilde{p}_{i-1}'(k) - \tilde{p}_{i+1}'(k) \right)$$

$$2 v_{i}'(k+1) = \gamma \left(h_{i-1}(k) + v_{i-1}'(k) \right) + 2 (1-\gamma)v_{i}'(k) + \gamma \left(-h_{i+1}(k) + v_{i+1}'(k) \right) - \Delta t' \left(\gamma \tilde{p}_{i-1}'(k) + 2 (1-\gamma)\tilde{p}_{i}'(k) + \gamma \tilde{p}_{i+1}'(k) \right)$$

$$2 \rho_{i}'(k+1) = \frac{1}{Fr^{2}} \left(\gamma \left((1-2Ma^{i})h_{i-1}(k) + v_{i-1}'(k) \right) + 2 (Ma^{i} - 1)\gamma h_{i}(k) + \gamma \left(\tilde{p}_{i-1}'(k) - \tilde{p}_{i+1}'(k) \right) \right) \right)$$

$$+ 2 \left(\gamma Ma^{i} - 1 \right) \gamma h_{i}(k) + \gamma \left(\tilde{p}_{i-1}'(k) - \tilde{p}_{i+1}'(k) \right) \right) + 2 \left(\gamma Ma^{i} \rho_{i-1}'(k) + (1-\gamma Ma^{i})\rho_{i}'(k) \right)$$

$$(23)$$

Let us introduce the state element vector

$$\mathbf{x_i} \equiv \begin{bmatrix} h_i \\ v'_i \\ \rho'_i \end{bmatrix}$$

Vector/matrix-notation leads to

$$2\mathbf{x}_{i}(k+1) = \mathbf{A}_{\gamma}^{+,i}\mathbf{x}_{i-1}(k) + 2\mathbf{A}_{1-\gamma}^{I,i}\mathbf{x}_{i}(k) + \mathbf{A}_{\gamma}^{-,i}\mathbf{x}_{i+1}(k) - \Delta t' \left(\mathbf{a}_{\gamma}^{+} \tilde{p}_{i-1}'(k) + 2 \mathbf{a}_{1-\gamma}^{0} \tilde{p}_{i}'(k) + \mathbf{a}_{\gamma}^{-} \tilde{p}_{i+1}'(k)\right)$$

with

$$\mathbf{A}_{\gamma}^{i+} \equiv \gamma \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \frac{1-2Ma^{i}}{Fr^{2}} & \frac{1}{Fr^{2}} & 2Ma^{i} \end{bmatrix} \\
\mathbf{A}_{\gamma}^{i-} \equiv \gamma \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ \frac{1}{Fr^{2}} & -\frac{1}{Fr^{2}} & 0 \end{bmatrix} \\
\mathbf{A}_{1-\gamma}^{i,I} \equiv \begin{bmatrix} 1-\gamma & 0 & 0 \\ 0 & 1-\gamma & 0 \\ \gamma \frac{Ma^{i}-1}{Fr^{2}} & 0 & 1-\gamma Ma^{i} \end{bmatrix} \\
\mathbf{a}_{\gamma}^{+} \equiv \gamma \begin{bmatrix} 1 \\ 1 \\ \frac{1}{Fr^{2}} \end{bmatrix} \\
\mathbf{a}_{\gamma}^{-} \equiv \gamma \begin{bmatrix} -1 \\ 1 \\ -\frac{1}{Fr^{2}} \end{bmatrix} \\
\mathbf{a}_{1-\gamma}^{0} \equiv (1-\gamma) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
(24)

Vectorization results in

$$2 \begin{bmatrix} \mathbf{x}_{1}(k+1) \\ \mathbf{x}_{2}(k+1) \\ \vdots \\ \mathbf{x}_{N-2}(k+1) \\ \mathbf{x}_{N-1}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{0}(k) \\ \mathbf{x}_{1}(k) \\ \vdots \\ \vdots \\ \mathbf{x}_{N}(k) \end{bmatrix} - \frac{\mathbf{A}_{1} \cdot \mathbf{0}}{\mathbf{A}_{2} \mathbf{1} \cdot \mathbf{0}} \begin{bmatrix} \tilde{p}_{0}(k) \\ \tilde{p}_{1}'(k) \\ \vdots \\ \vdots \\ \tilde{p}_{N-1}'(k) \\ \tilde{p}_{N}'(k) \end{bmatrix}$$
(25)

where because of the lack of space the matrices are written in a distributed form: **0** is the zero-matrix and

$$\mathbf{A_1} = \begin{bmatrix} \mathbf{A_{\gamma}^{1+} \ 2\mathbf{A_{1-\gamma}^{1,I}} \ \mathbf{A_{\gamma}^{1-} \ 0 \ \dots} \\ \mathbf{0} \ \mathbf{A_{\gamma}^{2+} \ 2\mathbf{A_{1-\gamma}^{2,I}} \ \mathbf{A_{\gamma}^{2-} \ \dots} \\ \vdots \ \vdots \ \ddots \ \ddots \ \ddots \ \vdots \\ \mathbf{A_2} = \begin{bmatrix} \ddots \ \ddots \ \ddots \ \vdots \ \vdots \\ \dots \ \mathbf{a_{\gamma}^{+} \ 2\mathbf{a_{1-\gamma}^{0}} \ \mathbf{a_{\gamma}^{-} \ 0} \\ \dots \ \mathbf{0} \ \mathbf{a_{\gamma}^{+} \ 2\mathbf{a_{1-\gamma}^{0} \ \mathbf{a_{\gamma}^{-} \ 0} } \end{bmatrix}$$

$$\mathbf{a_1} = \begin{bmatrix} \mathbf{a_{\gamma}^{+} \ 2\mathbf{a_{1-\gamma}^{0}} \ \mathbf{a_{\gamma}^{-} \ 0} \ \mathbf{0} \ \dots \\ \mathbf{0} \ \mathbf{a_{\gamma}^{+} \ 2\mathbf{a_{1-\gamma}^{0} \ \mathbf{a_{\gamma}^{-} \ 0} \ \dots \\ \vdots \ \vdots \ \ddots \ \ddots \ \ddots \ \ddots \end{bmatrix}$$

$$\mathbf{a_2} = \begin{bmatrix} & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & \dots & \mathbf{0} & \mathbf{a}_{\gamma}^+ & 2\mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- & \mathbf{0} \\ & \dots & \mathbf{0} & \mathbf{0} & \mathbf{a}_{\gamma}^+ & 2\mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- \end{bmatrix}$$

We now have to include the boundary conditions (17) and (18). The linear interpolation (21) for i = 0, N, respectively, yields

$$\begin{split} v_0'(k+1) &= -\left(\gamma h_1(k) + (1-\gamma)h_0(k)\right) + \\ &+ \left(\gamma v_1'(k) + (1-\gamma)v_0'(k)\right) - \\ &- \Delta t'\left(\gamma \tilde{p}_1'(k) + (1-\gamma)\tilde{p}_0'(k)\right) + h_0(k+1) \\ v_N'(k+1) &= \left(\gamma h_{N-1}(k) + (1-\gamma)h_N(k)\right) + \\ &+ \left(\gamma v_{N-1}'(k) + (1-\gamma)v_N'(k)\right) - \\ &- \Delta t'\left(\gamma \tilde{p}_{N-1}'(k) + (1-\gamma)\tilde{p}_N'(k)\right) - h_N(k+1) \\ \rho_N'(k+1) &= \frac{1}{Fr^2} \left(h_N(k+1) - \\ &- \left(\gamma Ma^N h_{N-1}(k) + (1-\gamma Ma^N)h_N(k)\right)\right) + \\ &+ \gamma Ma^N \rho_{N-1}'(k) + (1-\gamma Ma^N)\rho_N'(k) \end{split}$$

Applying similar procedures for the inlet and outlet and defining the following vectors:

• the linear part state vector $\mathbf{x} \in \Re^{2(N+1)}$ by

$$\mathbf{x} \equiv \begin{bmatrix} \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_N \end{bmatrix}$$
(26)

• the nonlinear part state vector $\mathbf{\tilde{p}}' \in \Re^{(N+1)}$ by

 $\tilde{\mathbf{p}}' \equiv \begin{bmatrix} \tilde{p}'_0 \\ \vdots \\ \tilde{p}'_N \end{bmatrix}$ (27)

• the input vector $\mathbf{\breve{u}} \in \Re^2$ by ¹

$$\mathbf{\breve{u}} \equiv \begin{bmatrix} h_0 \\ h_N \\ \rho'_0 \end{bmatrix} \tag{28}$$

• the output vector $\mathbf{y} \in \Re^4$ by

$$\equiv \begin{bmatrix} h_0 \\ v'_0 \\ \rho'_0 \\ h_N \\ v'_N \\ \rho'_N \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_N \end{bmatrix}$$
(29)

¹The input vector \mathbf{u} will be defined later.

у

we can complete (25), and obtain the *nonlinear state-space form*

$$\mathbf{x}(k+1) = \mathbf{A}_{\mathbf{L}} \mathbf{x}(k) - \mathbf{A}_{\mathbf{NL}} \, \tilde{\mathbf{p}}'(\mathbf{k}) + \mathbf{B} \, \breve{\mathbf{u}}(k+1)$$

$$\mathbf{y}(k+1) = \mathbf{C} \, \mathbf{x}(k+1)$$
(30)

with the $(3(N+1)) \times (3(N+1))$ linear part system matrix

$$\mathbf{A}_{\mathbf{L}} = \frac{1}{2} \begin{bmatrix} \mathbf{A}_{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\mathbf{4}} \end{bmatrix}$$
(31)

where because of the lack of space the matrix is written in a distributed form and

$$\mathbf{A_3} = \begin{bmatrix} 2 \, \mathbf{A}_{1-\gamma}^0 & 2 \, \mathbf{A}_{\gamma}^0 & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{A}_{\gamma}^{1+} & 2 \, \mathbf{A}_{1-\gamma}^{1,I} & \mathbf{A}_{\gamma}^{1-} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{A}_{\gamma}^{2+} & 2 \, \mathbf{A}_{1-\gamma}^{2,I} & \mathbf{A}_{\gamma}^{2-} & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

the $(3(N+1)) \times (N+1)$ nonlinear part system matrix

$$\mathbf{A_{NL}} = \frac{\Delta t'}{2} \begin{bmatrix} \mathbf{a_3} & \mathbf{0} \\ \mathbf{0} & \mathbf{a_4} \end{bmatrix}$$
$$\mathbf{a_3} = \begin{bmatrix} 2 \mathbf{a}_{1-\gamma}^0 & 2 \mathbf{a}_{\gamma}^0 & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{a}_{\gamma}^+ & 2 \mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{a}_{\gamma}^+ & 2 \mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$
$$\mathbf{a_4} = \begin{bmatrix} \ddots & \ddots & \ddots & \vdots & \vdots \\ \dots & \mathbf{a}_{\gamma}^+ & 2 \mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- & \mathbf{0} \\ \dots & \mathbf{0} & \mathbf{a}_{\gamma}^+ & 2 \mathbf{a}_{1-\gamma}^0 & \mathbf{a}_{\gamma}^- \\ \dots & \mathbf{0} & \mathbf{0} & 2 \mathbf{a}_{\gamma}^N & 2 \mathbf{a}_{1-\gamma}^N \end{bmatrix}$$

the $(3(N+1)) \times 3$ control matrix

$$\breve{\mathbf{B}} = \frac{1}{2} \begin{bmatrix} 2 \mathbf{U}_{\mathbf{0}} \\ \mathbf{0} \\ \vdots \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ 2 \mathbf{U}_{\mathbf{N}} \end{bmatrix}$$

and the $6 \times (3 (N + 1))$ measurement matrix

$$\mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

with the 3x3 identity matrix **I**.

The last thing to do is to investigate the $\tilde{\mathbf{p}}'$ in (30). Using (9), we obtain for the *i*th element of $\tilde{\mathbf{p}}'$

$$\tilde{p}'_{i} = \sin \alpha_{i} + R\lambda v'_{i} |v'_{i}| = \sin \alpha_{i} + R\lambda |\mathbf{x}'_{i}| \begin{bmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}_{i}$$
$$= \sin \alpha_{i} + R\lambda |\mathbf{x}'_{i}| \mathbf{A}^{0} \mathbf{x}_{i}$$

with

$$\mathbf{A}^{0} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and after vectorization

$$\tilde{\mathbf{p}}' = \sin\alpha + R\,\lambda\,|\mathbf{X}'|\,\mathbf{A_{NL}^{0}}\,\mathbf{x}$$
(32)

with the (N + 1) × 1 *inclination vector* and the $(3 (N + 1)) \times (N + 1)$ *state matrix*

$$\mathbf{sin}\boldsymbol{\alpha} \equiv \begin{bmatrix} \sin \alpha_0 \\ \vdots \\ \sin \alpha_N \end{bmatrix} \qquad \mathbf{X} \equiv \begin{bmatrix} \mathbf{x}_0 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{x}_N \end{bmatrix}$$

respectively and further the $(3\ (N+1))\times 3\ (N+1)$ non-linear part system matrix

$$\mathbf{A_{NL}^{0}} \equiv \begin{bmatrix} \mathbf{A}^{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{A}^{0} \end{bmatrix}$$

Defining

$$\begin{split} \Delta \mathbf{A}[\mathbf{x}(k)] &\equiv R \,\lambda \,\mathbf{A_{NL}} \, |\mathbf{X}'(k)| \,\mathbf{A_{NL}^{0}} \\ \mathbf{A}[\mathbf{x}(k)] &\equiv \mathbf{A_{L}} - \Delta \mathbf{A}[\mathbf{x}(k)] \\ \mathbf{u} &\equiv \mathbf{\breve{B}} \,\mathbf{\breve{u}} - \mathbf{A_{NL}} \,\mathbf{sin} \boldsymbol{\alpha} = \mathbf{\breve{B}} \,\mathbf{\breve{u}} - \mathbf{u}_{\alpha} \end{split}$$

with

$$\mathbf{u}_{lpha}\equiv \mathbf{A_{NL}}\,\mathbf{sin}oldsymbol{lpha}$$

we finally obtain

$$\mathbf{x}(k+1) = (\mathbf{A}_{\mathbf{L}} - \Delta \mathbf{A}[\mathbf{x}(k)]) \mathbf{x}(k) + \mathbf{u}(k+1)$$

= $\mathbf{A}[\mathbf{x}(k)] \mathbf{x}(k) + \mathbf{u}(k+1)$ (33)
 $\mathbf{y}(k+1) = \mathbf{C} \mathbf{x}(k+1)$

This is a slightly nonlinear system due to $\Delta \mathbf{A}[\mathbf{x}(k)]$.

5 Simulation and verification of the model

The models were verified on a real pipeline with the following data: length of the pipeline $L_p = 9854$ m, velocity of sound a = 1116 m/s, friction coefficient $\lambda = 0.0172$, gravity constant g = 9.81 m/s², diameter D = 0.2065 m, inclination $\alpha = -0.00256$ rad.

The fluid transients were generated for the experimental verification by closing a shunt valve at the outlet of the



Figure 2: Pressures at the inlet (upper line) and the outlet (lower line)

pipeline at t = 170s and opening it again at t = 503s. This leads to the rapid pressure increase and decrease, shown in Figure 2, causing fluid transients. There is no controller for the flow rate or the pump pressure. The flow transients are shown in Figures 3 and 4 for the inlet and outlet of the pipeline, respectively. The responses of the state-space model to the pressure changes shown in Figure 2 are depicted by a solid line whereas the measured velocities are indicated by a dashed line. Good agreement between the model's responses and the measured data can be observed. The average error in the steady-state responses is 0.0859%. These small deviations in the throttled flow velocities at the outlet of the pipeline are probably due to using a constant value for λ , which should, according to Colebrook, be dependent on the flow velocity.

The conformance during the transients is very good; the mean square errors during the transients at the inlet and outlet are 6.14e-4 and 5.26e-4, respectively, which is comparable to the variances of the corresponding measurements (2.06e-4 and 6.64e-5). This verifies the derived statespace model of the pipeline, especially its static and dynamic responses.

6 Conclusion

The state-space model of a pipeline was derived using the assumptions of compressible, viscous and adiabatic onedimensional flow. The model, in the form of partial differential equations, was solved numerically by the method of characteristics and the resulting model was represented in the state-space form. Normalized quantities were introduced in order to reduce the numerical sensitivity. The model was verified using data measured on a real pipeline. The good agreement between the measured and the simulated data, especially during the transients, validates the derived model. An improvement to the model with respect to the non-linear dependency of λ on the flow velocity (the



Figure 3: Flow velocity at the inlet of the pipeline



Figure 4: Flow velocity at the outlet of the pipeline

formula of Colebrook) is planned for the near future.

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