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The system theory of heat flux. One input problem

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Abstract: -This paper gives a detailed system theoretical treatment of the heat flux theory in the linear heat conduction based on the Laplace transformation method. By restricting the investigations to the simplest geometrical structures occurring in the practice, the authors prove the criteria guaranteeing the existence of the convolutional representations of the heat flux depending on the known temperature.

Keywords: -Heat conduction, heat flux, Laplace transform, district heat supply, differential and integral operators

1 Introduction

Let us consider the linear heat equation in one space variable x

$$\Delta\vartheta(x,t) = \frac{1}{\kappa}\frac{\partial\vartheta(x,t)}{\partial t}, \qquad t > 0,\ x \in I, \tag{1}$$

where I denotes a finite, or a semi-infinite interval, Δ, κ denote the Laplace operator, and the thermal diffusivity, respectively. We shall assume in the sequel that the initial condition equals to zero

$$\vartheta(x,0) = 0, \tag{2}$$

for every inner point of the interval I. The unicity of the solution of (1) is guaranteed by the initial condition (2) and the boundary conditions. However, from the view-point of the theory and applications of the heat flux, the knowledge of the boundary conditions is generally superfluous and uninteresting.

The main problems of the theory of the heat flux can be formulated as follows. Let an arbitrary linear heat conduction process be given satisfying (1), (2), moreover let x, x_0, x_1, x_2, $(x_1 \neq x_2)$ be arbitrary points of I.

Problem I. What is the connection between the heat flux at the point x, and the temperature at the point x_0 on the time interval $0<t<\infty$, provided that the temperature determines the heat flux uniquely. One input – problem.

Problem II. What is the connection between the heat flux at the point x, and the temperatures at the points x_1, x_2, on the time interval $0<t<\infty$, provided that the temperatures determine the heat flux uniquely. Two – inputs – problem.

Problem III. What is the connection between the heat flux at the point x, and the temperature and heat flux at the points x_1, x_2, respectively, on the time interval $0<t<\infty$, provided that the latter determine the previous quantity uniquely. Two – inputs – problem.

We shall call Problem II the pure problem and Problem III the mixed problem of theory of the heat flux, respectively.

The heat flux is by definition:

$$j(x,t) = -K\frac{\partial\vartheta(x,t)}{\partial x}, \tag{3}$$

where K denotes the thermal conductivity. In the sequel we assume that the quantities κ, K are constants not depending on position, time and temperature.

By restricting ourselves to the simplest geometrical structures, **we shall solve the above problem I.** by the application of the Laplace transformation method using a system theoretical treatment. We assume that the functions under consideration are Laplace transformable and that the time functions, which are obtained by the inverse Laplace transformation, describe the concrete heat flux problem.

2 The solution of problem I.

By transforming (1), and taking into account (2), we obtain

$$\Delta\Theta(x,s) - \frac{s}{\kappa}\Theta(x,s) = 0, \tag{4}$$

where

$$\Theta(x,s) = \int_0^\infty \vartheta(x,t)e^{-st}dt\,. \tag{5}$$

Let $\Theta_1(x,s), \Theta_2(x,s)$ two linearly independent solutions of (5). The general solution is of the for

$$\Theta(x,s) = \alpha(s)\Theta_1(x,s) + \beta(s)\Theta_2(x,s), \tag{6}$$

where $\alpha(s), \beta(s)$ are arbitrary functions of the complex variable s.

We have by (6)

$$\Theta(x_0,s) = \alpha(s)\Theta_1(x_0,s) + \beta(s)\Theta_2(x_0,s), \tag{7}$$

$$\Theta'(x,s) = \alpha(s)\Theta_1'(x,s) + \beta(s)\Theta_2'(x,s), \tag{8}$$

(' denotes the derivative $\frac{d}{dx}$). It is easily seen that the quantity $\Theta(x_0 s)$ does not determine uniquely the value of $\Theta'(x,s)$ in general. In this paragraph we shall restrict ourselves to such structures, where only one of the linearly independent solutions of (5) should be considered. Let us denote this solution by f (x, s). So we have

$$\begin{aligned} &\Theta(x,s) = \alpha(s)f(x,s), \\ &\Theta(x_0,s) = \alpha(s)f(x_0,s), \\ &\Theta'(x,s) = \alpha(s)f'(x,s), \end{aligned} \tag{9}$$

and

$$\Theta'(x,s) = \Theta(x_0,s)\frac{f'(x,s)}{f(x_0,s)}, \tag{10}$$

$$-K\Theta'(x,s) = -K\Theta(x_0,s)\frac{f'(x,s)}{f(x_0,s)}. \tag{11}$$

By introducing the notations

$$H(x,x_0,s) = -K\frac{f'(x,s)}{f(x_0,s)} \tag{12}$$

(11) can be written as

$$J(x,s) = \Theta(x_0,s)H(x,x_0,s)\,. \tag{13}$$

The equation (13) describes a transmission system, the scheme of which is illustrated in fig. 1.

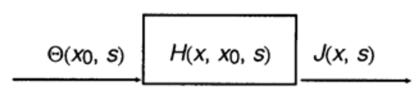

Fig. 1 Transmission system model of the heat flux

This scheme symbolises the connection between the input (temperature) and the output (heat flux). The function H (x, x_0, s) being the quotient of the Laplace transforms of the output and input, is called the transfer function of the system. (see Fodor [2], Kaplan [3]).

It follows from the convolution theorem of the Laplace transformation, that if there exists the time function h (x, x_0, t) having the Laplace transform H (x, x_0, s) then by inverting (1,6), the heat flux can be written in the form of the convolution integral.

$$j(x,t) = \int_0^t \vartheta(x_0, t-\tau)h(x,x_0,\tau)d\tau, \tag{14}$$

having a great practical importance.

If the transfer function has no inverse in the time domain, then as we shall see in special cases, the function

$$\frac{1}{H(x,x_0,s)}$$

will be invertable. Denoting its inverse by h*(x, x_0, t), (13) is equivalent to the following convolution type integral equation of the first kind.

$$\int_0^t j(x,\tau)h^* x,x_0,t-\tau)d\tau = \vartheta(x_0,t). \qquad (15)$$

We cannot give the explicit form of the solution of (15) in general, since (15) cannot be reduced to an integral equation of the second kind, the solution of which is represented by Neumann series. However, in special cases we give the explicit solution of (15), but not in the form of a convolution type integral.

So the knowledge of the criteria deciding about the two cases above is very important in the practice. We shall prove these simple criteria for the following geometrical structures

- The semi-infinite rod (or wall) $I=(0,\infty)$.
- The region bounded internally by a sphere $I=[a,\infty)$, a>0.
- The sphere $I=(0,a)$, a>0.
- The region bounded internally by an infinite circular cylinder $I=[a,\infty)$, a>0.
- The infinite circular cylinder $I=[0,a)$, a>0.

2.1 The semi-infinite rod (or wall)

$$f(x,s) = e^{-\sqrt{\frac{s}{k}}x}, \qquad (16)$$

(see Fodor [2], Doetsch [7]). We have by (12), that

$$H(x,x_0,s) = K\sqrt{\frac{s}{k}}e^{-\sqrt{\frac{s}{k}}(x-x_0)} \qquad (17)$$

holds. Let x> xo. Then

$$h(x,x_0,t) = \frac{K}{2t\sqrt{\pi kt}}\left[\frac{(x-x_0)^2}{2kt}-1\right]e^{-\frac{(x-x_0)^2}{4kt}}, \qquad (18)$$

see for example (Ditkin - Prudnikov [4]). It follows from (14), that

$$j(x,t) = \frac{K}{2}\int_0^t \vartheta(x_0,t-\tau)\frac{1}{\tau\sqrt{\pi k\tau}}\left[\frac{(x-x_0)^2}{2k\tau}-1\right]e^{-\frac{(x-x_0)^2}{4kt}d\tau}. \qquad (19)$$

Let $x \le x_0$. Then the inverse Laplace transform of (17) does not exists, since

$$\lim_{s\to\infty}\sqrt{\frac{s}{k}}e^{-\sqrt{\frac{s}{k}}(x-x_0)} \neq 0 \qquad \text{(see [3]).} \qquad (20)$$

The inverse of the function $\frac{1}{H(x,x_0 s)}$ exists. We have by [4]

$$h^*(x,x_0,t) = \frac{1}{K}\sqrt{\frac{k}{\pi t}}\exp\left[-\frac{(x-x_0)^2}{4kt}\right], \qquad (21)$$

and taking into account (15) the following integral equation will be obtained

$$\int_0^t j(x,\tau)\frac{\exp\left[-\frac{(x-x_0)^2}{4k(t-\tau)}\right]}{\sqrt{t-\tau}}d\tau = K\sqrt{\frac{\pi}{k}}\vartheta(x_0,t). \qquad (22)$$

The kernel of (8) and its derivatives of arbitrary high order vanish for t=0, if $x<x_0$. So (22) cannot be reduced to an integral equation of the second kind and the explicit solution of (22) cannot be given. (see Fenyő-Stolle [5]) For $x=x_0$ we obtain from (13), (15):

$$J(x_0,s) = K\sqrt{\frac{s}{k}}\Theta(x_0,s) = K\frac{1}{\sqrt{ks}}s\Theta(x_0,s). \qquad (23)$$

Let x_0 be an arbitrary inner point of the domain I. Since $\vartheta(x_0,t)$ is absolutely continuous and $\vartheta(x_0,0)=0$, by inverting (23) we obtain

$$j(x_0,t) = \frac{K}{\sqrt{\pi k}}\int_0^t \frac{\partial\vartheta(x_0,\tau)}{\partial\tau}\cdot\frac{1}{\sqrt{t-\tau}}d\tau. \qquad (24)$$

The convolution occurring on the right-hand side of (24) contains the derivative of the temperature (not the temperature itself). So we rewrite this formula as follows. Let $0<\varepsilon<t$

An integration by parts gives,

$$\int_0^t \frac{\frac{\partial \vartheta(x_0,\tau)}{\partial \tau}}{\sqrt{t-\tau}}d\tau = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\frac{\frac{\partial \vartheta(x_0,\alpha\tau)}{\partial \tau}}{\sqrt{t-\tau}}d\tau = \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)}{\sqrt{\varepsilon}} - \frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right] =$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)}{\sqrt{\varepsilon}} - \frac{1}{2}\vartheta(x_0,t)\int_0^{t-\varepsilon}\frac{d\tau}{(t-\tau)^{\frac{3}{2}}} + \frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right] =$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)-\vartheta(x_0,t)}{\sqrt{\varepsilon}} + \frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right] =$$

$$= \frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau. \qquad (25)$$

Finally we have

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right]. \qquad (26)$$

In other form

$$j(x,t) = \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x,t)}{\partial t^{\frac{1}{2}}}, \quad t>0. \qquad (27)$$

Let now $x_0=0$ and let $\vartheta(0,t)$ absolutely continuous. Then by inverting the formula (23)

$$j(0,t) = \frac{K}{\sqrt{\pi\kappa}}\int_0^t \frac{\partial \vartheta(0,\tau}{\partial \tau}\frac{1}{\sqrt{t-\tau}}d\tau + \frac{K\vartheta(0,0)}{\sqrt{\pi\kappa t}} \qquad (28)$$

is obtained. Analogously to the previous case a simple calculation shows that $\frac{K\vartheta(0,0)}{\sqrt{\pi\kappa t}}$ falls out and

$$j(0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(0,t)-\vartheta(0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right]. \qquad (29)$$

In other form

$$j(x,t) = \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x,t)}{\partial t^{\frac{1}{2}}}, \quad t>0. \qquad (30)$$

This formula can be found in Oldham - Spanier [6], the conditions of the validity of the above formula however, are not given in [6].

2.2 The region bounded internally by a sphere

$$f(x,s) = \frac{e^{-\sqrt{\frac{s}{\kappa}}x}}{x} \qquad (31)$$

(see [1]) and we obtain

$$H(x,x_0,s) = \frac{Kx_0\sqrt{\frac{s}{\kappa}}}{x}e^{-\sqrt{\frac{s}{\kappa}}(x-x_0)} + \frac{Kx_0}{x^2}e^{-\sqrt{\frac{s}{\kappa}}(x-x_0)}. \qquad (32)$$

We get from [4], that

$$h(x,x_0,t) = \frac{Kx_0}{2t\sqrt{\pi\kappa t}\,x}\left[\frac{(x-x_0)^2}{2\kappa t} - 1\right]e^{-\frac{(x-X_0)^2}{4\kappa t}} + \frac{Kx_0(x-x_0)}{2x^2t\sqrt{\kappa\pi t}}\exp\left[-\frac{(x-x_0)^2}{4\kappa t}\right], \ x>x_0, \qquad (33)$$

and

$$h^*(x,x_0,t) = \frac{\sqrt{\kappa}x}{Kx_0\sqrt{\pi t}}\exp\left[-\frac{(x-x_0)^2}{4\kappa t}\right] - \frac{2\kappa}{K\sqrt{\pi}x_0}\exp\left(\frac{x_0}{x} - 1 + \frac{\kappa t}{x^2}\right)\int_{\frac{x_0-x}{2\sqrt{\kappa t}}+\frac{\sqrt{\kappa t}}{x}}^{\infty} e^{-u^2}du, \ x_0 \ge x \qquad (34)$$

hold. By the aid of (34) we obtain the corresponding integral equation related to the heat flux.

We have by (32), that

$$H(x_0,x_0,s) = K\sqrt{\frac{s}{\kappa}} + \frac{K}{x_0} \qquad (35)$$

holds. By taking into account (13)

$$J(x_0,s) = K\sqrt{\frac{s}{\kappa}}\Theta(x_0 s) + \frac{K}{x_0}\Theta(x_0,s) =$$
$$K\frac{s}{\sqrt{\kappa s}}\Theta(x_0 s) + \frac{K}{x_0}\Theta(x_0 s) \tag{36}$$

will be obtained. Finally, by an inverse Laplace transformation we get the formula

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}}\int_0^t \frac{\frac{\partial\vartheta(x_0,\tau)}{\partial\tau}}{\sqrt{t-\tau}}d\tau + \frac{K}{x_0}\vartheta(x_0,t) \tag{37}$$

for every inner point x_0 of the domain.

Moreover, by (24), (26)

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right] +$$
$$\frac{K}{x_0}\vartheta(x_0,t) \tag{38}$$

holds. Similarly we obtain, that if (a,t) is absolutely continuous, then the validity of (38) holds true also for the limit point x_0=a.

2.3 The sphere

$$f(x,s) = \frac{\operatorname{sh}\sqrt{\frac{s}{\kappa}}x}{x}, \tag{39}$$

and

$$H(x,x_0,s) = \frac{Kx_0\left(\operatorname{sh}\sqrt{\frac{s}{\kappa}}x - \sqrt{\frac{s}{\kappa}}x\operatorname{ch}\sqrt{\frac{s}{\kappa}}x\right)}{x^2\operatorname{sh}\sqrt{\frac{s}{\kappa}}x_0}. \tag{40}$$

The case x=0 can be excluded from the discussion, since the heat flux equals to zero for x=0. (40) has no inverse for $x \geq x_0$ since $\lim_{s\to\infty} H \neq 0$. The inverse of (40) exists for $x<x_0$. Applying

$$\left(1-e^{-2\sqrt{\frac{s}{\kappa}}x_0}\right)^{-1} = \sum_{\nu=0}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}, \tag{41}$$

we have

$$H(x,x_0,s) = K\frac{x_0}{x^2}\left[\left(1-\sqrt{\frac{s}{\kappa}}x\right)\sum_{\nu=0}^{\infty}\exp\left(-\sqrt{\frac{s}{\kappa}}\left[(1+2\nu)x_0 - x\right]\right) - \right.$$
$$\left. -\left(1+\sqrt{\frac{s}{\kappa}}x\right)\sum_{\nu=0}^{\infty}\exp\left(-\sqrt{\frac{s}{\kappa}}\left[(1+2\nu)x_0 + x\right]\right)\right]. \tag{42}$$

By the application of a theorem of Doetsch [7] (page 206) it is easily seen that the term by term inversion (42) is admissible.

So applying [4] we get

$$h(x,x_0,t) = \frac{-Kx_0}{2xt\sqrt{\pi\kappa t}}\left[\sum_{\nu=0}^{\infty}\left(\frac{\left[(1+2\nu)x_0 - x\right]^2}{2\kappa t} - 1\right)e^{-\frac{\left[(1+2\nu)x_0-x\right]^2}{4\kappa t}}\right.$$
$$\left. + \sum_{\nu=0}^{\infty}\left(\frac{\left[(1+2\nu)x_0 + x\right]}{2\kappa t} - 1\right)e^{-\frac{\left[(1+2\nu)x_0+x\right]^2}{4\kappa t}}\right]$$
$$-\frac{Kx_0}{2x^2 t\sqrt{\pi\kappa t}}\left[\sum_{\nu=0}^{\infty}((1+2\nu)x_0 + x)e^{-\frac{\left[(1+2\nu)x_0+x\right]^2}{4\kappa t}} - \right.$$
$$\left. -\sum_{\nu=0}^{\infty}((1+2\nu)x_0 - x)e^{-\frac{\left[(1+2\nu)x_0-x\right]^2}{4\kappa t}}\right]. \tag{43}$$

For $x \geq x_0$ we apply Heaviside's Expansion Theorem and obtain

$$h^*(x,x_0,t) = \frac{-2\kappa}{Kx_0}\sum_{n=0}^{\infty}\frac{\sin\alpha_n\frac{x_0}{x}}{\sin\alpha_n}e^{-\frac{\alpha_n^2}{x^2}\kappa t}, t \geq 0, x_0 \neq 0, \tag{44}$$

(see Carslaw-Jaeger [1]). I here α_n denotes the n-th positive root of the equation

$$\alpha = \operatorname{tg}\alpha. \tag{45}$$

Important special cases:

$$x = x_0 \neq 0, \quad h^*(x_0,x_0,t) = \frac{-2\kappa}{Kx_0}\sum_{n=1}^{\infty}e^{-\frac{\alpha_n^2}{x^2}\kappa t}, \tag{46}$$

t>0

$$x_0 = 0, \quad h^*(x,0,t) = -\frac{2\kappa}{Kx}\sum_{n=1}^{\infty}\frac{\alpha_n e^{-\frac{\alpha_n^2}{x^2}\kappa t}}{\sin\alpha_n}. \tag{47}$$

Let $x=x_0 \neq 0$. The explicit form of the heat flux can be obtained in the following way. By (40) we have

$$H(x_0,x_0,s)=\frac{K}{x_0}-K\sqrt{\frac{s}{k}}\frac{\mathrm{ch}\sqrt{\frac{s}{\kappa}}x_0}{\mathrm{sh}\sqrt{\frac{s}{\kappa}}x_0} \tag{48}$$

and

$$H(x_0,x_0,s)=\frac{K}{x_0}-K\sqrt{\frac{s}{\kappa}}\frac{1+e^{-2\sqrt{\frac{s}{\kappa}}x_0}}{1-e^{-2\sqrt{\frac{s}{\kappa}}x_0}}=$$

$$=\frac{K}{x_0}-K\sqrt{\frac{s}{\kappa}}\left(1+e^{-2\sqrt{\frac{s}{\kappa}}x_0}\right)\sum_{\nu=0}^{\infty}e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}=$$

$$=\frac{K}{x_0}-K\sqrt{\frac{s}{\kappa}}-K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty}e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}-K\sqrt{\frac{s}{\kappa}}\sum_{\nu=0}^{\infty}e^{-2(\nu+1)\sqrt{\frac{s}{\kappa}}x_0}=$$

$$=\frac{K}{x_0}-K\sqrt{\frac{s}{\kappa}}-2K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty}e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}\quad. \tag{49}$$

(13) gives

$$J(x_0,s)=\frac{K}{x_0}\Theta(x_0,s)-K\sqrt{\frac{s}{\kappa}}\Theta(x_0,s)-2K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty}e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0\Theta(x_0,s)}. \tag{50}$$

Taking into account (36), (38) and applying [4], we obtain by the application of a Laplace invertation the formula

$$j(x_0,t)=\frac{K\vartheta(x_0,t)}{x_0}-\frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0,t)}{\sqrt{t}}+\frac{1}{2}\int_0^t\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}d\tau\right]-$$

$$-\vartheta(x_0,t)*\frac{K}{t}\sum_{\nu=1}^{\infty}\frac{1}{\sqrt{\pi\kappa t}}e^{-\frac{\nu^2x_0^2}{\kappa t}}\left(\frac{2\nu^2x_0^2}{\kappa t}-1\right) \tag{51}$$

provided that either x_0 is an inner point of I or x_0=a and $\vartheta(a,t)$ is absolutely continuous. (We denoted here the convolution by *

In other form

$$j(x_0,t)=\frac{K\vartheta(x_0,t)}{x_0}-\frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x,t)}{\partial t^{\frac{1}{2}}}-\vartheta(x_0,t)*\frac{K}{t}\sum_{\nu=1}^{\infty}\frac{1}{\sqrt{\pi\kappa t}}e^{-\frac{\nu^2x_0^2}{\kappa t}\left(\frac{2\nu^2x_0^2}{\kappa t}-1\right)}. \tag{52}$$

2.4 The region bounded internally by an infinite circular cylinder

We have

$$f(x,s)=K_0\left(\sqrt{\frac{s}{\kappa}}x\right), \tag{53}$$

where K_0 denotes the modified Bessel function of the second kind of order zero. So it is

$$H(x,x_0,s)=K\sqrt{\frac{s}{\kappa}}\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)}, \tag{54}$$

where K_1 denotes the modified first order Bessel function of the second kind. From the asymptotic expansion of the Bessel functions it follows that

$$\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)}\sim\sqrt{\frac{x_0}{x}}e^{\sqrt{\frac{s}{\kappa}}(x_0-x)}, \tag{55}$$

holds for $s\to\infty$.

If $x\le x_0$ then $\lim\limits_{s\to\infty}=\infty$ and (54) has no inverse Laplace transform. We show that (54) has the inverse for $x>x_0$ and we determine this.

Eq. (54) has the following properties for $x>x_0$. Let $\gamma>0$ be arbitrary. Then

1. $H(x,x_0,s)$ is analytic in the half plane Re $s\ge\gamma$.
2. $\int_{\gamma-i\infty}^{\gamma+i\infty}\left|H(x,x_0,s)\right|ds<\infty$. (56)
3. In the half plane Re $s\ge\gamma$ $H(x,x_0,s)$ tends uniformly to zero with respect to arg s if $|s|\to\infty$.

Then an easy application of a theorem in Doetsch [3] (page 236) or Berg [8] (page 27) shows that $H(x,x_0,s)$ has its inverse in the above half plane and

$$h(x,x_0,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}H(x,x_0,s)e^{st}ds, \tag{57}$$

moreover, $h(x, x_0, t)$ is a continuous function of t and $h(x, x_0, 0)$. Applying the Fourier-Mellin inversion integral

$$h(x, x_0, t) = \frac{K}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \sqrt{\frac{\lambda}{\kappa}} \frac{K_1\left(\sqrt{\frac{\lambda}{\kappa}} x\right)}{K_0\left(\sqrt{\frac{\lambda}{\kappa}} x_0\right)} d\lambda , \quad (58)$$

The integrand has a branch point in $\lambda = 0$, so we choose the following contour on the complex plane (see fig. 2.).

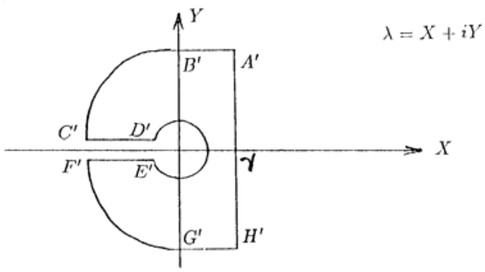

Fig. 2 Applying of Fourier-Mellin inversion integral on the complex plane

By omitting the details, the evaluation of the inversion integral above gives the following results.

$$h(x, x_0, t) = \frac{2\kappa K}{\pi} \int_0^\infty e^{-\kappa u^2 t} u^2$$

$$\times \frac{J_1(xu) Y_0(x_0 u) - Y_1(xu) J_0(x_0 u)}{J_0^2(x_0 u) + Y_0^2(x_0 u)} du \text{ , t>0,}$$

$$h(x, x_0, t) = 0 . \quad (59)$$

Here J_0, Y_0 denote the nullth order Bessel functions of the first and second kind, J_1, Y_1 denote the first order Bessel functions of the first and second kind, respectively.

Finally let $x \le x_0$. If $x < x_0$, then the above properties 1,2,3 are satisfied for the function $\frac{1}{H}$

However, the inverse of $\frac{1}{H}$ also exists for $x = x_0$ (see the procedure in Carslaw-Jaeger [1] page 388).

The following results are obtained:

$$h^{\otimes}(x, x_0, t) = \frac{2\kappa}{K\pi} \int_0^\infty e^{-\kappa u^2 t}$$

$$\frac{J_1(xu) Y_0(x_0 u) - Y_1(xu) J_0(x_0 u)}{J_1^2(xu) + Y_1^2(xu)} du \text{ , t>0,} \quad (60)$$

$$h^{\otimes}(x, x_0, 0) = 0 . \quad (61)$$

For $x=x_0$

$$h^{\otimes}(x_0, x_0, 0) = \infty \quad (62)$$

$$h^{\otimes}(x_0, x_0, t) =$$

$$\frac{4\kappa}{K\pi^2 x_0} \int_0^\infty \frac{e^{-\kappa u^2 t} du}{u\left[J_1^2(x_0 u) + Y_1^2(x_0 u)\right]} \text{, t>0,} \quad (63)$$

which follows from, (60) by the application of the relation

$$J_1(z) Y_0(z) - J_0(z) Y_1(z) = \frac{2}{\pi z} . \quad (64)$$

Goldstein [10] proves, that the inverse of

$$s^{\alpha} K_{\mu}\left(\sqrt{s}\lambda\right) \quad (65)$$

exists and can be represented by the aid of Whittaker functions. So, for $\alpha = \frac{1}{2}$, $\mu = 1$, our result can be considered as a generalisation of [10].

2.5 The infinite circular cylinder

We have

$$H(x, x_0, s) = -K \sqrt{\frac{s}{\kappa}} \frac{I_1\left(\sqrt{\frac{s}{\kappa}} x\right)}{I_0\left(\sqrt{\frac{s}{\kappa}} x\right)}, \quad (66)$$

where I_0, I_1 denote the modified nullth, and first order Bessel functions of the first kind, respectively. By the application of the inversion formula we obtain the following:

Let $x<x_0$, then

$$h(x, x_0, t) = \frac{2\kappa K}{x_0} \sum_{n=1}^{\infty} \alpha_n^2 e^{-\kappa \alpha_n^2 t} \frac{J_1(\alpha_n x)}{J_1(\alpha_n x_0)}, \text{ t>0,} \quad (67)$$

$$h\left(x, x_0, 0\right) = 0, \tag{68}$$

where α_n denote the positive roots of the equation

$$J_0\left(\alpha x_0\right) = 0. \tag{69}$$

Let x > x_0, then

$$h^*\left(x, x_0 t\right) = -\frac{2\kappa}{Kx}\left(1 + \sum_{n=1}^{\infty} \frac{J_0\left(\beta_n x_0\right)}{J_0\left(\beta_n x\right)} e^{-\kappa\beta_n^2 t}\right), \ t>0, \tag{70}$$

$$h^{\otimes}\left(x, x_0, 0\right) = 0\,. \tag{71}$$

Let x = x_0, then

$$h^{\otimes}\left(x, x_0, 0\right) = -\infty\,, \tag{72}$$

$$h^*\left(x_0, x_0 t\right) = -\frac{2\kappa}{Kx_0}\left(1 + \sum_{n=1}^{\infty} e^{-\kappa\beta_n^2 t}\right), \ t>0, \tag{73}$$

where β_n denote the positive roots of the equation

$$J_1\left(\beta x\right) = 0\,. \tag{74}$$

The following statement holds:
Statement. Let us consider the cases A, B, D. The heat flux can be represented as a convolution integral is and only if x > x_0. For $x \le x_0$ the heat flux satisfied a convolution type integral equation of the first kind. Let us consider the cases C, E. The heat flux can be represented as a convolution integral if and only if $x < x_0$. For $x \ge x_0$, the heat flux satisfies a convolution type integral equation of the first kind.

Moreover, if $x = x_0$, then the solutions of the corresponding integral equations can be given in explicit forms in the cases A, B, C provided that the point x_0 is either an inner point of the domain *I*, or is the limit point of *I*, where the temperature is absolutely continuous.

Remarks. 1.) In the discussion of the case of a region bounded internally by an infinite circular cylinder, we obtained

$$J(x,s) = K\sqrt{\frac{s}{\kappa}}\,\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)}\,\Theta(x_0 s). \tag{75}$$

Garbai [11] gets an integral equation for the heat flux as follows. Since

$$K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right) J(x,s) = K\sqrt{\frac{s}{\kappa}} K_1\left(\sqrt{\frac{s}{\kappa}}x\right) \Theta(x_0, s). \tag{76}$$

By inverting both sides of this equation and applying the convolution theorem of the Laplace transformation, the integral equation.

$$\int_0^t j(x,\tau)\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{t-\tau}d\tau = \frac{Kx}{2\kappa}\int_0^t \vartheta(x_0,\tau)\,\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{(t-\tau)^2}d\tau, \tag{77}$$

is obtained. (77) holds for every pair (x, x_0) and its kernel function is more simpler than the corresponding ones given by (60), (63). The disadvantage of (77) lies in the fact that there occurs a convolution on the right-hand side of it.

It is surprising that (77) has no analogue in the case of the infinite circular cylinder.

2.) Our results can be well applied in the practice, if the heat flux has a convolutional representation. Then by measuring the temperature in discrete time intervals, the convolution can be evaluated by known numerical methods. On the other hand, there are numerical methods also for solving convolutional integral equations, we shall deal with these methods in a following paper.

3.) The condition of the absolute continuity of the temperature in the limit points is a sufficient condition, which holds in the practice. It is however not necessary.

3 Harmonic processes

It follows from the theory of the linear systems, that the results related to the harmonic processes are simple consequences of our results discussed above (see [2], [3]). If we substitute $s = i\omega$ in (13), where ω is the angular frequency of the harmonic oscillation, and replace the Laplace transforms by the notations $\overline{\Theta}(x, i\omega)$, $\bar{J}(x, i\omega)$ then the equation

$$\bar{J}(x, i\omega) = H(x, x_0, i\omega)\overline{\Theta}(x_0, i\omega) \tag{78}$$

will be obtained. $\bar{\Theta}(x, i\omega)$, $\bar{J}(x, i\omega)$ are the complex amplitudes of the harmonic input (temperature), and harmonic output (heat flux), respectively. $H(x, x_0, i\omega)$ is the complex transfer characteristics of the system. Equation (78) describes this transmission system, the scheme of which is illustrated in figure 3.

$$\xrightarrow{\bar{\Theta}(x_0, i\omega)} \boxed{H(x, x_0, i\omega)} \xrightarrow{\bar{J}(x, i\omega)}$$

Fig. 3. Transmission system model of heat flux for harmonic processes

Practically, the most important quantity is the amplitude characteristics. $A(x, x_0, \omega)$ being the absolute value of the transfer characteristics $H(x, x_0, i\omega)$.

The amplitude characteristics describes the frequence dependency of the quotient of the amplitudes of the output and input (reasonance curve). The results are presented on the basic geometrical structures in the article [15].

4 Conclusion

The system theoretical treatment given in this paper presents a new approach of the heat flux problem and the results can be well applied in the engineering practice. In those simpler cases when the heat flux can he given explicitly lay convolutional or other type of integrals, these integrals may be computed by the application of well-known numerical techniques. On the other hand, in the cases when the determination of the heat flux is reduced to the solution of convolutional integral equations, simple approximate methods are available in the mathematical literature.

Symbols:

c	specific heat,
j	heat flux,
s	complex variable,
t	time,
K	heat conduction factor,
ρ	density,
ϑ	temperature,
T	time
κ	thermal diffusivity
J	Laplace transformed
Δ	Laplace operator

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