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Transient heat conduction in composite systems

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Abstract: - This paper presents a new method for the determination of the transient heat conducting processes occurring in composite systems of different heat conducting layers. The problems of this type are treated by the aid of the Laplace transformation in the literature [3], [4]. The classical inversion methods are no applicable for N > 2 [1]. This new method is based on the Papoulis-Berg inversion method [2], [5]. The most advantage of the method lies in the fact that it is applicable for linear composite heat conducting systems of any number of layers, for constant heat conduction coefficient.

Key-Words: - Transient heat conduction, composite systems, different heat conducting layers, Laplace transformation, Papoulis-Berg inversion method, heat flux.

1 Introduction

In technical practice one often encounters transient heat conduction problems in composite systems consisting of solid layers, e.g., walls of buildings, walls of furnaces, heat insulation of pipelines, etc.

Investigating these involves solving the simultaneous system of differential equations

$$\Delta\vartheta_i = \frac{1}{\kappa_i}\frac{\partial\vartheta_i}{\partial t}\text{, i=1, 2, …, N.} \tag{1}$$

under prescribed initial and boundary conditions, where:

ϑ_i temperature in the i-th heat conductor; Δ the Laplace-operator; N the total number of layers in the system; t time; K_i the thermal diffusivity of the i-th substance.

If we assume that the temperature depends only one space coordinate, x, in addition to time, and that the temperature of the system at the time t=0 was zero, then problems of the type indicated above can be redefined mathematically in the following way:

From among the solutions of the system of heat equations

$$\Delta\vartheta_i(x,t) = \frac{1}{\kappa_i}\frac{\partial\vartheta_i}{\partial t},\quad l_{i-1} < x < l_i < \infty,$$

$$\text{i=1, 2, …, N}, \cancel{t \geq 0} \tag{2}$$

we are to determine the one that satisfies the zero initial condition at the time t=0 and the continuity conditions at the separating surfaces (or points) with co-ordinates l_i

$$\vartheta_i(l_i,t)=\vartheta_{i+1}(l_i,t),$$
$$-K_i\left.\frac{\partial\vartheta_i(x,t)}{\partial x}\right|_{x=l_i}=-K_{i+1}\left.\frac{\partial\vartheta_{i+1}(x,t)}{\partial x}\right|_{x=l_i}$$
i=1, 2, …, N-1,

where K_i is the thermal conductivity of the i-th conductor and the boundary conditions applicable to the system [3].

The simplest composite systems in practice are the following:

I. Composite plane walls of N layers.
II . Composite hollow spheres of N layers .
III. Composite hollow circular cylinders of N layers.

The following figure indicates the domain x for each of the three cases.

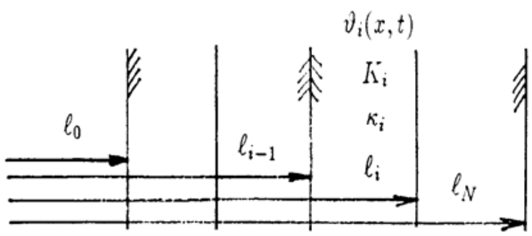

Fig. 1

We offer solutions for these three basic problems in our paper. In the first structure we choose $l_o=0$, but in structures II. and III., $l_o=a>0$, as we do not consider the process in the interior of the system in the domain ($0\le x\le l_o$). Temperatures will only be considered in the points l_i dividing the individual conductors, at the beginning of the system (l_o) and at its end (l_N), which are important special cases in technical practice, and whose investigation, as we are going to see, makes a clear system-theoretic approach possible.

Problems of the types shown above are discussed in the literature using the method of Laplace transformation [3], [4]. However, even in the case N=2, the Laplace transforms of the temperatures become complex expressions whose inversion - with the exception of some special cases - poses insurmountable difficulties. This fact was also stated by Jaeger in [1].

In this paper we are going to present the Papoulis-Berg inversion method [2], [5] in the system theoretic investigation of heat conduction problems of the type shown above. The greatest advantage of this method is that it is fairly easy to apply for arbitrarily large values of N, so the number of heat conducting layers with different physical properties is not limited.

The introduced method provides an interesting alternative on describing of the transient heat conduction and temperature field in multi-layered composite bodies.

2 Determining the Laplace transform of temperatures

After applying the Laplace transformation to equation (1), considering the zero initial condition, we obtain the following transformed expression

$$\Delta\Theta_i(x,s)=q_i^2\Theta_i(x,s),\quad q_i\sqrt{\frac{s}{\kappa_i}}\ ,$$
i=1, 2, …, N. (3)

In the special structures investigated by us:

I. Composite plane walls of N layers:

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2}=q_i^2\Theta_i(x,s). \qquad (4)$$

II. Composite hollow spheres of N layers:

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2}+\frac{2}{x}\frac{\partial\Theta_i(x,s)}{\partial x}=q_i^2\Theta_i(x,s). \qquad (5)$$

III. Composite hollow cylinders of N layers:

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2}+\frac{1}{x}\frac{\partial\Theta_i(x,s)}{\partial x}=q_i^2\Theta_i(x,s). \qquad (6)$$

[3].

Let use introduce the Laplace transforms of the heat fluxes

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$$j_i(x,t) = -K_i \frac{\partial \Theta_i(x,t)}{\partial x},$$

$$J_i(x,s) = \int_0^{\infty} j_i(x,t) e^{-st} dt, \ i=1, 2, \ldots, N. \quad (7)$$

For the sake of simplicity, let us use the notations $\vartheta(l_i, t), j(l_i, t), \Theta(l_i), J(l_i)$ for the temperatures, heat fluxes and their Laplace transforms in the points l_i. It can be demonstrated that the relationship between the transforms of the temperatures and heat fluxes occurring at the input with coordinate l_{i-1} and exit with coordinate l_i of the i-th heat conductor can be established using the so-called transfer matrix

$$\underline{\underline{A_i(s)}} = \begin{pmatrix} A_i(s) & B_i(s) \\ C_i(s) & D_i(s) \end{pmatrix},$$

of the i-th layer in the following form:

$$\begin{pmatrix} \Theta(l_i) \\ J(l_i) \end{pmatrix} = \begin{pmatrix} A_i(s) & B_i(s) \\ C_i(s) & D_i(s) \end{pmatrix} \begin{pmatrix} \Theta(l_{i-1}) \\ J(l_{i-1}) \end{pmatrix}$$

$$i=1, 2, \ldots, N. \quad (8)$$

For structure I. the entries of the transfer matrix can be fond in [3]. For structures II. and III., we have computed the values, and we will get back to them later.

Let us now consider the system consisting of N heat conducting layers. Then the following matrix relationship prevails between the temperatures and heat fluxes at the input of the system and at the exit of the i-th heat conductor:

$$\begin{pmatrix} \Theta(l_i) \\ J(l_i) \end{pmatrix} = \begin{pmatrix} \overline{A}_i(s) & \overline{B}_i(s) \\ \overline{C}_i(s) & \overline{D}_i(s) \end{pmatrix} \begin{pmatrix} \Theta(l_0) \\ J(l_0) \end{pmatrix}, \quad (9)$$

where

$$\underline{\underline{H_i(s)}} = \begin{pmatrix} \overline{A}_i(s) & \overline{B}_i(s) \\ \overline{C}_i(s) & \overline{D}_i(s) \end{pmatrix} = \prod_{j=0}^{i-1} A_{i-j}(s),$$

$$i=1, 2, \ldots, N, \quad (10)$$

and $\underline{\underline{H_1(s)}} = A_1(s)$, and in particular, for i=N

$$\begin{pmatrix} \Theta(l_N) \\ J(l_N) \end{pmatrix} = \begin{pmatrix} \overline{A}_N(s) & \overline{B}_N(s) \\ \overline{C}_N(s) & \overline{D}_N(s) \end{pmatrix} \begin{pmatrix} \Theta(l_0) \\ J(l_0) \end{pmatrix}, \quad (11)$$

which describes the operator relationship between the input and exit of the system of heat conductors, and which can be schematically represented as a linear transmission system as follows:

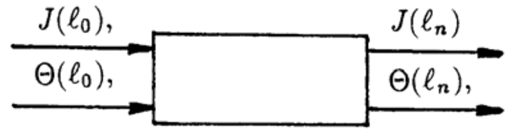

Fig. 2

It is apparent that the combined transfer matrix of the two systems is equal to the product of the two transfer matrices. Two boundary conditions must be given in order to solve the problem. On of the two pertains to the beginning (input) of the system of heat conductors, the other to the end (output). If two of the operators $\Theta(l_0), \Theta(l_N), J(l_0), J(l_N)$, are known, than the other two can be determined from (11).

Now we are going to write the Laplace transforms of the temperature we are most interested in far the most important boundary condition occurring in technical practice.

1. Temperature is given at both ends of the system. Then

$$\Theta(l_i) = \Theta(l_0) \frac{\overline{A}_i \overline{B}_N - \overline{B}_i \overline{A}_N}{\overline{B}_N} + \Theta(l_N) \frac{\overline{B}_i}{\overline{B}_N},$$

$$i=1, 2, \ldots, N. \quad (12)$$

2. Heat flux is given at both ends of the system. Then from (1.9)

$$\Theta(l_0) = -J(l_0) \frac{\overline{D}_N}{\overline{C}_n} + \frac{J(l_N)}{\overline{C}_N},$$

$$\Theta(l_i) = J(l_0) \frac{\overline{B}_i \overline{C}_N - \overline{A}_i \overline{D}_N}{\overline{C}_N} + J(l_N) \frac{\overline{A}_i}{\overline{C}_N},$$

$$i=1, 2, \ldots, N. \quad (13)$$

3. Temperature is given at l_0, the input of the system, and the heat flux is given at l_N, the output.

Similarly to the preceding cases,

$$\Theta(l_i)=\Theta(l_0)\frac{\overline{A}_i\overline{D}_N-\overline{B}_i\overline{C}_N}{\overline{D}_N}+J(l_N)\frac{\overline{B}_i}{\overline{D}_N},$$

i=1, 2, …, N. (14)

The case involving the roles of l_0 and l_N reversed can be written analogously.

4. Temperature is given at l_0, the input of the system, and the heat flux is proportional to the temperature at l_N, the output.

The latter boundary condition in the form of a Laplace transform is (I):

$$J(l_N)=\gamma\Theta(l_N). \quad (15)$$

Then

$$\Theta(l_i)=\Theta(l_0)\frac{\gamma\left(\overline{B}_i\overline{A}_N-\overline{A}_i\overline{B}_N\right)+\overline{A}_i\overline{D}_N-\overline{B}_i\overline{C}_N}{\overline{D}_N-\gamma\overline{B}_N},$$

i=1, 2, …, N. (16)

5. The heat flux is given at l_0, the input of the system, and the heat flux us proportional to the temperature at (l_N), the output.

In this case

$$\Theta(l_0)=J(l_0)\frac{\overline{D}_N-\gamma\overline{B}_N}{\gamma\overline{A}_N-\overline{C}_N}, \quad (17)$$

$$\Theta(l_i)=J(l_0)\frac{\gamma\left(\overline{A}_i\overline{B}_N-\overline{B}_i\overline{A}_N\right)-\overline{A}_i\overline{D}_N+\overline{B}_i\overline{C}_N}{\overline{C}_N-\gamma\overline{A}_N}. \quad (18)$$

Let us now write the entries of the transition matrix $\underline{\underline{A_i(s)}}$ for structures I, II and III:

I. Composite plane walls of N layers:
Then, according to Carslaw-Jaeger [1]

$$\varphi_i^{(1)}=\mathrm{ch}\,q_il_i, \qquad \varphi_i^{(2)}=\mathrm{sh}\,q_il_i,$$

$$A_i(s)=D_i(s)=\mathrm{ch}(l_i-l_{i-1})q_i,$$

$$B_i(s)=-\frac{1}{K_iq_i}\mathrm{sh}(l_i-l_{i-1})q_i,$$

$$C_i(s)=-K_iq_i\,\mathrm{sh}(l_i-l_{i-1})q_i,$$

i=1, 2, …, N. (19)

and $\det\underline{\underline{A_i(s)}}=1$,

which implies $\det\underline{\underline{H_i(s)}}=1$.

II. Composite hollow spheres of N layers:

$$\varphi_i^{(1)}=\frac{\mathrm{ch}\,q_il_i}{l_i}, \qquad \varphi_i^{(2)}=\frac{\mathrm{sh}\,q_il_i}{l_i}.$$

Computing the entries of the transfer matrix, we obtain

$$A_i(s)=\frac{l_{i-1}}{l_i}\mathrm{ch}q_i(l_i-l_{i-1})+\frac{1}{q_il_i}\mathrm{ch}q_i(l_i-l_{i-1}),$$

We assume that the surrounding temperature is zero $\vartheta(x,t)=0$, if $x<l_0$, or $x>l_N$.

$$B_i(s)=-\frac{l_{i-1}}{K_iq_il_i}shq_i(l_i-l_{i-1}),$$

$$C_i(s)=-K_i\frac{l_i-l_{i-1}}{l_i^2}chq_i(l_i-l_{i-1})+$$

$$+K_i\left(\frac{1}{q_il_i^2}-\frac{l_{i-1}}{l_i}q_i\right)shq_i(l_i-l_{i-1}),$$

$$D_i(s)=\frac{l_{i-1}}{l_i}chq_i(l_i-l_{i-1})-\frac{l_{i-1}}{q_il_i^2}shq_i(l_i-l_{i-1}),$$

$$\det\underline{\underline{A_i(s)}}=\frac{l_{i-1}^2}{l_i^2},$$

thus

$$\det\underline{\underline{H_i(s)}}=\frac{l_0^2}{l_i^2}, \qquad i=1, 2, \dots, N. \quad (20)$$

III. Composite hollow circular cylinders of N layers:
In this case

$$\varphi_i^{(1)}=I_0(q_il_i), \qquad \varphi_i^{(2)}=K_0(q_il_i).$$

where I_0 and K_0 are the Bessel functions of the first and second kind, of order 0. The entries of the transfer matrix are found to be

$$A_i(s)=l_{i-1}q_i\left[I_0(q_il_i)K_1(q_il_{i-1})+K_0(q_il_i)I_1(q_il_{i-1})\right],$$

$$B_i(s)=\frac{l_{i-1}}{K_i}\left[I_0(q_i l_{i-1})K_0(q_i l_i)-K_0(q_i l_{i-1})I_0(q_i l_i)\right],$$

$$C_i(s)=K_i l_{i-1} q_i^2\left[K_1(q_i l_i)I_1(q_i l_{i-1})-I_1(q_i l_i)K_1(q_i l_{i-1})\right],$$

$$D_i(s)=l_{i-1} q_i\left[I_0(q_i l_{i-1})K_1(q_i l_i)+K_0(q_i l_{i-1})I_1(q_i l_i)\right], \tag{21}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind, of order one. Furthermore,

$$\det \underline{\underline{A_i}} = \frac{l_{i-1}}{l_i}, \qquad i=1, 2, \ldots, N,$$

consequently $\det \underline{\underline{H_i}}(s) = \frac{l_0}{l_i}$.

Now the Laplace transforms of the temperatures at the points l_i can be written explicitly with the help of the formulae obtained thus far. Unfortunately, owing to the products of the function matrices in (20), the entries of the matrix $\underline{\underline{H_i}}(s)$ and, consequently, the formulae generating transforms of the temperatures are so complex even for N>2 that inverting them with classical methods is practically impossible.

The following Papoulis-Berg inversion method fixes this problem. Next, we are going to explain the method briefly, then we are going to invert the Laplace transforms of the temperatures using the method.

3 Applying the Papoulis-Berg inversion method to solving the heat conduction problem

Let f(t) be a continuous function of bounded variation defined for $t \geq 0$ and Laplace transformable, f(0)=0 and let

$$F(s)=\int_0^\infty f(t)e^{-st}dt, \qquad s > s_0. \tag{22}$$

Papoulis [4] obtains the inverse of F(s) without applying the Fourier-Mellin inversion integral in the following way.

Let $\sigma > s_0$ be an arbitrary positive number and let us substitute

$$e^{-\sigma t} = \cos(x) \tag{23}$$

into (22). Then by denoting f(t)=g(x) , we obtain

$$F(s)=\frac{1}{\sigma}\int_0^{\pi/2} g(x)\cos^{\frac{s}{\sigma}-1} x \cdot \sin x dx. \tag{24}$$

Let

$$s=(2\nu+1)\sigma, \qquad \nu = 1,2,3,\ldots,$$

then

$$F[(2\nu+1)\sigma]=\frac{1}{\sigma}\int_0^{\pi/2} g(x)\cos^{2\nu} x \cdot \sin x dx. \tag{25}$$

Let us now define the function g(x) to the domain $\frac{\pi}{2} < x \leq \pi$ by way of the formula $g(x)=g(\pi - x)$. From the theory of Fourier series it is known that the function f(t) can be expanded into a Fourier series of the form

$$f(t)=\sum_{n=0}^\infty c_n \sin(2n+1)x = $$

$$=\sum_{n=0}^\infty c_n \sin\left[(2n+1)\arccos\left(e^{-\sigma t}\right)\right], \tag{26}$$

$$c_n=\frac{4}{\pi}\int_0^{\pi/2} g(x)\sin(2n+1)x dx. \tag{27}$$

The following formula is easily verified by mathematical induction:

$$\sin(2n+1)x=\sin x\sum_{\nu=0}^{n}(-1)^{n-\nu}4^\nu\binom{n+\nu}{n-\nu}\cos^{2\nu}x. \tag{28}$$

Substituting this into (27), taking (25) into account, we obtain

$$c_n=\frac{\sigma}{\pi}\sum_{\nu=0}^{n}(-1)^{n-\nu}4^{\nu+1}\binom{n+\nu}{n-\nu}F[\sigma(2\nu+1)]. \tag{29}$$

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Thereby the inverse Laplace transform of F(s) obtained. Indeed, the coefficients c_n are easy to compute from (29) once F(s) is known, so f(t) can be determined from (26) for an arbitrary t. For fixed values of t, the speed of convergence of (26) depends on the choice of σ, of course.

Considering that, Berg [5] has suggested that, in order to accelerate the convergence of (26), one should not use a constant value of σ, but rather - given the asymptotic relationship between f(t) and F(s) - the product σt should be chosen to be constant (i.e., for small values of t, σ should be large and vice versa). From (23) it is apparent that the value of x is constant this way. Choosing this constant to be the midpoint of the basic interval $\left(0,\frac{\pi}{2}\right)$, $x=\frac{\pi}{4}$ the following formulae obtain:

$$\sigma=\frac{\log 2}{2t}, \qquad \sin(2n+1)\frac{\pi}{4}=\frac{(-1)^{\left[\frac{n}{2}\right]}}{\sqrt{2}},$$

and

$$f(t)=\frac{\sqrt{2}\log 2}{\pi t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\times$$
$$\times\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}F\left[\frac{\log 2}{2t}(2\nu+1)\right], \quad (30)$$

here $\left[\frac{n}{2}\right]$ is the least integer of n and t>0. (see Berg [5].)

It is apparent that the time t takes the role of the complex variable s in the Laplace transform F(s) in the form (30) of the inverse transform f(t), meaning that it is to be evaluated numerically for arbitrary fixed t.

Let us now apply formula (30) for inverting the Laplace transforms of the temperatures determined in the preceding section. For the sake of better understanding, let us introduce the following notations:

$$\Theta(l_N,s)=\Theta(l_N), \Theta(l_0,s)=\Theta(l_0), J(l_N,s)=$$
$$=J(l_N), J(l_0,s)=J(l_0),$$
$$A_i^{(\nu)}=A_i^{(\nu)}(t)=\overline{A}_i\left[\frac{\log 2}{2t}(1+2\nu)\right],$$
$$B_i^{(\nu)}=B_i^{(\nu)}(t)=\overline{B}_i\left[\frac{\log 2}{2t}(1+2\nu)\right],$$
$$C_i^{(\nu)}=C_i^{(\nu)}(t)=\overline{C}_i\left[\frac{\log 2}{2t}(1+2\nu)\right],$$
$$D_i^{(\nu)}=D_i^{(\nu)}(t)=\overline{D}_i\left[\frac{\log 2}{2t}(1+2\nu)\right]. \quad (31)$$

Then, on the basis of (30), we obtain the following formulae for the individual inverses.

The inverse of (12):

$$\vartheta(l_i,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times$$
$$\times\left\{\Theta\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\cdot\frac{A_i^{(\nu)}B_N^{(\nu)}-B_i^{(\nu)}A_N^{(\nu)}}{B_N^{(\nu)}}+\right.$$
$$\left.+\Theta\left[l_N,\frac{\log 2}{2t}(2\nu+1)\right]\frac{B_i^{(\nu)}}{B_N^{(\nu)}}\right\}. \quad (32)$$

The inverse of (13):

$$\vartheta(l_i,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times\cdot$$
$$\times\left\{J\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\cdot\frac{B_i^{(\nu)}C_N^{(\nu)}-A_i^{(\nu)}D_N^{(\nu)}}{C_N^{(\nu)}}+\right.$$
$$\left.\cdot+J\left[l_N,\frac{\log 2}{2t}(2\nu+1)\right]\frac{A_i^{(\nu)}}{C_N^{(\nu)}}\right\}. \quad (33)$$

The inverse of (14):

$$\vartheta(l_i,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times\cdot$$
$$\times\left\{\Theta\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\cdot\frac{A_i^{(\nu)}D_N^{(\nu)}-B_i^{(\nu)}C_N^{(\nu)}}{D_N^{(\nu)}}+\right.$$
$$\left.+J\left[l_N,\frac{\log 2}{2t}(2\nu+1)\right]\frac{B_i^{(\nu)}}{D_N^{(\nu)}}\right\}. \quad (34)$$

The inverse of (15):

$$\vartheta(l_i,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times$$
$$\times\Theta\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\times$$

$$\times\frac{\gamma\left(\mathrm{B}_{\mathrm{i}}^{(\nu)}\mathrm{A}_{\mathrm{N}}^{(\nu)}-\mathrm{A}_{\mathrm{i}}^{(\nu)}\mathrm{B}_{\mathrm{N}}^{(\nu)}\right)+\mathrm{A}_{\mathrm{i}}^{(\nu)}\mathrm{D}_{\mathrm{N}}^{(\nu)}-\mathrm{B}_{\mathrm{i}}^{(\nu)}\mathrm{C}_{\mathrm{N}}^{(\nu)}}{\mathrm{D}_{\mathrm{N}}^{(\nu)}-\gamma\mathrm{B}_{\mathrm{N}}^{(\nu)}}. \tag{35}$$

The inverse of (16):

$$\vartheta(l_o,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times$$
$$\times J\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\times\frac{\gamma B_N^{(\nu)}-D_N^{(\nu)}}{C_N^{(\nu)}-\gamma A_N^{(\nu)}}. \tag{36}$$

The inverse of (17):

$$\vartheta(l_i,t)=\frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]}\sum_{\nu=0}^{n}(-1)^{\nu}4^{\nu}\binom{n+\nu}{n-\nu}\times$$
$$\times J\left[l_0,\frac{\log 2}{2t}(2\nu+1)\right]\times$$

$$\times\frac{\gamma\left(\mathrm{A}_{\mathrm{i}}^{(\nu)}\mathrm{B}_{\mathrm{N}}^{(\nu)}-\mathrm{B}_{\mathrm{i}}^{(\nu)}\mathrm{A}_{\mathrm{N}}^{(\nu)}\right)-\mathrm{A}_{\mathrm{i}}^{(\nu)}\mathrm{D}_{\mathrm{N}}^{(\nu)}-\mathrm{B}_{\mathrm{i}}^{(\nu)}\mathrm{C}_{\mathrm{N}}^{(\nu)}}{\mathrm{C}_{\mathrm{N}}^{(\nu)}-\gamma\mathrm{A}_{\mathrm{N}}^{(\nu)}}. \tag{37}$$

From formulae (17), (19) and (22) it is apparent that

$$\begin{pmatrix} A_i^{(\nu)}(t) & B_i^{(\nu)}(t) \\ C_i^{(\nu)}(t) & D_i^{(\nu)}(t) \end{pmatrix}=$$
$$\prod_{j=0}^{i-1}\begin{pmatrix} A_{i-j}\left[\frac{\log 2}{2t}(1+2\nu)\right] & B_{i-j}\left[\frac{\log 2}{2t}(1+2\nu)\right] \\ C_{i-j}\left[\frac{\log 2}{2t}(1+2\nu)\right] & D_{i-j}\left[\frac{\log 2}{2t}(1+2\nu)\right] \end{pmatrix}. \tag{38}$$

where the quantities A_{i-j} , B_{i-j} , C_{i-j} , D_{i-j} in the matrix product on the right-hand side of the equation can be obtained from formulae (18), (19) or (20), depending on the geometric structure being investigated, putting the expression $\frac{\log 2}{2\mathrm{t}}(1+2\nu)$ in the place of s.

Thus, if we wish to compute the temperatures (22), . . . (27) for a fixed time, t> 0, then we must substitute the numerical values of the time t into (28), whereby the multiplication of function matrices is reduced to the multiplication of numeric matrices.

This is the main advantage of the Papoulis-Berg inversion, which is the consequence of the fact that the time t replaces the complex variable s in the Laplace transform of formula (30).

4 Conclusion

In the paper a mathematical model and calculation procedure is presented, which is capable of solving transient heat conduction problems in multi-layer composite systems. It is proved that the procedure makes the calculations more effective and quick and may substitute the finite difference and finite elements methods. The method may be utilised well for the calculation of the transient temperature field in case building structure elements, walls, energetic appliances, insulations, furnace walls and district heating pipelines. The method is capable of solving non-linear heat conduction problems for temperature dependent thermal conductivity.

Symbols:

j	--	heat flux
J	--	Laplace transformed form of the head flux (j)
K	--	thermal conductivity
s	--	complex variable
t	--	time
x	--	space variable
Δ	--	Laplace operator
ϑ	--	temperature
κ	--	thermal diffusivity
τ	--	time

References:

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