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Flow Between Two Almost Rigidly Rotating Spheres

VASSILIOS C. LOUKOPOULOS

Department of Physics
University of Patras
Patras 26500
GREECE

Abstract: **-** The flow of a viscous incompressible fluid contained between two concentric spheres that rotate with almost equal angular speeds is studied. For the numerical solution of the problem, which is described by a coupled and non linear system of PDEs, like the Navier-Stokes equations are, with their appropriate boundary conditions, the stream function-vorticity formulation is adopted and the solution of the problem is obtained developing an efficient numerical technique based on the central finite differences of second order accuracy. Our interest is focused on the secondary flow pattern and the radial and meridional components of velocity. It is found that the flow is bounded from the cylindrical surface that touches the inner sphere.

Keywords: - Rigid rotation, Spherical shell, Finite-difference method.

1 Introduction

The flow in spherical shells is an important problem in fluid dynamics and geophysics. Besides some engineering applications, this simple system represents a parade example for study of structure formation. Furthermore, spherical flow is often used to investigate dynamics of large-scale geophysical and astrophysical motions in planetary interior and atmospheres [1]. In this respect, a wide spherical shell seems generally more relevant as, e.g. the whole mantle thickness of the major terrestrial planets (Earth, Venus, and Mars) is believed to be about half of the outer sphere.

Proudman [2] and Stewartson [3, 4] studied the fluid motion in the absence of heat and of a gravitational field when both spheres rotate with almost equal angular speeds, at large Reynolds numbers of flow. In [2, 3] it was shown that the fluid can be divided into two regions, the one that is outer to the cylinder circumscribing the inner sphere and the other, that is enclosed by this cylinder. The outer portion rotates as a rigid body with the angular velocity of the outer sphere. In the inner portion, centrifugal forces create a secondary circulatory motion (Mager [5]).

Munson and Joseph [6] and Dennis and Singh [7] worked on the same problem considering spheres rotating with different angular velocities. The obtained approximate solutions where valid for $\mathrm{Re} \leq 500$ in [5] and for $\mathrm{Re} \leq 2000$ in [6], where $\mathrm{Re} = \Omega \mathrm{R}^2 / \nu$ is the Reynolds number of the flow. In the previous notation Ω is a typical angular velocity, R is the radius of the outer sphere and ν is the coefficient of the kinematic viscosity of the fluid.

Karahalios [8] studied the problem of two coaxial surfaces of revolution, which rotate about the common axis of symmetry with almost the same angular velocity. At large Reynolds numbers, three singular surfaces were found. One over each surface of revolution and a cylindrical shear layer touching the outer surface. Outside the cylinder the velocity was determined by the velocity distribution over the two boundary layers, while inside the cylinder the fluid was at rest.

Hollerback [9] investigated numerically the flow of an electrically conducting fluid in a differentially rotating spherical shell, in the presence of an imposed magnetic field. For a very weak field the flow was seen to consist of an Ekman layer on the inner and outer spherical boundaries, and a Stewartson layer on the cylinder circumscribing the inner sphere and parallel to the axis of rotation, in agreement with the classical non-magnetic analysis.

In the present work we simulate the flow between two almost rigidly rotating spheres and solve the equations of motion numerically for various values of the Reynolds number. In the numerical treatment we use a finite difference scheme, described by Loukopoulos *et al.* [10] and extended by Loukopoulos [11], of second order accuracy and convert the equations of motion into a system of linear algebraic equations. The solution of this system is obtained by an iterative procedure and the method converges for $\mathrm{Re} \leq 1000$.

2 Equations of Motion

We consider the steady flow of a viscous incompressible fluid between two concentric spheres, rotating about a common diameter with almost equal angular velocities Ω_1 and $\Omega_2 = \Omega_1(1+\varepsilon)\Omega_2$, where $\varepsilon \ll 1$ and subscripts 1 and 2 correspond to the inner and to the outer sphere. Let in addition $r_1^* = R/k$ and $r_2^* = R$ be their radii where $k>1$. The flow is symmetric about the axis of rotation and hence all quantities are independent of the azimuthal angle φ. If (u^*, v^*, w^*) are the velocity components in spherical polar coordinates (r^*, θ, φ), Fig. 1, the equations of motion and continuity are

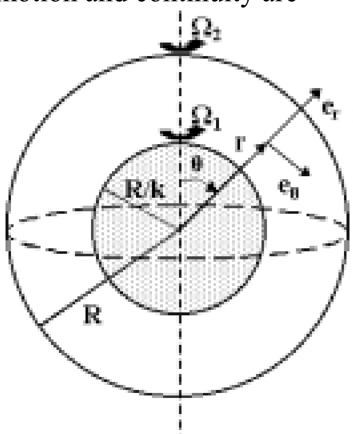


Fig. 1. Spherical annulus.

$$u^*\frac{\partial u^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial u^*}{\partial \theta}-\frac{v^{*2}+w^{*2}}{r^*}=-\frac{1}{\rho}\frac{\partial p}{\partial r^*}+\nu\left(\nabla^{*2}u^*-\frac{2u^*}{r^{*2}}-\frac{2}{r^{*2}}\frac{\partial v^*}{\partial \theta}-\frac{2v^*\cot\theta}{r^{*2}}\right), \quad (1)$$

$$u^*\frac{\partial v^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial v^*}{\partial \theta}+\frac{u^*v^*}{r^*}-\frac{w^{*2}\cot\theta}{r^*}=-\frac{1}{\rho r^*}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^{*2}v^*+\frac{2}{r^{*2}}\frac{\partial u^*}{\partial \theta}-\frac{v^*}{r^{*2}\sin^2\theta}\right), \quad (2)$$

$$u^*\frac{\partial w^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial w^*}{\partial \theta}+\frac{u^*w^*}{r^*}+\frac{v^*w^*\cot\theta}{r^*}=\nu\left(\nabla^{*2}w^*-\frac{w^*}{r^{*2}\sin^2\theta}\right), \quad (3)$$

$$\frac{\partial u^*}{\partial r^*}+\frac{2u^*}{r^*}+\frac{1}{r^*}\frac{\partial v^*}{\partial \theta}+\frac{v^*\cot\theta}{r^*}=0, \quad (4)$$

where ρ is the density and

$$\nabla^{*2}=\frac{\partial^2}{\partial r^{*2}}+\frac{2}{r^*}\frac{\partial}{\partial r^*}+\frac{\cos\theta}{r^{*2}\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^{*2}}\frac{\partial^2}{\partial\theta^2}.$$

The boundary conditions of the problem are:

Inner sphere $r^*=R/k$: $u^*=0$, $v^*=0$.

Outer sphere $r^*=R$: $u^*=0$, $v^*=0$.

Eliminating the pressure terms between (1) and (2), introducing the stream function Ψ^* and the axial velocity function χ^* by the relations

$$u^*=\frac{1}{r^{*2}\sin\theta}\frac{\partial\Psi^*}{\partial\theta},\quad v^*=-\frac{1}{r^*\sin\theta}\frac{\partial\Psi^*}{\partial r^*},\quad w^*=\frac{\chi^*}{r^*\sin\theta} \quad (5)$$

and using the following non dimensional variables

$$\Psi^*=R^3\Omega_0\Psi,\ \chi^*=R^2\Omega_0\chi,\ r^*=Rr, \quad (6)$$

the following equations are obtained:

Stream function equation

$$D^2\Psi=-J, \quad (7)$$

where $J=(r\sin\theta)\zeta$ is a function of the third component of the vorticity $\boldsymbol{\omega}=(\xi, n, \zeta)$.

Vorticity transport equation

$$D^2J=\frac{Re}{r^2\sin\theta}\left[-\frac{\partial(\Psi,J)}{\partial(r,\theta)}+\frac{2J}{r\sin\theta}\left(\frac{\partial\Psi}{\partial r}r\cos\theta-\frac{\partial\Psi}{\partial\theta}\sin\theta\right)\right]-\frac{Re\,2\chi}{r^3\sin^2\theta}\left(\frac{\partial\chi}{\partial r}r\cos\theta-\frac{\partial\chi}{\partial\theta}\sin\theta\right). \quad (8)$$

Axial velocity function equation

$$D^2\chi=-\frac{Re}{r^2\sin\theta}\frac{\partial(\Psi,\chi)}{\partial(r,\theta)}, \quad (9)$$

where $D^2=\frac{\partial^2}{\partial r^2}-\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$,

$\frac{\partial(a,b)}{\partial(x,y)}=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, $Re=\Omega_0R^2/\nu$ is the Reynolds number of the flow and Ω_0 is a typical angular velocity. In the present case we have taken $\Omega_1=\Omega_0$.

The boundary conditions of the flow are

$$\Psi=\frac{\partial\Psi}{\partial\theta}=0 \quad \text{on both boundaries}, \quad (10)$$

$$\chi=\frac{1}{k^2}\sin^2\theta\,\frac{\Omega_1}{\Omega_0} \quad \text{at} \quad r=1/k \ \text{(inner boundary)},$$

$$\chi=\sin^2\theta\,\frac{\Omega_2}{\Omega_0} \quad \text{at} \quad r=1 \ \text{(outer boundary)}, \tag{11}$$

The boundary conditions governing J is found to be of $O(\Delta r^2)$ following similar procedure with Wood [12] and they are

$$J(1/k,\theta)=-\frac{3\Psi(1/k+\Delta r,\theta)}{\Delta r^2}-\frac{J(1/k+\Delta r,\theta)}{2} \quad \text{at} \quad r=1/k,$$

$$J(1,\theta)=-\frac{3\Psi(1-\Delta r,\theta)}{\Delta r^2}-\frac{J(1-\Delta r,\theta)}{2} \quad \text{at} \quad r=1. \tag{12}$$

3 Method of Solution

In the present work equations (7) to (9) have been solved numerically, using a numerical technique described by Loukopoulos *et al.* [10] and extended by Loukopoulos [11], as follows. We construct a grid of mesh points, Fig. 2, with constant radial and angular mesh sizes $\Delta r=(1-1/k)/N$ and $\Delta\theta=\pi/M$ where N and M are integers indicating the density of the grid of mesh points. We next denote all quantities at a typical set of grid points (r_0,θ_0), $(r_0+\Delta r,\theta_0)$, $(r_0,\theta_0+\Delta\theta)$, $(r_0-\Delta r,\theta_0)$ and $(r_0,\theta_0-\Delta\theta)$ by the subscripts 0, 1, 2, 3 and 4 respectively and replace equations (7) to (9) by central differences. We are therefore led to an algebraic system of equations in which the matrix of the coefficients of the unknowns may be diagonally dominant, like (7) is. Following the same steps like Loukopoulos *et al.* [10] we are led to systems of linear equations for all unknowns functions where the matrices of coefficients of unknowns are diagonally dominant so that they can be solved by iterative methods.

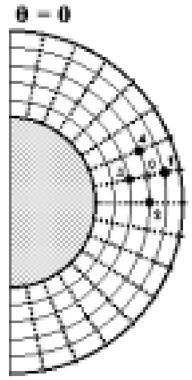


Fig. 2. Mesh points in r – θ plane.

In this way equations (7) to (9) take the form

$$a_1J_1+a_2J_2+a_3J_3+a_4J_4+a_0J_0=-2h^2G(r_0,\theta_0), \tag{13}$$

$$n_1\chi_1+n_2\chi_2+n_3\chi_3+n_4\chi_4+n_0\chi_0=0, \tag{14}$$

$$m_1\Psi_1+m_2\Psi_2+m_3\Psi_3+m_4\Psi_4+m_0\Psi_0=-h^2J_0, \tag{15}$$

with the boundary conditions given by (10), (11) and (12). The coefficients a_i, n_i, m_i (i = 0, 1, 2, 3, 4) of the unknowns J, χ and Ψ and the term $G(r_0,\theta_0)$ are presented in [10].

The linear systems (13), (14) and (15) are sparse and the matrices associated with them are always diagonally dominant since the coefficients of the unknowns satisfy the conditions (Atkinson [13])

$$\sum_{\substack{j=1\\j\neq i}}^{N\times M}|a_{ij}|\le|a_{ii}|, \quad \sum_{\substack{j=1\\j\neq i}}^{N\times M}|n_{ij}|\le|n_{ii}|, \quad \sum_{\substack{j=1\\j\le i}}^{N\times M}|m_{ij}|\le|m_{ii}|,$$

$i=1,2,\ldots,N\times M$, $j=1,2,\ldots,N\times M$.

The solution of all these equations is obtained by employing the under - relaxation method at all internal points of the annular region $1/k<r<1$, $0<\theta<\pi$ subject to boundary conditions that are either known *a priori* (equations (10)–(11)) or are calculated by equations (12). The iterative scheme is repeated until an adopted criterion of accuracy for the variables χ, J and Ψ is satisfied. This is

$$\max\left|1-\frac{g^{(s)}(r,\theta)}{g^{(s+1)}(r,\theta)}\right|\le 5\times10^{-5}.$$

In the previous notation g stands for anyone of the variables J, χ, Ψ and s is the number of iterations. The iterative sequence terminates when all quantities have converged to limits and satisfies the criterion of accuracy.

4 Results and Discussion

In order to study the flow of a viscous incompressible fluid contained between two concentric spheres that rotate with almost equal angular speeds, the system of equations (7)-(9) was solved, under the appropriate boundary conditions. The results have been obtained for radius ratio $k=3,4,5$, angular velocities $\Omega_1=\Omega_0$, $\Omega_2=0.99\Omega_0$ and for Reynolds number $Re\le1000$.

Figure 3 depicts the zero vortex mode at $Re=500$ for radius ration $k=3,4,5$. The figure is a typical two dimensional projection of the flow on the (r,θ) plane at a fixed azimuthal angle φ. Counter-clockwise circulation is shown as solid line, clockwise circulation is shown as dashed line. Owing to the Ekman pumping at the poles, fluid is thrown outward along the rotating inner sphere and forms inward radial jets at the poles and outward radial jets at the equator. The flow is reflection-symmetric about the equator. As the Reynolds number is increased, the flow is restricted in the region defined from the inner sphere and the cylinder touching the inner sphere, as predicted by Proudman [2] for infinite Re.

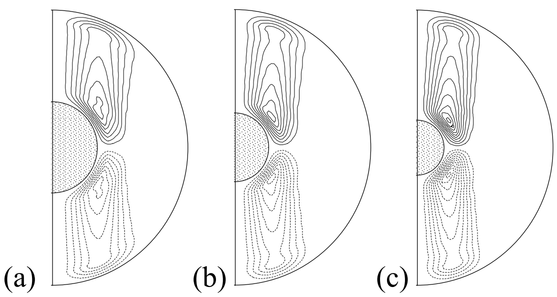


Fig. 3. Streamlines of the motion for $Re = 500$ and (a) $k = 3$; (b) $k = 4$; (c) $k = 5$.

In Figures 4(a) and 5(a) the secondary flow streamline pattern on a meridional plane is presented when both spheres are rotated with almost equal angular velocities for $k=5$, $Re=100$ and $Re=800$ respectively, whereas in Figures 4(b, c, d) and 5(b, c, d) the function J of the vorticity, the radial component of velocity and the meridional component of velocity patterns are shown. In figures 4 and 5 solid lines show positive velocities and dashed lines show negative velocities.

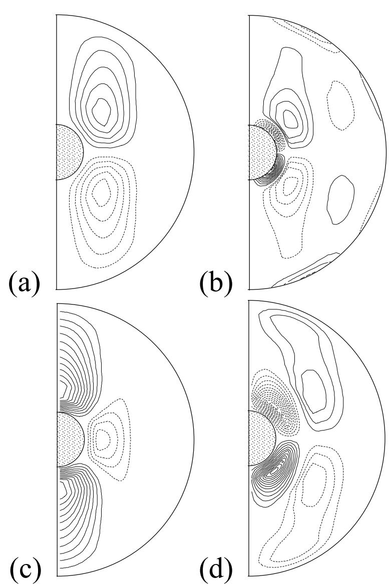


Fig. 4. Streamlines of the motion, function J of the vorticity, radial velocity and meridional velocity for $Re = 100$ and $k = 5$.

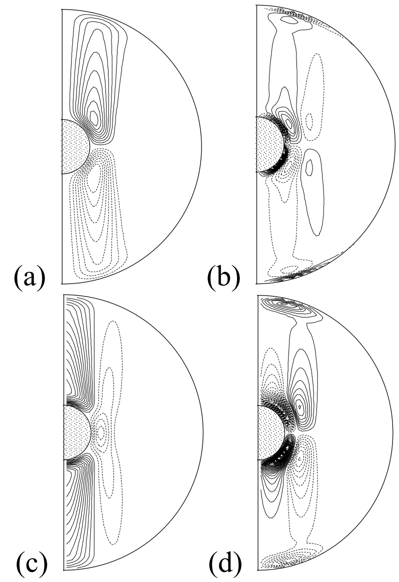
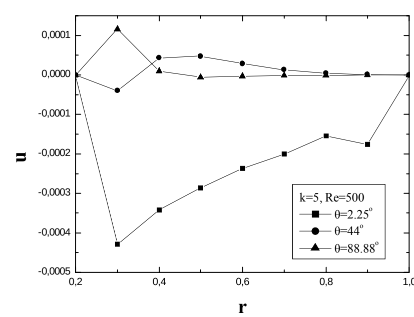


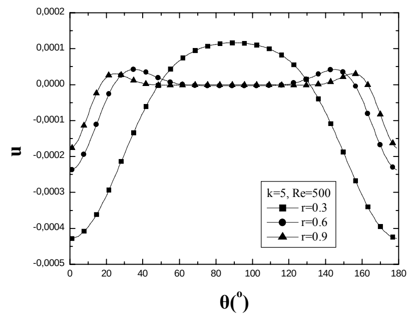
Fig. 5. Streamlines of the motion, function J of the vorticity, radial component of velocity and meridional component of velocity for $Re = 800$ and $k = 5$.

The velocity distributions are calculated from the solutions of the stream function Ψ. The velocity distribution depends on the radial and meridional coordinates and on the mode of the flow. In the equatorial zone, the radial velocity is zero or nearly zero, as expected.

The radial velocity u [(θ = 2.25°, 44°, 88.88°), r] as function of the radial coordinate r is plotted in Figure 6(a) for different values of the angle θ. In Figure 6(b) the radial velocity u [(r = 0.3, 0.6, 0.9), θ] as a function of the meridional coordinate θ is plotted for different values of distance r. Near the poles the flow is directed inward, so that the radial velocity component is negative in this region.



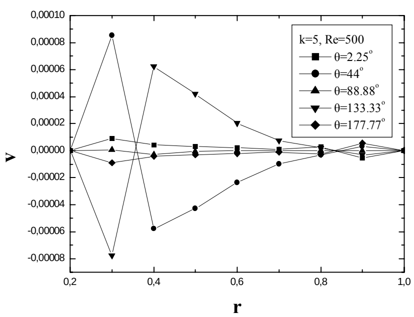
(a)



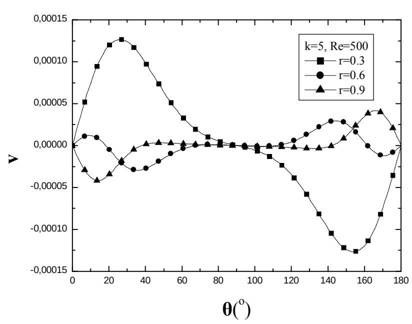
(b)

Fig.6. (a) Variation of radial component of velocity with radial distance at $\theta = 2.25^o$, 44^o, 88.88^o for $Re = 500$ and $k = 5$; (b) Variation of radial component of velocity with meridional coordinate θ at r = 0.3, 0.6, 0.9 for $Re = 500$ and $k = 5$.

The meridional velocity distribution v [($\theta = 2.25^o$, 44^o, 88.88^o), r] as function of the radial coordinate r is plotted in Figure 7(a) for different values of angle θ. The meridional velocity distribution v [(r = 0.3, 0.6, 0.9), θ] as function of the meridional coordinate θ is plotted in Figure 7(b) for different values of distance r. In the equatorial zone, the meridional velocity is zero or nearly zero, as expected.



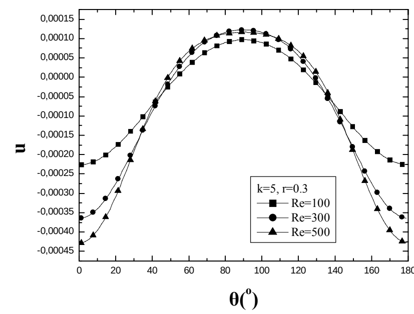
(a)



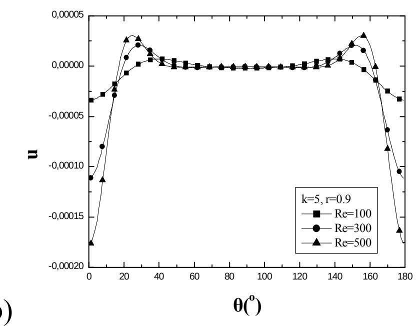
(b)

Fig. 7. (a) Variation of meridional component of velocity with radial distance at $\theta = 2.25^o$, 44^o, 88.88^o, 133.33^o, 177.77^o for $Re = 500$ and $k = 5$; (b) Variation of meridional component of velocity with meridional coordinate θ at r = 0.3, 0.6, 0.9 for $Re = 500$ and $k = 5$.

The radial velocity u near the inner boundary (r = 0.3) as function of the meridional coordinate θ is plotted in Figure 8(a) for different values of Reynolds number (Re = 100, 300, 500). The radial velocity u near the outer boundary (r = 0.9) as function of the meridional coordinate θ is plotted in Figure 8(b) for different values of Reynolds number (Re = 100, 300, 500).



(a)



(b)

Fig. 8. (a) Variation of radial component of velocity with meridional coordinate near the inner boundary ($r = 0.3$) for $Re = 100, 300, 500$ and $k = 5$; (b) Variation of radial component of velocity with meridional coordinate near the outer boundary ($r = 0.9$) for $Re = 100, 300, 500$ and $k = 5$.

The meridional velocity distribution v near the inner boundary ($r = 0.3$) as function of the meridional coordinate θ is plotted in Figure 9(a) for different values of Reynolds number ($Re = 100, 300, 500$). The meridional velocity distribution v near the outer boundary ($r = 0.9$) as function of the meridional coordinate θ is plotted in Figure 9(b) for different values of Reynolds number ($Re = 100, 300, 500$).

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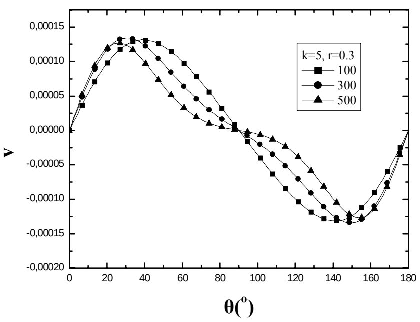
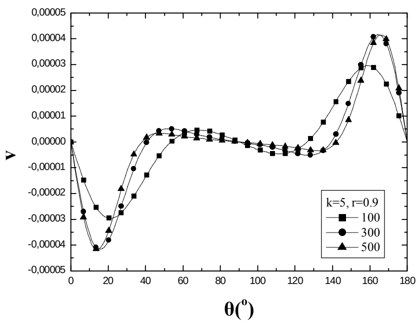


(a)



(b)

Fig. 9. (a) Variation of meridional component of velocity with meridional coordinate near the inner boundary ($r = 0.3$) for $\mathrm{Re} = 100, 300, 500$ and $k = 5$; (b) Variation of meridional component of velocity with meridional coordinate near the outer boundary ($r = 0.9$) for $\mathrm{Re} = 100, 300, 500$ and $k = 5$.

5 Conclusions

It is found that the cylindrical surface that touches the inner sphere (the axis being the axis of rotation) is a singular surface in which velocity gradients are very large. Everywhere outside this cylinder, the fluid rotates as a rigid body with the same angular velocity as the outer sphere. Inside the cylinder, the velocity distribution in the central (inviscid) core of the motion is shown to be determined by the velocity distribution in the boundary layers over the spheres, and explicit solutions are obtained for all these velocity distributions.

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