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Spine Fin Efficiency – A Three Sided Pyramidal Fin of Equilateral Triangular Cross-Sectional Area: Analytical Solution

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Abstract: An analytical, closed form solution is presented for determining the fin efficiency of a spine fin with the geometry of a three sided pyramid of equilateral triangular cross-sectional area (from here on referred to as TSPECA). A solution to this problem is presented in an earlier work and provides a numerical solution using finite differences. This work, however, is completely analytical and makes use of Bessel functions.

Key Words: fin efficiency, finite differences, spine fin, numerical analysis, Bessel function.

1 Introduction

The fin efficiency of a TSPECA is presented in an analytical, closed form equation. Though the solutions to such problems are usually obtained quite easily using numerical methods, an analytical solution is, for the most part, of the highest priority.

Carranza [1] in a previous work provides a numerical solution for the fin efficiency of a TSPECA. The solution is obtained using finite differences. The fin has a variable cross-sectional area. It is assumed that the fin tip is the same temperature as the fluid used for convection. Please refer to Carranza [1] for a complete description of the spine fin geometry and a detailed explanation of all transport equations.

This problem is particularly difficult because the governing equation is a second order differential equation with variable coefficients. If the coefficients were constant, the solution would be readily available. Due to the complexity of the problem, Carranza [1] employs numerical methods.

In contrast, this work explores an analytical solution. A closed form solution is here presented to the problem posed by Carranza [1] – Bessel functions are now employed. Thus, this paper is, in actuality, a companion paper, to the work presented earlier by Carranza [1].

1.1 Literature Review

Holtzapple et al. [2] is the first to make a somewhat rigorous attempt to determine the fin efficiency of a spine fin. He simplifies the problem by assuming the fin is of constant cross-sectional area. This assumption allows the spine fin efficiency to be determined using well established equations readily found in the literature [3]. The straight fin of constant cross-sectional area is a classical problem in the field of heat transfer. Prior to Holtzapple, researchers in the area assumed the spine fin efficiency to be 100%.

Carranza [4] takes the same spine fin geometry used by Holtzapple and this time assumes the spine is pyramidal; not of constant cross-sectional area. The problem requires numerical methods, finite differences, to be solved. The only draw back is that Carranza [4] restricts the scope of the work to the exact geometries used by Holtzapple and does not derive a robust solution for the fin efficiency of a spine fin of variable cross-sectional area. Carranza [4] finds that the spine fin efficiency varies significantly depending on whether the geometry is assumed to be of constant or variable cross-sectional area: as much as 30%. The spine geometry is not a TSPECA.

Kern and Krause [5] present fin efficiencies for a great variety of different fin geometries. Although their work is very extensive, they make no mention of a TSPECA. Thus, to date, Carranza [1] is the only investigator known to study the fin efficiency of a TSPECA.

2 Spine Fin Temperature Profile

After applying all relevant transport and geometrical relationships relating to TSPECA, Carranza [1] derives the following equation:

$$(\mathrm{X}-1)\frac{d^2\Theta}{d\mathrm{X}^2}+2\frac{d\Theta}{d\mathrm{X}}+\alpha\Theta=0 \tag{1}$$

where $\Theta(0) = 1$ and $\Theta(1) = 0$. Equation 1 is the governing equation for TSPECA; and when solved, generates a temperature profile throughout the length of the fin.

The dimensionless variables in Equation 1 are defined in Equations 2 and 3.

$$\Theta=\frac{T-T_\infty}{T_b-T_\infty} \tag{2}$$

$$\mathrm{X}=\frac{x}{b} \tag{3}$$

Equation 1 is solved using Bessel functions; however, first it must be put into the proper form. This is done through a transformation of variables that uses the following relationship:

$$\Theta=\frac{u}{\sqrt{X-1}} \tag{4}$$

The substitution of Equation 4 into Equation 1 yields Equation 5:

$$\begin{aligned}&\frac{d^2u}{dX^2}\sqrt{X-1}+\frac{du}{dX}\frac{1}{\sqrt{X-1}}+\\&\frac{(-1+4\alpha X-4\alpha)u}{4\sqrt{(X-1)^3}}=0\end{aligned} \tag{5}$$

Now, one final substitution is required to reformulate Equation 1 in the form of a Bessel equation:

$$X=1+\frac{y^2}{4\alpha} \tag{6}$$

After Equation 6 is substituted into Equation 5, a Bessel equation of the first order is derived (Bowman [6]).

$$\frac{d^2u}{dy^2}+\frac{1}{y}\frac{du}{dy}+\left(1-\frac{1}{y^2}\right)u=0 \tag{7}$$

Thus, the solution to Equation 1 is given in Equation 8, with the help of Bessel functions:

$$\Theta = C_1 \frac{J_1(2\sqrt{\alpha}\sqrt{X-1})}{\sqrt{X-1}} + C_2 \frac{Y_1(2\sqrt{\alpha}\sqrt{X-1})}{\sqrt{X-1}} \tag{8}$$

It is now time to apply the boundary conditions. If $\Theta(1) = 0$, then C_2 must equal 0 since $Y_1(0)$ = infinity. Applying the second boundary condition, $\Theta(0) = 1$, C_1 is defined.

$$C_1 = \frac{1}{I_1(2\sqrt{\alpha})} \tag{9}$$

Thus, the following is the solution for Equation 1:

$$\Theta = \frac{J_1(2\sqrt{\alpha}\sqrt{X-1})}{I_1(2\sqrt{\alpha})\sqrt{X-1}} \tag{10}$$

3 Spine fin efficiency

Again, using the appropriate transport and geometrical relationships, Carranza [1] derives an equation for determining the fin efficiency of a TSPECA. Please refer to Equation 11 (for more details on Equation 11, see Carranza [1]):

$$\eta = 2\int_0^1 \Theta(1-\mathrm{X})d\mathrm{X} \tag{11}$$

Equation 10 is subsequently substituted into Equation 11. The result yields an integral defined in terms of Bessel functions.

$$\eta = \frac{2}{I_1(2\sqrt{\alpha})} \int_0^1 \frac{J_1(2\sqrt{\alpha}\sqrt{X-1})(1-X)}{\sqrt{X-1}} dX \tag{12}$$

Finally, integration yields the complete equation for the fin efficiency of a TSPECA.

$$\eta = -\frac{2}{\alpha} + \frac{2I_0(2\sqrt{\alpha})}{I_1(2\sqrt{\alpha})\sqrt{\alpha}} \tag{13}$$

With the aid of MAPLE, Equations 10 and 13 reproduce the work of Carranza [1]. In Carranza's paper, Figures 3 and 4 are verified with respect to the fin temperature profile and the fin efficiency.

4 Conclusion

A closed form, analytical solution is found for the work done by Carranza [1]. Originally, Carranza produces a solution to Equation 1 using finite differences – a numerical solution.

The work here, however, now presents a solution to Equation 1 that is in an equational, analytical form: completely continuous, congruent and differentiable. Furthermore, results here also reproduce all the conclusions by Carranza [1].

It should be noted that Equation 1 cannot be solved without the aid of Equations 4 and 6. Equation 1 must be put into the form of a Bessel equation and this is done through a transformation of variables, which Equations 4 and 6 provide.

Guessing what form Equations 4 and 6 must be to transform Equation 1 into a Bessel equation is unbelievably difficult. Thus, MAPLE is used to solve Equation 1. Once the final solution is known, Equations 4 and 6 are inferred from the final solution.

Trail and error is then used. Guesses are made at Equations 4 and 6 and again MAPLE is used to speed up the solutions. After considerable effort, the final forms of Equations 4 and 6 are deduced.

MAPLE is an instrument of immeasurable value. Nevertheless, all results presented here are verified with traditional hand calculations and derivations.

5 Nomenclature

C	integration constant
I	modified Bessel function
J	Bessel function
T	temperature
Y	Bessel function of the second kind
b	spine fin length
u	dummy variable
x	spatial coordinate
y	dummy variable
Θ	dimensionless temperature
X	dimensionless length
α	ratio of conductance due to convection per conductance due to conduction
η	fin efficiency

5.1 Subscripts

b	base of fin
∞	bulk fluid conditions

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