

# Synchronization of Two Chaotic Duffing-type Electrical Oscillators

Ch. K. VOLOS, I. M. KYPRIANIDIS, and I. N. STOUBOULOS

Physics Department  
 Aristotle University of Thessaloniki  
 Thessaloniki, 54124  
 GREECE

**Abstract:** - We have studied experimentally the inverse system approach, which is used when we want to synchronize two chaotic systems (master – slave). We apply this method to a second order nonlinear circuit, which is described by a Duffing equation. We present two different implementations of the slave circuit with the inverse method and we demonstrate the possibility of synchronization when the two circuits have the same parameter values. We have also coupled the two circuits via a linear resistor  $R_x$  and we discovered, that synchronization holds for small values of  $R_x$ . Finally, we found out that the two circuits must be identical, otherwise the synchronization collapses.

**Key-Words:** - Chaos, Duffing equation, Synchronization, Inverse system, Resistive coupling, Master-Slave circuits.

## 1. Introduction

In recent years chaos synchronization has become a topic of great interest. Since the pioneering work of Pecora and Carroll on synchronization of two coupled chaotic systems [1-3], many researchers have discussed the stability of this type of dynamics. Synchronization of chaotic systems plays an important role in several research areas. For example, neural signals in the brain are observed to be chaotic and it is worth to consider further their possible synchronization [4]. Other interesting examples may be seen from the working artificial neural networks [5], biological networks [6], coupled chaotic neurons [7], multimode lasers [8], coupled map lattices [9, 10], and coupled electric oscillators [11]. Also, the topic of synchronization has arised great interest as a potential means for communication [12, 13]. The last few years, a considerable effort has been devoted to extend the chaotic communication applications to the field of secure communications. Accordingly, a number of cryptosystems based on chaos has been proposed [14-16].

Many synchronization schemes have been proposed and pursued. Three of them are the most common to synchronize chaotic systems, i.e. linear feedback [17-19], decomposition into subsystems [20], and the inverse system [21, 22]. The last method can be used to synchronize non-autonomous systems. In this method, the basic idea is to construct an inverse nonlinear system (slave), which will reproduce the input signal by using the output signal from the original system (master), as shown in Fig.1. In general, finding an inverse of a nonlinear system (inverse of a nonlinear operator realized by a circuitry), is a very

difficult task. Several approaches for solving this problem are known from the automatic control literature, however, there is no general solution to this problem.

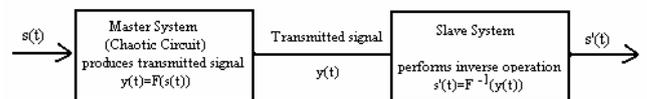


Fig.1. Schematic explanation of the inverse system approach

In the present paper we have shown, the experimental synchronization between two Duffing-type circuits, with the inverse system approach. In Section 2, the Duffing-type circuit is presented. The confirmation of the synchronization of two identical Duffing-type circuits is presented in section 3. Also, in section 3, we present the results of the coupling of two mismatched circuits and also, of the coupling via a linear resistor.

## 2. The Duffing-type Circuit

Duffing's equation,

$$\frac{d^2x_1}{dt^2} + \varepsilon \cdot \frac{dx_1}{dt} + a \cdot x_1 + b \cdot x_1^3 = B \cdot \cos(\omega \cdot t) \quad (1)$$

is one of the most famous and well studied nonlinear non-autonomous equations, exhibiting various dynamic behaviors, including chaos and bifurcations. One of the simplest implementations of the Duffing equation has been presented by Leuciuc [23]. It is a

second order nonlinear circuit, which is excited by a sinusoidal voltage source and contains two op-amps (LF411) operating in the linear region Fig.2. This circuit has also a very simple nonlinear element, implementing a cubic function of the form

$$i(v) = p \cdot v + q \cdot v^3 \quad (2)$$

which is shown in Fig.3.

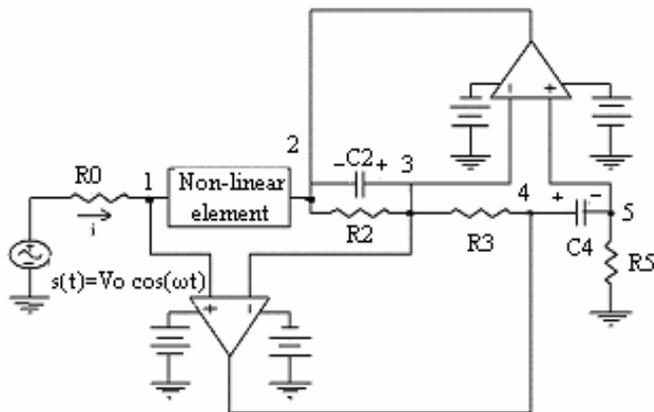


Fig.2. The electric circuit obeying Duffing's equation

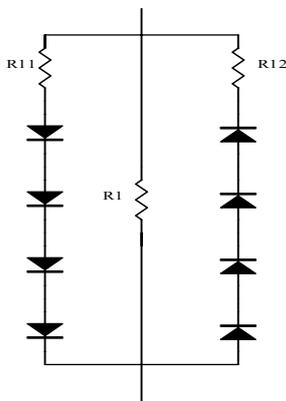


Fig.3. The nonlinear element implementing the cubic function of the form  $i(v) = p \cdot v + q \cdot v^3$

Denoting by  $x_1$  and  $x_2$  the voltages across capacitors  $C_2$  and  $C_4$  respectively, we have the following state equations.

$$\frac{dx_1}{dt} = -\frac{1}{C_2 \cdot R_2} \cdot x_1 + \frac{1}{C_2 \cdot R_3} \cdot x_2 \quad (3)$$

$$\frac{dx_2}{dt} = -\frac{R_0}{C_4 \cdot R_5} \cdot f(x_1) + \frac{V_0}{C_4 \cdot R_5} \cdot \cos(\omega \cdot t) \quad (4)$$

where,  $f(x_1) = p \cdot x_1 + q \cdot x_1^3$ , is a cubic function.

Finally, from equations (3) and (4), we take the Duffing equation (1), where,

$$\begin{aligned} \varepsilon &= \frac{1}{C_2 \cdot R_2}, & a &= \frac{p \cdot R_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5} \\ b &= \frac{r \cdot R_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5}, & B &= \frac{V_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5} \end{aligned} \quad (5)$$

The values of circuit parameters are  $R_0=2.05k\Omega$ ,  $R_2=5.248k\Omega$ ,  $R_3=R_5=1k\Omega$ ,  $R_{11}=R_{12}=0.557k\Omega$ ,  $R_1=8.11k\Omega$ ,  $C_2=105.9nF$ ,  $C_4=9.79nF$ ,  $V_0=2V$  and  $f=1.273kHz$ , so the normalized parameters take the following values  $a=0.25$ ,  $b=1$ ,  $\varepsilon=0.18$ ,  $\omega=0.8$  and  $B=20$ . The phase portrait of  $x_2$  vs.  $x_1$  is shown in Fig.4, where we can see that the circuit has a chaotic behaviour.

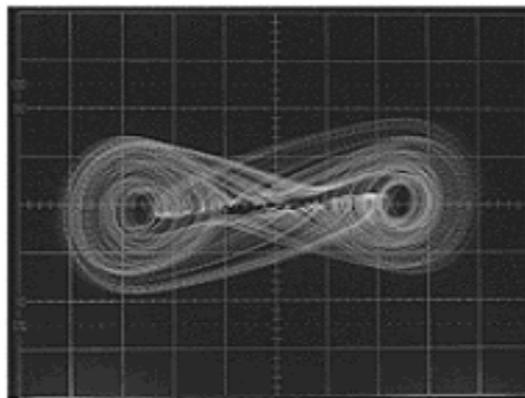


Fig.4. Experimental phase portrait of  $x_2$  vs.  $x_1$  for  $a=0.25$ ,  $b=1$ ,  $\varepsilon=0.18$ ,  $\omega=0.8$  and  $B=20$  (Horiz.: 1V/div., Vert.: 5V/div.)

## 2.1 The Coupled Circuits

We have constructed two different coupled schemes, and we have studied two different slave circuits with the inverse method.

### 2.1.1 Coupling via nodes-4

In the first case the circuits (master-slave), are coupled via nodes-4, (Fig.5). The slave circuit is described by the following set of equations.

$$\begin{aligned} \frac{dx_1'}{dt} &= -\frac{1}{C_2 \cdot R_2} \cdot x_1' + \frac{1}{C_2 \cdot R_3} \cdot x_2' \\ \frac{dx_2'}{dt} &= -\frac{1}{C_4 \cdot R_5} \cdot x_2' + \frac{1}{C_4 \cdot R_5} \cdot y_1 \end{aligned} \quad (6)$$

$$s' = R_0 \cdot f(x_1') - x_2' + y_1$$

where,  $x_1', x_2'$ , are the voltages across the capacitors  $C_2$  and  $C_4$  respectively of the slave circuit.

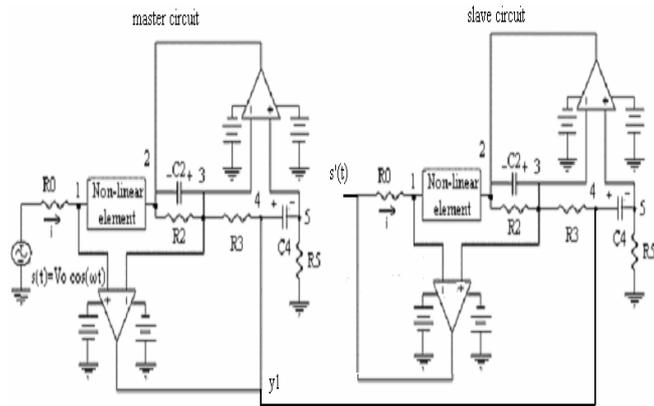


Fig.5. The system when the variable  $y_1$  is transmitted

Also,  $y_1 = x_2 - R_0 \cdot f(x_1) + V_0 \cdot \cos(\omega \cdot t)$ ,  $y_1$  is the transmitted signal from the master circuit. So, from the above equations we take,  $s' = (x_2 - x_2') + R_0 \cdot (f(x_1') - f(x_1)) + V_0 \cdot \cos(\omega \cdot t)$  (7) In the case of synchronization,  $s' = s = V_0 \cdot \cos(\omega \cdot t)$ .

**2.1.2 Coupling via nodes-2**

At the second case, we have coupled the circuits (master-slave) via nodes-2, (Fig.6).

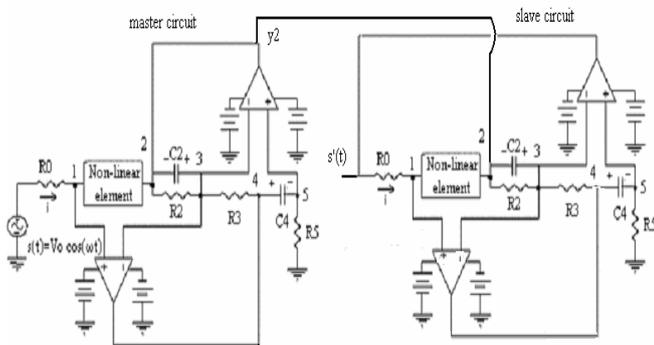


Fig. 6. The system when the variable  $y_2$  is transmitted

The slave circuit is described by the following set of equations.

$$\frac{dx_1'}{dt} = -\frac{1}{C_2 \cdot R_2} \cdot x_1' + \frac{1}{C_2 \cdot R_3} \cdot x_2'$$

$$\frac{dx_2'}{dt} = \frac{1}{C_4 \cdot R_5} \cdot x_1' + \frac{1}{C_4 \cdot R_5} \cdot y_2 \tag{8}$$

$$s' = R_0 \cdot f(x_1') + x_1' + y_2$$

where,  $y_2 = -x_1 - R_0 \cdot f(x_1) + V_0 \cdot \cos(\omega \cdot t)$ ,  $y_2$  is the transmitted signal from the master circuit. So, from the above equations we take,  $s' = (x_1' - x_1) + R_0 \cdot (f(x_1') - f(x_1)) + V_0 \cdot \cos(\omega \cdot t)$  (9) In the case of synchronization,  $s' = s = V_0 \cdot \cos(\omega \cdot t)$ .

**3. Experimental Results**

We have studied this method of synchronization for the first case of coupling (§ 2.1.1), when the two circuits

- are identical,
- coupled via a linear resistor and
- have different values of elements.

The same results apply also at the second case of coupling.

**3.1 Identical Circuits**

In this case, the two coupled circuits have exactly the same values of elements, so the two coupled circuits have the same parameters,  $a=0.25$ ,  $b=1$ ,  $\epsilon=0.18$ ,  $\omega=0.8$  and  $B=20$ . In Fig.7, the transmitted signal  $y_1$  is shown. It is a chaotic signal.

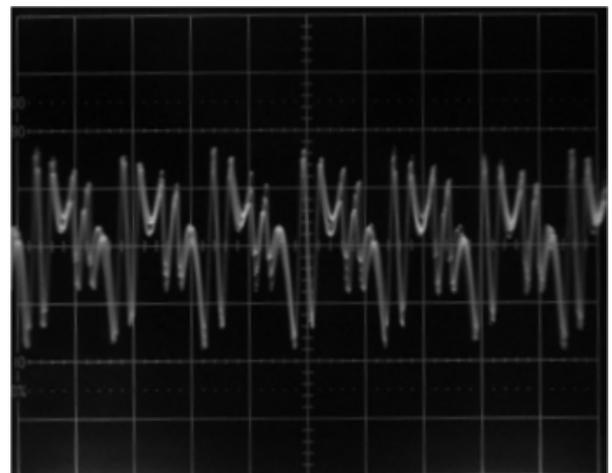
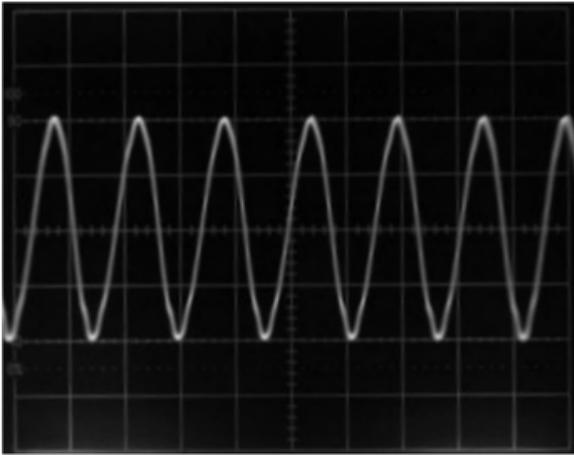
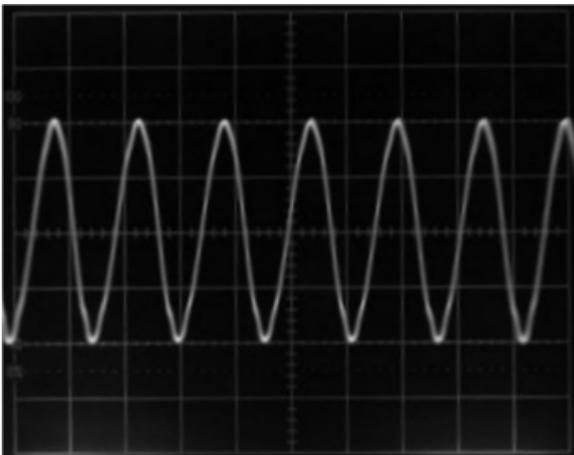


Fig.7. Waveforms of the transmitted signals  $y_1$  (Horiz.: 0.5 msec/div., Vert.: 1V/div.)

The synchronization of two circuits is perfect as we observe from the comparison of the waveforms of the information  $s(t)$  and recovered  $s'(t)$  signals, (Fig.8), and from the plot of  $s'(t)$  vs.  $s(t)$ , (Fig.9).



(a)



(b)

Fig.8. Waveforms of (a) information signal  $s(t)$ , (b) recovered signal  $s'(t)$ , (Horiz.: 0.5 msec/div., Vert.: 1V/div.)

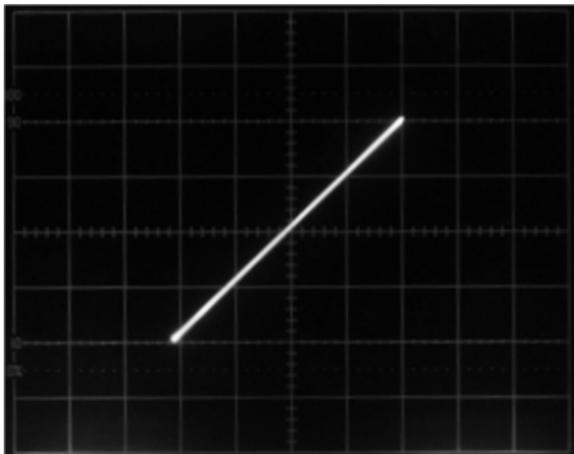
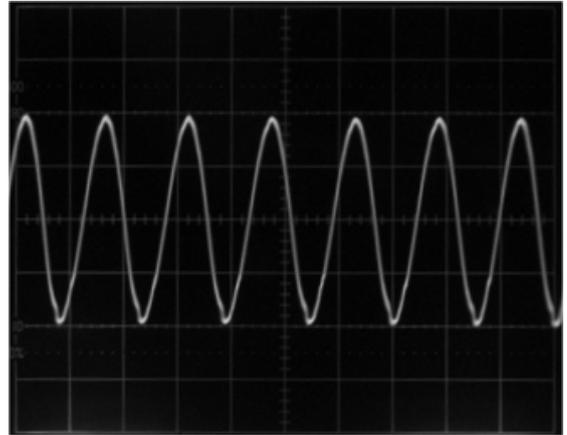


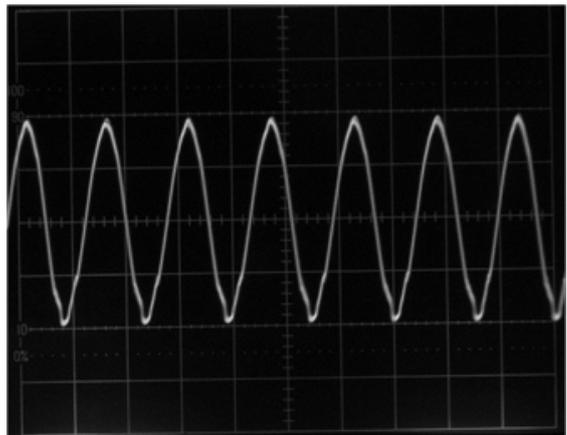
Fig.9. Plot of  $s'(t)$  vs.  $s(t)$  (Horiz.: 1V/div., Vert.: 1V/div.). Synchronization is observed.

### 3.2 Resistive coupling

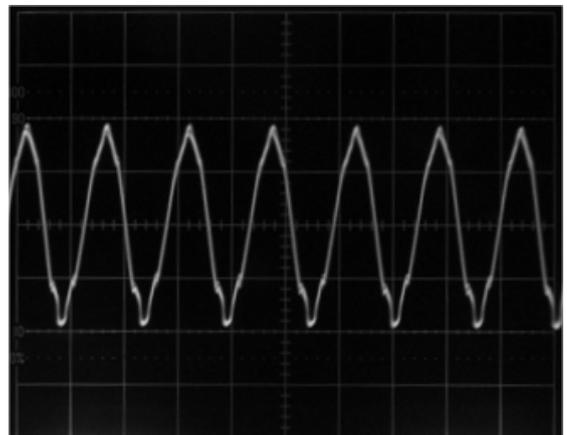
If the two circuits are coupled via a linear resistor  $R_x$ , we observe that, as the value of  $R_x$  increases, synchronization collapses, (Fig.10). Specifically, as we can see in Fig.10(a), for  $R_x=10\Omega$  the synchronization holds, but for  $R_x=20\Omega$ , (Fig.10(b)), the recovered signal is not further identical to the transmitted.



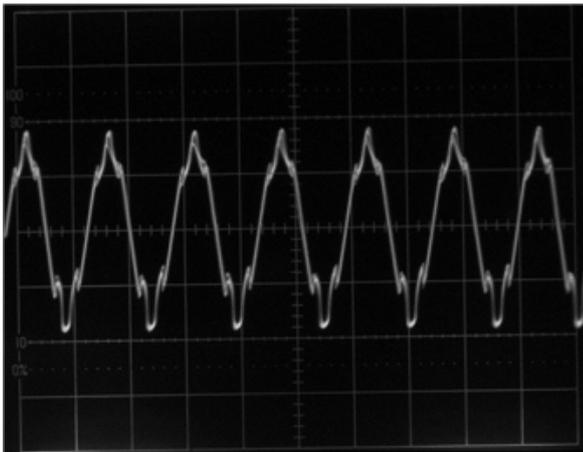
(a)



(b)



(c)



(d)

Fig.10. Waveforms of recovered signal  $s'(t)$ , (a)  $R_x=10\Omega$ , (b)  $R_x=20\Omega$ , (c)  $R_x=40\Omega$ , (d)  $R_x=80\Omega$  (Horiz.: 0.5 msec/div., Vert.: 1V/div.)

Finally, for greater values of  $R_x$ , that is for  $R_x=40\Omega$ , (Fig.10(c)), and  $R_x=80\Omega$ , (Fig.10(d)) the deformation is obvious.

### 3.3 Different Values of Elements

In this case, we have chosen a bit different values for the capacitors of the slave circuit,  $C_2=106.2\text{nF}$ ,  $C_4=9.76\text{nF}$ , and also the resistor  $R_2=5.233\Omega$ , so we have again the same values of parameters  $a=0.25$ ,  $b=1$ ,  $\epsilon=0.18$ ,  $\omega=0.8$  and  $B=20$ . As we can see from the Fig.11, the information signal and the recovered signal have a slight divergence at the peaks. The same result turns up from Fig.12, in which we see that the diagram diverges from the straight line near the edges.

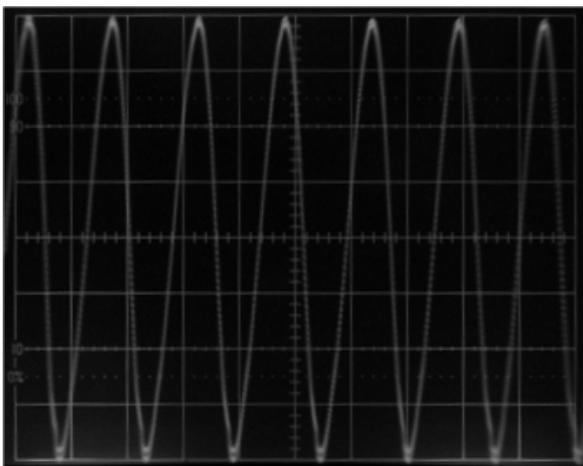


Fig.11. Waveforms of information signal  $s(t)$ , recovered signal  $s'(t)$ , (Horiz.: 0.5 msec/div., Vert.: 1V/div.)

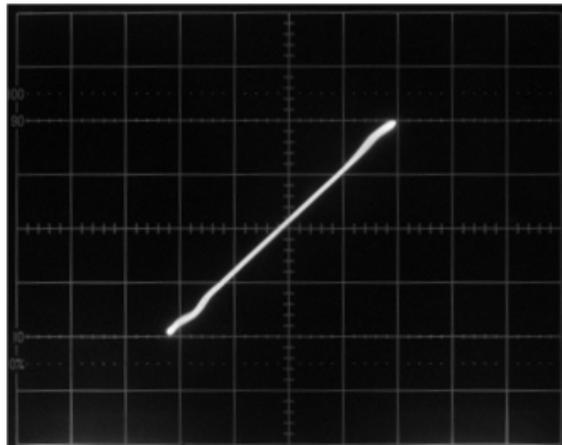


Fig.12. Plot of  $s'(t)$  vs.  $s(t)$  (Horiz.: 1V/div., Vert.: 1V/div.)

### 4. Conclusions

In this paper we have studied the inverse system approach of synchronization. We have applied this method to a Duffing-type circuit, and we have presented two different implementations of the slave circuit.

The proposed method seems to work well, when the two circuits (master – slave), have exactly the same values of elements (identical). We have checked this method in the case of coupling the two circuits via a resistor,  $R_x$ . We have observed that the synchronization holds for small values of  $R_x$ . Also, we discovered that if we choose a bit different values of elements, we have a slight divergence between the two signals (information – recovered). So, we conclude that this method of synchronization is very sensitive at the variation of the values of the elements and of the external conditions.

### Acknowledgements

This work has been supported by the research program “EPEAEK II, PYTHAGORAS II”, with code number 80831, of the Greek Ministry of Education and E.U.

### References:

[1] Pecora L. M., Carroll T.L., Synchronization in Chaotic Systems, *Phys. Rev. Lett.*, Vol. 64,1990, pp. 821-824.  
 [2] Pecora L. M., Carroll T.L., Driving Systems with Chaotic Signals, *Phys. Rev. A*, Vol. 44, 1991, pp. 2374-2383.  
 [3] Carroll T.L., Pecora L.M., Synchronizing Chaotic Circuits, *IEEE Trans. Circ. Syst. CAS*, Vol. 38, 1991, pp. 453-456.

- [4] Tass P., Roseblum M.G., Weule M.G., Kurths J., Pikovsky A., Volkman J., Schnitzler A., Freund H.J., Detection of n:m Phase Locking from Noise Data: Application to Magnetoencephalography, *Phys. Rev. Lett.*, Vol. 81, 1998, pp. 3291-3294.
- [5] Schafer C., Roseblum M.G., Kurths J., Abel H.H., Heartbeat Synchronized with Ventilation, *Nature*, Vol. 392, 1998, pp.239-240.
- [6] Neiman A., Xing Pei, Russell D., Wojtenek W., Wilkens L., Moss F., Braun H. A., Huber M. T., and Voigt K., Synchronization of the Noisy Electrosensitive Cells in the Paddlefish, *Phys. Rev. Lett.*, Vol. 82, 1999, pp. 660-663.
- [7] Bazhenov M., Huerta R., Rabinovich M. I. and Sejnowski T., Cooperative Behavior of a Chain Synaptically Coupled Chaotic Neurons, *Physica D*, Vol. 116, 1998, pp. 392-400.
- [8] Van Wiggeren G. D. and Roy R., Communication with Chaotic Lasers, *Science*, Vol. 279, 1998, pp. 1198-1200.
- [9] Meng Zhan, Gang Hu, and Junzhong Yang, Synchronization of chaos in coupled systems, *Phys. Rev. E*, Vol. 62, 2000, pp. 2963-2966.
- [10] Wang Jinlan, Chen Guangzhi, Qin Tuanfa, Ni Wansun, and Wang Xuming, Synchronizing spatiotemporal chaos in coupled map lattices via active-passive decomposition, *Phys. Rev. E*, Vol. 58, 1998, pp. 3017-3021.
- [11] Kyprianidis I.M., Stouboulos I.N., Chaotic Synchronization of Two Resistively Coupled Nonautonomous and Hyperchaotic Oscillators, *Chaos Solitons and Fractals*, Vol. 17, 2003, pp. 317-325.
- [12] Dedieu H., Kennedy M.P. and Hasler M., Chaos Shift Keying: Modulation and Demodulation of a chaotic Carrier Using Self-Synchronizing Chua's Circuits, *IEEE Trans Circ. Syst-II*, Vol. 40, 1993, pp. 634-642.
- [13] Tse C.K. and Lau F., *Chaos-based Digital Communication Systems: Operating Principles, Analysis Methods, and Performance Evaluation*, Berlin, New York: Springer Verlag, 2003.
- [14] Cuomo K.M. and Oppenheim A.V., Circuit Implementation of Synchronized Chaos with Applications to Communications, *Phys. Rev. Lett.*, Vol. 71, 1993, pp. 65-68.
- [15] Wu, Chai Wah, Chua, L. O., A Simple Way to Synchronize Chaotic Systems with Applications to Secure Communication Systems, *Int. J. Bif. Chaos*, Vol. 6, 1993, pp.1617-1627.
- [16] Lozi R. nad Chua L.O., Secure Communications via Chaotic Synchronization. II: Noise Reduction by Cascading two Identical Receivers, *Int. J. Bif. Chaos*, Vol. 3, 1993, pp.1319-1325.
- [17] Orgozalek M.J., Taming Chaos Part I: Synchronization, *IEEE Trans. Circ. Sys. I*, Vol. 40, 1993, pp. 693-699.
- [18] Orgozalek M.J., Taming Chaos Part II: Control, *IEEE Trans. Circ. Sys. I*, Vol. 40, 1993, pp. 700-706.
- [19] I. M. Kyprianidis I. M., Ch. K. Volos and Stouboulos I. N., Suppression of Chaos by Linear Resistive Coupling, *WSEAS Trans. Circ. Syst.*, Vol 4, 2005, pp. 527-534.
- [20] Carroll T.L., Communicating by Use of Filtered, Synchronized Chaotic Signals, *IEEE Trans. Circ. Syst. I*, Vol. 42, 1995, pp. 105-110.
- [21] Feldmann U., Hasler M. and Schwartz W., Communication by Chaotic Signals: The Inverse System Approach, *Int. J. Circ. Theory Appl.*, Vol. 24, 1996, pp. 551-579.
- [22] Bohme F. and Schwartz W., The Chaotizer-dechaotizer channel, *IEEE Trans. Circ. Syst.*, Vol. 43, 1996, pp. 569-599.
- [23] Leuciuc A., The Realization of Inverse System for Circuits Containing Nullors with application in Chaos Synchronization, *Int. J. Circ. Theory Appl.*, Vol. 26, 1998, pp. 1-12.