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Calculation of Velocity Field for Ideal Fluid, Induced by Vortex Curves in a Finite Cylinder

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Abstract: - In this paper we calculate the velocity field and distribution of stream function for ideal incompressible fluid, induced by a different system of vortex threads in a finite frustum of the cone. An original method was used to calculate the radial and axial components of the velocity vectors. Such a procedure allows us to calculate the velocity fields inside the cylinder depending on the arrangement, on the intensity and on the radius of circular vortex lines. In this paper we have developed original mathematical model for new type of ecologically clean and energetically effective device for producing electricity by the wind power.

Key-Words: -incompressible fluid, vortex curves, finite cylinder, finite cone, velocity field.

1 Introduction

In new technological applications it is important to use vortex distributions in area for obtaining large values of velocity. The effective use of vortex energy in production of strong velocity fields by different device is one of the modern areas of applications, developed during the last decade. Such processes are ecologically clean; there is no environment pollution. Although, on the other hand the aspect of energy is very important: the transformation process should be organized in such way that vortex energy is effectively transformed into heat or mechanical energy. In our previous papers [1] – [3] we have mathematically modeled the process how to transform the alternating electrical current into heat energy.

The goal of this paper is to develop the mathematical models for new type of ecologically clean and energetically effective devices for producing electricity by the wind power [4]. Such type of devices firstly was developed by I. Rechenberg. Now the continuator of the work is one of authors J. Schatz. The devices of such type can be considered as the energy source of the new generation because of several reasons. Firstly, they are completely ecologically clean. Secondly, they are very compact (if their size is compared to their power). Thirdly, the idea of the structure of the devices is based on the processes in nature; in this case it is hurricane.

2 Mathematical Notations and the General Formulation of the Problem

Let the ideal incompressible fluid occupy a finite frustum of cone $\tilde{\Omega}(\varepsilon)=\{(r,z,\varphi):\ 0<r<a-\varepsilon z,$ $0<z<Z, 0<\varphi<2\pi\}$, where the parameter ε fulfills the condition: $0\le \varepsilon Z<a$. If $\varepsilon=0$, the cone transforms to circular cylinder with the radius a.

We will start with some geometrical descriptions of placement of the vortexes. We will consider the situation, when N discrete circular vortexes L_i, where $L_i=\{(r,z): r=a_i, z=z_i\}, i=\overline{1,N}$ with intensity $\Gamma_i\left(\frac{m^2}{s}\right)$ and radii $a_i\left(m\right)$ are placed in the cylinder. The system of circular vortexes creates the radial v_r and axial v_z components of the velocity field in ideal incompressible liquid.

Similarly can be considered the system of N discrete spiral vortex threads ($i=\overline{1,N}$) $S_i=\{(r,z,\varphi): r=a-\varepsilon t, z=a\tau t, \varphi=t+i\delta\}$ with parameters $\delta=\frac{2\pi}{N}, \tau=\frac{Z}{2\pi aM}$. The argument φ fulfills the following enclosure: $\varphi\in\left[\frac{2\pi}{N}, 2\pi(M+1)\right]$, parameter $t\in[0,2\pi M]$, where M is the number of circulation periods and τ is the rise (step) of the vortex threads. The system of vortex threads creates the radial v_r, axial v_z and azimuthal v_φ components of the velocity field in ideal incompressible liquid.

The vector potential A will be determined from the equations of vortex motion of ideal incompressible fluid

$$\begin{cases} div\, v = 0, \\ rot\, v = \Omega. \end{cases} \quad (1)$$

in following form:

$$\Delta A = -\Omega . \quad (2)$$

Here

$$v = rot\, A , \quad (3)$$

where v, Ω are the vectors of velocity and vortex fields and Δ is the Laplace operator.

It is well known that if we replace the velocity vector v with the magnet field induction vector B and the vortex vector Ω with the electrical current vector j then the system of equations (1) is identical with steady-state Maxwell's equations. It means that all our mathematical results could be applicable to some electromagnetic processes as it was done in our papers [1]-[3]. In spite of this now we will apply our mathematical investigations to see how the vortexes act to the distribution of velocity field of incompressible fluid. The aim of this investigation is to try to understand the process in devices of Hurricane Energy Transformer [4] type.

3 The Description of the Problem in the Case of the Finite Cone

Applying the Biot-Savart law [5],[6] we receive the following representation form for the vector potential created by the vortex thread $W_i = S_i$ or $W_i = L_i$:

$$A(P)_i = \frac{\Gamma_i}{4\pi} \int_{W_i} \frac{dl}{R(NP)_i} , \quad (4)$$

where dl is an element of the curve W_i, $P = P(x, y, z)$ is the fixed point in the fluid, $N = N(\xi, \eta, \varsigma) \in W_i$ is the integration point,

$$R(NP)_i = R_i = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2}. \quad (5)$$

For the **circular** vortex $W_i = L_i$ in cylindrical coordinates we have:

$$x = r\cos\varphi,\ y = r\sin\varphi, z = z_i . \quad (6)$$

For the **spiral** vortex $W_i = S_i$ we have expressions:

$$\xi = a_*(t)\cos(t + i\delta), \eta = a_*(t)\sin(t + i\delta),$$
$$\varsigma = bt . \quad (7)$$

Here $a_*(t) = a - \varepsilon t,\ b = a\tau,\ t \in [0, 2\pi M]$.

This gives following expressions for the components of vector potential:

$$A_{x,i} = \frac{\Gamma_i}{4\pi}\int_{S_i}\frac{d\xi}{R_i}, A_{y,i} = \frac{\Gamma_i}{4\pi}\int_{S_i}\frac{d\eta}{R_i}, A_{z,i} = \frac{\Gamma_i}{4\pi}\int_{S_i}\frac{d\varsigma}{R_i} .$$

For the spiral vortex we have from (5) and (7):

$$d\xi = -[a_*(t)\sin(t + i\delta) + \varepsilon\cos(t + i\delta)],$$
$$d\eta = a_*(t)\cos(t + i\delta) - \varepsilon\sin(t + i\delta),$$
$$d\varsigma = bdt,$$
$$R_i = \\ = \sqrt{r^2 + a_*(t)^2 - 2a_*(t)r\cos(\varphi - t - i\delta) + (z - bt)^2}$$

Therefore

$$A_{x,i} = -\frac{\Gamma_i}{4\pi}\int_0^{2\pi M}\frac{a_*(t)\sin(t + i\delta) + \varepsilon\cos(t + i\delta)}{R_i}dt ,$$

$$A_{y,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}\frac{a_*(t)\cos(t + i\delta) - \varepsilon\sin(t + i\delta)}{R_i}dt ,$$

$$A_{z,i} = \frac{\Gamma_i b}{4\pi}\int_0^{2\pi M}\frac{dt}{R_i} .$$

In accordance with formulae (2) we have following expressions for the components of velocity field:

$$\begin{cases} v_{r,i} = -\dfrac{\partial A_{\varphi,i}}{\partial z} + \dfrac{\partial A_{z,i}}{r\partial\varphi}, \\ v_{z,i} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rA_{\varphi,i}\right) - \dfrac{1}{r}\dfrac{\partial A_{r,i}}{\partial\varphi}, \\ v_{\varphi,i} = \dfrac{\partial A_{r,i}}{\partial z} - \dfrac{\partial A_{z,i}}{\partial r}. \end{cases} \quad (8)$$

Further we have from (6):

$$A_{r,i} = A_{x,i}\cos\varphi + A_{y,i}\sin\varphi,$$
$$A_{\varphi,i} = -A_{x,i}\sin\varphi + A_{y,i}\cos\varphi.$$

It gives following expressions for last two components of vector potential:

$$A_{r,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}\frac{a_*(t)\sin(\psi(t)) - \varepsilon\cos(\psi(t))}{R_i}dt ,$$

$$A_{\varphi,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M} \frac{a_*(t)\cos(\psi(t)) + \varepsilon\sin(\psi(t))}{R_i}dt.$$

Here we have introduced short notation $\psi(t) = \varphi - t - i\delta$. For the partial derivatives of the distance R_i we obtain following expressions:

$$\frac{\partial R_i}{\partial r} = \frac{r - a_*(t)\cos(\psi(t))}{R_i},$$

$$\frac{\partial R_i}{\partial z} = \frac{z - bt}{R_i},$$

$$\frac{\partial R_i}{\partial \varphi} = \frac{a_*(t)\sin(\psi(t))}{R_i}.$$

For the components of the velocity field we have representations as follows:

$$v_{r,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}[(z-bt)(a_*(t)\cos(\psi(t)) \tag{9}$$

$$+\varepsilon\sin(\psi(t))) - ba_*(t)\sin(\psi(t))]\frac{dt}{R_i^3},$$

$$v_{z,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}[a_*(t)(a_*(t) - \tag{10}$$

$$r\cos(\psi(t))) - \varepsilon r\sin(\psi(t))]\frac{dt}{R_i^3},$$

$$v_{\varphi,i} = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}[b(r - a_*(t)\cos(\psi(t))) - \tag{11}$$

$$(z-bt)(a_*(t)\sin(\psi(t)) + \varepsilon\cos(\psi(t)))]\frac{dt}{R_i^3}.$$

4 Solution of the Problem for the Finite Cylinder

Further in this paper we will concentrate our attention to the case of the circular cylinder ($\varepsilon = 0$). It is easy to proof that for the cylinder with the radius a all components of velocity are a even functions according to middle point $z = \frac{Z}{2}$ of the cylinder, i.e.:

$$v_i(r, \frac{Z}{2} - z, \varphi) = v_i(r, \frac{Z}{2} + z, \varphi). \tag{12}$$

The representations for the components of vector potential in case of cylinder take a simplified form:

$$A_{x,i} = -\frac{\Gamma_i a}{4\pi}\int_0^{2\pi M}\frac{\sin(t + i\delta)}{R_i}dt,$$

$$A_{y,i} = \frac{\Gamma_i a}{4\pi}\int_0^{2\pi M}\frac{\cos(t + i\delta)}{R_i}dt,$$

$$A_{z,i} = \frac{\Gamma_i b}{4\pi}\int_0^{2\pi M}\frac{dt}{R_i}.$$

Respectively simplifies the both components of the vector potential in cylindrical coordinates:

$$A_{r,i} = \frac{\Gamma_i a}{4\pi}\int_0^{2\pi M}\frac{\sin(\psi(t))}{R_i}dt,$$

$$A_{\varphi,i} = \frac{\Gamma_i a}{4\pi}\int_0^{2\pi M}\frac{\cos(\psi(t))}{R_i}dt.$$

The components of the velocity field look now as follows:

$$v_{r,i}(r,z,\varphi) = \frac{\Gamma_i a}{4\pi}\times \tag{13}$$

$$\int_0^{2\pi M}[(z-bt)\cos(\psi(t)) - b\sin(\psi(t))]\frac{dt}{R_i^3},$$

$$v_{z,i}(r,z,\varphi) = \frac{\Gamma_i a}{4\pi}\times \tag{14}$$

$$\int_0^{2\pi M}[a - r\cos(\psi(t))]\frac{dt}{R_i^3},$$

$$v_{\varphi,i}(r,z,\varphi) = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M}[b(r - a\cos(\psi(t)) \tag{15}$$

$$-a(z-bt)\sin(\psi(t))]\frac{dt}{R_i^3}.$$

On the axis of the cylinder the second component (14) of velocity reduced to simple expression:

$$v_{z,i}(0,z,\varphi) = \frac{\Gamma_i a^2}{4\pi}\int_0^{2\pi M}\frac{dt}{\left[a^2 + (z-bt)^2\right]^{3/2}}.$$

This representation easy can be written in closed form:

$$v_{z,i}(0,z) = \frac{\Gamma_i M}{2Z}\left[\frac{z}{\sqrt{a^2 + z^2}} + \frac{Z - z}{\sqrt{a^2 + (Z-z)^2}}\right].$$

This function takes it's maximal value in middle point of cylinder axis $z = Z/2$ [8]:

$$v_{z,i}(0, Z/2) = \frac{\Gamma_i M}{2a\sqrt{1 + (Z/(2a))^2}}. \tag{16}$$

The minimal values of the z component of the velocity we obtain in two end points of cylinder axis [8]:

$$v_{z,i}(0,0) = v_{z,i}(0,Z) = \frac{\Gamma_i M}{2a\sqrt{1 + (Z/a)^2}}. \tag{17}$$

We can calculate the integral averaged value of the axial velocity component:

$$v_{av,i} = \frac{1}{Z}\int_0^Z v_{z,i}(0,z)dz .$$

This value is:

$$v_{av,i} = \frac{\Gamma_i M}{2a}\frac{2}{1+\sqrt{1+(Z/a)^2}} . \tag{18}$$

The whole solution can be written now as the sum of separate vortexes:

$$\begin{aligned} v_r(r,z,\varphi) &= \sum_{i=1}^N v_{r,i}(r,z,\varphi), \\ v_z(r,z,\varphi) &= \sum_{i=1}^N v_{z,i}(r,z,\varphi), \\ v_\varphi(r,z,\varphi) &= \sum_{i=1}^N v_{\varphi,i}(r,z,\varphi), \\ A_\varphi(r,z,\varphi) &= \sum_{i=1}^N A_{\varphi,i}(r,z,\varphi). \end{aligned} \tag{19}$$

In general case we calculated all needed integrals with the trapezoid formulas.

In case of circular vortex we have following expressions instead of (7):

$$\xi = a_i \cos\alpha, \eta = a_i \sin\alpha, \varsigma = z_i . \tag{20}$$

Therefore

$$A_{z,i} = 0. \tag{21}$$

The circular vortex originate axially-symmetric conditions; at $\varphi = 0$ we have

$$A_{x,i} = 0. \tag{22}$$

It follows then that

$$A_{y,i} = A_{\varphi,i} = A_i(z,r) = \frac{\Gamma_i a_i}{4\pi} I_i ,$$

where

$$I_i = \int_0^{2\pi} \frac{\cos\alpha d\alpha}{\sqrt{(z-z_i)^2 + a_i^2 + r^2 - 2a_i r\cos\alpha}} . \tag{23}$$

We have (see, e.g. [6]):

$$I_i = \int_0^{\pi/2} \frac{(1-2\sin^2 t)dt}{\sqrt{(z-z_i)^2+(r+a_i)^2}\sqrt{1-k_i^2\sin^2 t}} =$$

$$\frac{2}{\sqrt{ra_i}}\left[\left(\frac{2}{k_i}-k_i\right)K(k_i) - \frac{2}{k_i}E(k_i)\right].$$

Here

$$t = (\alpha - \pi)/2 ,$$

$$k_i = 2\sqrt{ar}/c_i , \tag{24}$$

$$c_i = (z-z_i)^2 + (r+a_i)^2 . \tag{25}$$

Further $K(k)$ and $E(k)$ are the total elliptical integral of first, respectively second kind:

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}dt.$$

Therefore the azimuthal component of vector potential A_i induced by the circular vortex L_i is:

$$A_i(r,z) = \frac{\Gamma_i}{2\pi}\sqrt{\frac{a_i}{r}}F(k_i) . \tag{26}$$

Here

$$F(k_i) = \left[\left(\frac{2}{k_i}-k_i\right)K(k_i) - \frac{2}{k_i}E(k_i)\right]. \tag{27}$$

The two non-zero components of the velocity field for the circular vortex according the formulas (8) reduce to expressions:

$$v_{r,i} = -\frac{\partial A_i}{\partial z},\ v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_i\right). \tag{28}$$

We have from formulas (24),(25) and (27):

$$\frac{\partial k_i}{\partial z} = -\frac{k_i(z-z_i)}{c_i^2}, \frac{\partial k_i}{\partial r} = \sqrt{\frac{a_i}{r}}\frac{(z-z_i)^2+a_i^2-r^2}{c_i^3},$$

$$\frac{dK}{dk_i} = \frac{1}{k_i}\left[\frac{E}{1-k_i^2} - K\right], \frac{dE}{dk_i} = \frac{1}{k_i}\left[E-K\right]$$

and

$$\frac{dF}{dk_i} = \frac{2}{k_i^2}\left[E(k_i)\frac{a_i^2+r^2+(z-z_i)^2}{(a_i-r)^2+(z-z_i)^2} - K(k_i)\right].$$

These expressions give finally:

$$\begin{aligned} v_{r,i}(r,z) &= \frac{\Gamma_i}{2\pi r}\frac{z-z_i}{c_i}\times \\ &\left[E(k_i)\frac{a_i^2+r^2+(z-z_i)^2}{(a_i-r)^2+(z-z_i)^2} - K(k_i)\right], \end{aligned} \tag{29}$$

$$\begin{aligned} v_{z,i}(r,z) &= \frac{\Gamma_i}{2\pi c_i}\times \\ &\left[K(k_i) - \frac{a_i^2-r^2-(z-z_i)^2}{(a_i-r)^2+(z-z_i)^2}E(k_i)\right]. \end{aligned} \tag{30}$$

Both elliptical integrals can be calculated numerically with the accuracy to five decimal places [9] (see our paper [10] also) as follow:

$\tilde{K} = 1.3863 + 0.112\kappa + 0.0725\kappa^2 -$
$(0.5 + 0.1213\kappa + 0.0289\kappa^2)\ln\kappa,$
$\tilde{E} = 1.0 + 0.463\kappa + 0.1078\kappa^2 -$
$(0.2453\kappa + 0.0412\kappa^2)\ln\kappa.$

Few examples:

$$k = 0 : K = E = \frac{\pi}{2} = 1.570800, \tilde{K} = \tilde{E} = 1.570796;$$

$$k = 0.1 : K = 1.574746, \tilde{K} = 1.574754,$$
$$E = 1.566862, \tilde{E} = 1.566871;$$
$$k = 0.9 : K \equiv 2.280549, \tilde{K} = 2.280570,$$
$$E = 1.171697, \tilde{E} = 1.171733.$$

We have on the axis of the cylinder:

$$v_{z,i}(0,z) = \frac{\Gamma_i a_i^2}{2\left[a_i^2 + (z - z_i)^2\right]^{3/2}}.$$

This component of the velocity has the maximal value by $z = z_i, a_i = a$ on the axis and it is as follows:

$$v_{z,i}(0,z_i) = \frac{\Gamma_i}{2a}.$$

In the middle point of the z axis we obtain the value (for $a = a_i$):

$$v_{z,i}(0,Z/2) = \frac{\Gamma_i}{D\left[1 + (\frac{Z}{2} - z_i)^2 / a^2\right]^{3/2}}.$$

For the integral averaged value of the axial velocity component we have following formula:

$$v_{av,i} = \frac{\Gamma_i a}{DZ} \times \left[\frac{(Z - z_i)/a}{\sqrt{1 + ((Z - z_i)/a)^2}} + \frac{z_i / a}{\sqrt{1 + (z_i / a)^2}}\right]. \quad (31)$$

From here we have in the middle point $z_i = Z/2$:

$$v_{av,i} = \frac{\Gamma_i}{D} \frac{1}{\sqrt{1 + (Z/D)^2}}.$$

The total velocity field of all the circular vortexes and the vector potential A_φ we have as the sum of :

$$v_r(r,z) = \sum_{i=1}^{N} v_{r,i}(r,z),$$
$$v_z(r,z) = \sum_{i=1}^{N} v_{z,i}(r,z),$$
$$A_\varphi(r,z) = \sum_{i=1}^{N} A_{\varphi,i}(r,z).$$

The hydrodynamic stream function $\psi = \psi(r,z)$ is given by relations:

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}.$$

Then we have from (28):

$$\psi(r,z) = rA_\varphi(r,z). \quad (32)$$

Important attribute of the process is the amount Q of substance which flows through the cross section $[z = z_0, 0 \le r \le a_0]$ of the cylinder, which is given by the integral:

$$Q(a_0, z_0) = \int_0^{2\pi} d\varphi \int_0^{a_0} v_z(r, z_0) r dr.$$

It is very easy to calculate the quantity:

$$Q(a_0, z_0) = 2\pi a_0 A_\varphi(a_0, z_0) = 2\pi\psi(a_0, z_0). \quad (33)$$

Then the amount Q_T of substance which flows trough the whole cylindrical domain is equal to:

$$Q_T(a_0) = \int_0^Z Q(a_0, z) dz = 2\pi \int_0^Z \psi(a_0, z) dz. \quad (34)$$

As the reader can see, the proposed method allows to calculate the velocity field for arbitrary number and location of circular vortexes or vortex threads in a finite cylinder. This approach is different from the usual methods, e.g., in book [11].

5 Conclusion

Velocity field of ideal incompressible fluid influenced by circular vortex field in a finite cylinder is investigated. The maximal value of the velocity induced by the spiral vortexes is in the middle of the cylinder. The behavior of circular vortexes in the ideal incompressible flow depends on the number, location and on the orientation of the vortexes. This approach can be generalized for the vortex threads on the surface of finite frustum of the cone. It will be investigated in separate paper, where the results of the calculations will also be given.

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