Numerical Study of Travelling Wave Solutions in Competition-Diffusion Systems

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Abstract: - In this work we numerically investigate the behaviour of travelling wave solutions of PDE depending on coefficients of equations and initial conditions. We apply the operator splitting methods for numerical calculations and illustrate some theoretical findings and hypotheses.

Key-Words: travelling waves, splitting methods, propagating speed, reaction-diffusion equations.

1 Introduction

Many biological phenomena can be modeled by nonlinear Partial Differential Equations (PDEs) exhibiting wave solutions. Steady propagating solutions in the form of front travelling without change of form and speed are of special interest.

To find the form of propagating front some authors first reduce the PDEs to Ordinary Differential Equations (ODEs) by introducing a moving frame, and then apply numerical methods suitable for ODEs to find the propagating front. For example, similar way was used in [1]. After reducing PDEs to ODEs the shooting methods were applied and finally improved by relaxation methods. This way has some numerical difficulties. In this work we use another approach. Application of operator splitting methods gives accurate and stable results and allows to numerically investigate some interesting features of travelling waves depending on parameters of the PDEs and initial conditions.

For a system of two competing species, it is also interesting problem to predict a final state of the system. When competition between two species is bitter, the two species cannot coexist in the same habitat. Namely only one species can survive and the other species becomes extinct due to the competition. Such a phenomenon is called "the competition exclusion principle." In a monostable system, the weaker species will become extinct no matter how the initial state is. However, when the system is bistable, which species will become extinct depends on an initial state. We study this question here. Another point of interest is the speed of propagating front.

2 Competition - Diffusion System

Here we consider the bistable diffusive Lotka-Volterra system for two competing species:

$$u_t = d_1 \Delta u + (a_1 - b_1 u - c_1 v)u,$$
$$v_t = d_2 \Delta v + (a_2 - b_2 u - c_2 v)v.$$

where u, v are the population densities of two competing species, a_1, a_2 - the intrinsic growth rates, b_1, c_2 - the intraspecific competition rates, c_1, b_2 - the interspecific competition rates, and d_1, d_2 - the diffusivity constants.

By suitable normalization, the above system reduces to

$$\begin{aligned} u_t &= \Delta u + (1 - u - cv)u, \\ v_t &= d\Delta v + (a - bu - v)v, \end{aligned} \tag{1}$$

where a, b, c, d are positive constants.

In one-dimensional case the system (1) is written as

$$\begin{aligned} u_t &= u_{xx} + (1 - u - cv)u, \\ v_t &= dv_{xx} + (a - bu - v)v, \end{aligned} \qquad x \in \mathbb{R}. \qquad (2)$$

In particular, we are interested in the case where a, b, c satisfy the inequalities.

$$\frac{1}{c} < a < b$$

Then the system becomes bistable so that the system possesses two locally stable spatially homogeneous equilibria

$$(u, v) = (1, 0), \ (0, a).$$

We consider this system in the entire space $\mathbb{R}^1$ with initial conditions $(u(x,0), v(x,0))$:

$$\text{(i)} \qquad \begin{cases} u_0 = 1 - 0.8\, exp(-0.5\, |x|), \\ v_0 = 0.8\, exp(-0.5\, |x|), \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} u_0 = 0.8\, exp(-0.5\, |x|), \\ v_0 = a - 0.8\, exp(-0.5\, |x|), \end{cases}$$

$$\text{(iii)} \qquad \begin{cases} u_0 = 1 - \left(\frac{1}{\pi}\, arctg\, (x) + 0.5\right), \\ v_0 = a\, \left(\frac{1}{\pi}\, arctg\, (x) + 0.5\right), \end{cases}$$

$$\text{(iv)} \qquad \begin{cases} u_0 = \frac{1}{\pi}\, arctg\, (x) + 0.5, \\ v_0 = a - a\, \left(\frac{1}{\pi}\, arctg\, (x) + 0.5\right), \end{cases}$$

We also consider the travelling front solution

$$(u, v) = (\varphi(z), \psi(z)), \ z = x - \alpha t, \qquad (3)$$

where $z \in \mathbb{R}^1$,

$$\begin{aligned} \varphi'' + \alpha\varphi' + (1 - \varphi - c\psi)\varphi = 0, \\ d\psi'' + \alpha\psi' + (a - b\varphi - \psi)\psi = 0, \end{aligned} \qquad (4)$$

and

$$\begin{aligned} \varphi(-\infty), \psi(-\infty)) = (1, 0), \\ (\varphi(+\infty), \psi(+\infty)) = (0, a). \end{aligned} \qquad (5)$$

3 Numerical Scheme

The model equations (2) are solved numerically using the operator splitting method [7] which has been successfully applied previously by the authors [5, 6]. The differential equation system is split into a pair of nonlinear reaction equations

$$\begin{cases} \frac{1}{2}\, u_t = (1 - u - cv)u, \\ \frac{1}{2}\, v_t = (a - bu - v)v, \end{cases} \qquad (6)$$

which are used for the first half of the time step, and a pair of linear diffusion equations

$$\begin{cases} \frac{1}{2}\, u_t = u_{xx}, \\ \frac{1}{2}\, v_t = dv_{xx}, \end{cases} \qquad (7)$$

which are used for the second half of the time step.

The numerical method used for the reaction equations and the diffusion equations is the forward Euler scheme. Then equations (6) become

$$\begin{cases} \bar{u}_j^{n+\frac{1}{2}} = u_j^n + \Delta t u_j^n \left(1 - u_j^n - cv_j^n\right), \\ \bar{v}_j^{n+\frac{1}{2}} = v_j^n + \Delta t v_j^n \left(a - bu_j^n - v_j^n\right), \end{cases}. \qquad (8)$$

where u_j^n and v_j^n indicate the approximate values of u and v at the positions $x_j = X_1 + j\Delta x$, $j = 0, 1, \cdots$ and time $t_n = n\Delta t$, $n = 0, 1, \cdots$, and $\bar{u}_j^{n+1/2}$ and $\bar{v}_j^{n+1/2}$ indicate the representative values at the half time step, $[X_1, X_2]$ is the interval of numerical calculations.

Similarly, equations (7) become

$$\begin{cases} u_j^{n+1} = \bar{u}_j^{n+\frac{1}{2}} + \dfrac{\Delta t}{(\Delta x)^2} \times \\ \qquad \times \left(\bar{u}_{j-1}^{n+\frac{1}{2}} - 2\bar{u}_j^{n+\frac{1}{2}} + \bar{u}_{j+1}^{n+\frac{1}{2}}\right), \\ \\ v_j^{n+1} = \bar{v}_j^{n+\frac{1}{2}} + \dfrac{d\Delta t}{(\Delta x)^2} \times \\ \qquad \times \left(\bar{v}_{j-1}^{n+\frac{1}{2}} - 2\bar{v}_j^{n+\frac{1}{2}} + \bar{v}_{j+1}^{n+\frac{1}{2}}\right). \end{cases}. \qquad (9)$$

The numerical schemes (9) for the diffusion equations give stable solutions provided

$$\frac{\Delta t}{(\Delta x)^2} \le 0.5, \ \frac{d\Delta t}{(\Delta x)^2} \le 0.5.$$

These conditions are satisfied for the calculations of all cases of this paper.

4 Results and Discussion

In the calculations presented below we numerically show the existence of travelling front solution (3) and explore features of the travelling front speed α given in the system (4). We also show the existence of spatially inhomogeneous stationary solution of (2), and its role as a separator.

These results are given in the form of mathematical hypothesis and confirmed by numerical computations. For all results below coefficient a is given, $b = 2.2, c = 10, d = 1$ if not specified. For almost all cases numerical examples given as time evolution of a solution $(u(x, 0), v(x, 0))$ of the system (2) in time intervals equal to 10, otherwise the time intervals are specified.

4.1 Existence of the Solution

It is known that there exists a unique $a = a^*(b, c, d) \in (1/c, b)$ such that (4) with $\alpha = 0$ and (5) has a solution (i.e., stationary solution) (Example 4 - Fig. 5 - 6). Kan-on [2] investigated the existence of travelling front solutions and showed the following:

(i) For every $a \in (1/c, a^*)$, there exists a unique $\alpha > 0$ such that (4) with (5) has a solution (Example 3 - Fig. 2 - 4).

(ii) For every $a \in (a^*, b)$, there exists a unique $\alpha < 0$ such that (4) with (5) has a solution (Example 5 - Fig. 7 - 8).

Moreover, in [3], Kan-on showed the following result on the existence of spatially inhomogeneous stationary solution $(u, v) = (\Phi(x), \Psi(x))$:

(i) For every $a \in (1/c, a^*)$, there exists a spatially inhomogeneous stationary solution of (2) with $(\Phi(\pm\infty), \Psi(\pm\infty)) = (0, a)$ (Example 1 - Fig. 3).

(ii) For every $a \in (a^*, b)$, there exists a spatially inhomogeneous stationary solution of (2) with $(\Phi(\pm\infty), \Psi(\pm\infty)) = (1, 0)$ (Example 2 - Fig. 1).

We confirm this claim with the following numerical examples.

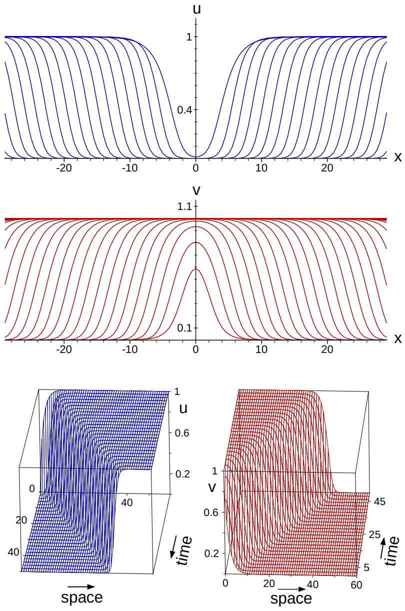

Fig. 1. Solution for $a \in (a^*, b),\ a = 1$.

4.1.1 Numerical Examples 1-5

For all the examples the numerically calculated value of $a^* = a^*(b, c, d) = 0.42243$ as it depends on parameters b, c, d, which are not changed through this paper.

For all cases calculated and just shortly given here the propagating speed α is defined by the parameter a. For $a \in (1/c, a^*)$ the absolute value of α decreases with increasing of a while it reaches the value of zero with with $a = a^*$. And then for $a \in (a^*, b)$ it also increases with increasing of a. This process is clearly presented on Fig. 2 - Fig. 8. The confirmation of the statements above is given below as a list of Examples.

Example 1 - the existence of a spatially inhomogeneous solution of system (2) with $(\Phi(\pm\infty), \Psi(\pm\infty)) = (0, a)$ is shown in Fig. 3 with $a < a^*$ and Initial Conditions (ii).

Example 2 - the existence of a spatially inhomogeneous stationary solution of system (2) with $(\Phi(\pm\infty), \Psi(\pm\infty)) = (1, 0)$ - Fig. 1 with $a > a^*$ and Initial Conditions (i).

Example 3 - the existence of a unique α for each $a \in (1/c, a^*)$ is shown on Fig. 2 - 4 with $a < a^*$ and Initial Conditions (ii).

Example 4 - the existence of a unique $a = a^*(b, c, d) \in (1/c, b)$ such that $\alpha = 0$ and the sta-

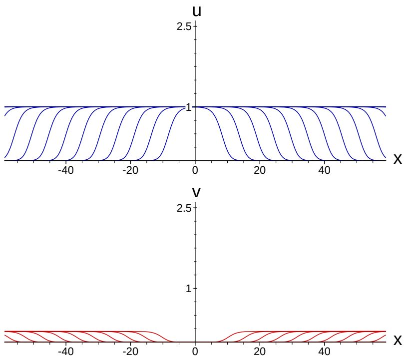

Fig. 2. Solution for $a \in (1/c, a^*),\ a = 0.2.$

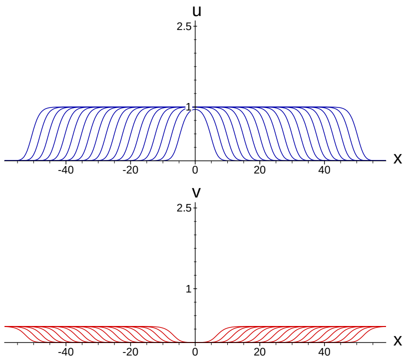

Fig. 3. Solution for $a \in (1/c, a^*),\ a = 0.3.$

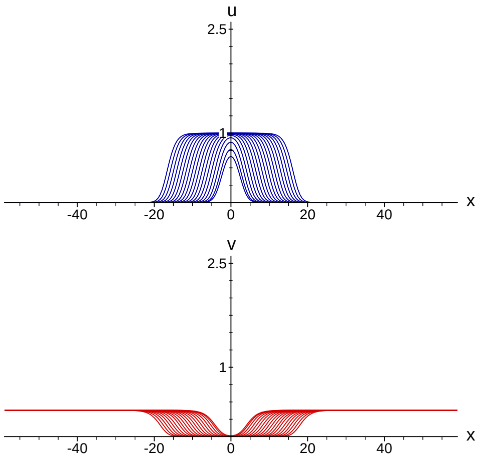

Fig. 4. Solution for $a \in (1/c, a^*),\ a = 0.38.$

tionary solution exists is presented on Fig. 5 - 6 with Initial Conditions (iii) and (iv). Dependence

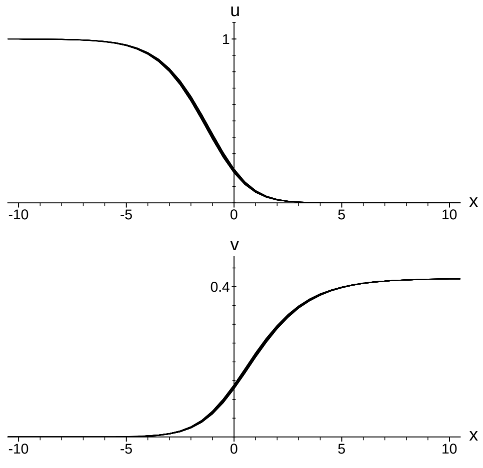

Fig. 5. Solution for $a \approx a^*(b, c, d) \in (1/c, b),\ a = 0.422.$

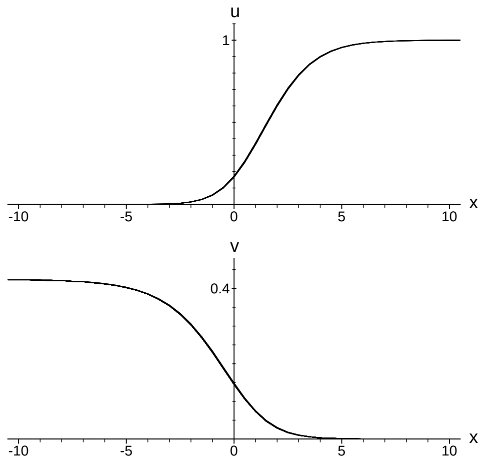

Fig. 6. Solution for $a \approx a^*(b, c, d) \in (1/c, b),\ a = 0.424.$

of the final solution on the choice of initial condition is clear.

Example 5 - the existence of a unique α for each $a \in (a^*, b)$ is shown on Fig. 7 - 8 with Initial Condition (i).

4.2 Stationary Solution is a Separator

We expect that the stationary solution is unstable [4], and plays a role of a separator. Namely, we claim the following:

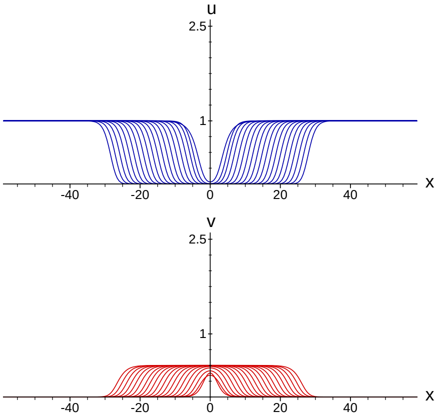

Fig. 7. Solution for $a \in (a^*, b),\ a = 0.5$.

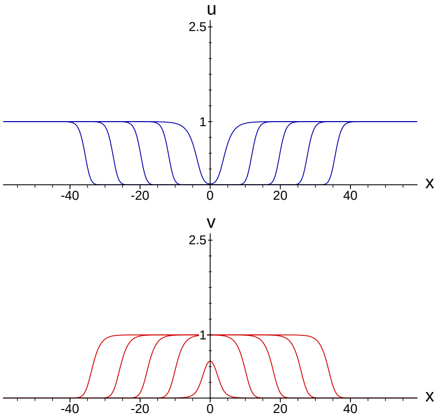

Fig. 8. Solution for $a \in (a^*, b),\ a = 1$.

- (i) Let $a \in (1/c, a^*)$. If $u_0(x) > \Phi(x)$ and $v_0(x) < \Psi(x)$ for all $x \in \mathbb{R}$, then the solution of (2) converges to $(1, 0)$ as $t \to \infty$ locally uniformly in $x \in \mathbb{R}$. Conversely, if $u_0(x) < \Phi(x)$ and $v_0(x) > \Psi(x)$ for all $x \in \mathbb{R}$, then the solution of (2) converges to $(0, a)$ as $t \to \infty$ uniformly in $x \in \mathbb{R}$. (Example 6 - Fig. 4 and Fig. 9.)

- (ii) Let $a \in (a^*, b)$. If $u_0(x) > \Phi(x)$ and $v_0(x) < \Psi(x)$ for all $x \in \mathbb{R}$, then the solution of (2) converges to $(1, 0)$ as $t \to \infty$ uniformly in $x \in \mathbb{R}$. Conversely, if $u_0(x) < \Phi(x)$ and $v_0(x) > \Psi(x)$ for all $x \in \mathbb{R}$, then the solution of (2) converges to $(0, a)$ as $t \to \infty$ locally uniformly in $x \in \mathbb{R}$. (Example 6 - Fig. 7 and Fig. 10).

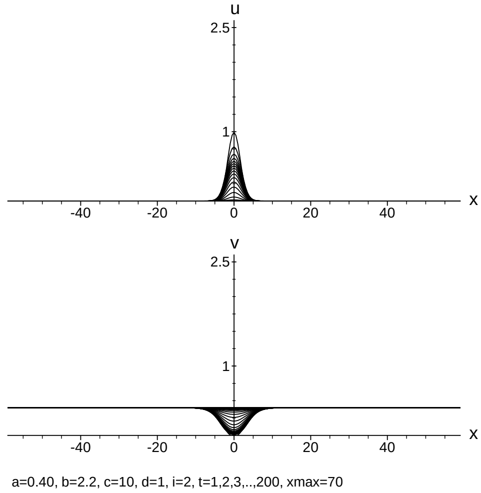

Fig. 9. Solution for $a \in (1/c, a^*),\ a = 0.40$.

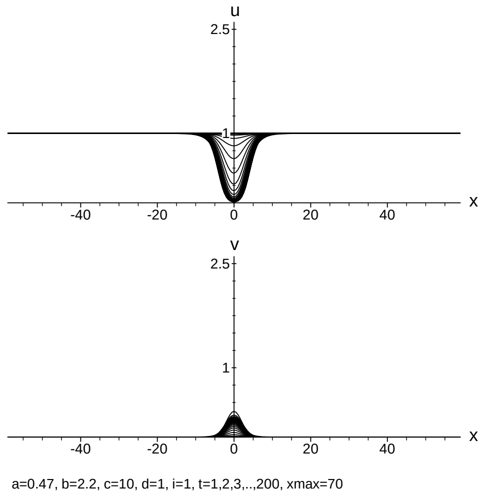

Fig. 10. Solution for $a \in (a^*, b),\ a = 0.47$.

The numerical simulations confirm this claim. Some numerical Examples above and below show this confirmation.

4.2.1 Numerical Example 6

Example 6 - stationary solution as a separator and dependence on a and initial conditions is also could be seen when comparing Fig. 4 with Fig. 9 and Fig. 7 with Fig. 10.

Fig. 4 and Fig. 9 both present the case when $a \in (1/c, a^*)$ and differ just a bit in value of a and hence in initial condition, even of the same type

- Initial Condition (ii). But results are completely different - both show the part (i) of the statement and both present opposite cases of (i)

The similar situation we have on Fig. 7 and Fig. 10. Both present the part (ii) of the statement with $a \in (a^*, b)$ and both illustrate the opposite cases of (ii) while calculated with the same type of initial conditions (Initial Conditions (i)) and small difference in a.

5 Conclusion

In this work we have investigated some very interesting properties of the travelling wave solutions of reaction-diffusion equations using powerful and simple operator splitting methods. The method was successful without any numerical difficulties and highlighted the important properties of travelling wave solutions, its dependence on parameters and initial conditions. It also was possible to find critical value of parameter a with the speed of front equals to zero.

We numerically confirm the statements given in the paper with this method, which we believe was never used with the problems of travelling waves.

This numerical method has shown the simplicity and quick convergence. We believe it could be used with other, more complicated systems of PDEs, in 2 and more dimensions.

6 Acknowledgements

We would like to thank the Japanese Governments for support. The provision of a financial support from the Japan Society for the Promotion of Science are greatly appreciated.

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