A Modified Theory of Laminar Flow Near a Rotating Disk

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Abstract: - Scale-invariant forms of mass, energy, and linear momentum conservation equations in chemically - reactive fields are described. The modified equation of motion is then solved for the classical problems of laminar flow near a rotating disk. The predicted velocity profiles obtained from analytical solutions are shown to be in excellent agreement with the exact numerical calculations of *Cochran* based on the *von Kármán* classical theory.

Key-Words: - Theory of laminar flow near a rotating disk.

1 Introduction

The universality of turbulent phenomena from stochastic quantum fields to classical hydrodynamic fields resulted in recent introduction of a scale-invariant model of statistical mechanics and its application to the field of thermodynamics [4]. The implications of the model to the study of transport phenomena and invariant forms of conservation equations have also been addressed [5, 6]. In the present study, following the classical theory of *von Kármán*, the modified equation of motion is solved for the problem of laminar flow near a rotating disk. The predicted velocity profiles are found to be in excellent agreement with the numerical calculations of the classical theory.

2 Scale-Invariant Form of the Conservation Equations for Reactive Fields

Following the classical methods [1-3], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [4]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \tag{2}$$

Also, the invariant definitions of the peculiar and the diffusion velocities are given as [4]

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta = \mathbf{V}'_{\beta+1} \tag{3}$$

Next, following the classical methods [1-3], the scale-invariant forms of mass, thermal energy, and linear momentum conservation equations at scale β are given as [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla . (\rho_\beta \mathbf{v}_\beta) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla . (\varepsilon_\beta \mathbf{v}_\beta) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla . (\mathbf{p}_\beta \mathbf{v}_\beta) = 0 \tag{6}$$

involving the *volumetric density* of thermal energy $\varepsilon_\beta = \rho_\beta h_\beta$ and linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$. Also, Ω_β is the chemical reaction rate and h_β is the absolute enthalpy.

The local velocity $\mathbf{v}_\beta$ in (4)-(6) is expressed as the sum of convective $\mathbf{w}_\beta = \langle \mathbf{v}_\beta \rangle$ and diffusive $\mathbf{V}_\beta$ velocities [5]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{7a}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{7b}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \tag{7c}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$) are respectively the diffusive, the thermo-diffusive, the linear hydro-diffusiv velocities. For unity *Schmidt* and *Prandtl* numbers, one may express

$$\mathbf{V}_{\beta tg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta t} \ , \quad \mathbf{V}_{\beta t} = -\alpha_\beta \nabla \ln(h_\beta) \tag{8a}$$

$$\mathbf{V}_{\beta hg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta h} \ , \quad \mathbf{V}_{\beta h} = -\nu_\beta \nabla \ln(\mathbf{v}_\beta) \tag{8b}$$

that involve the thermal $\mathbf{V}_{\beta t}$, and linear hydrodynamic $\mathbf{V}_{\beta h}$ diffusion velocities [5]. Since for an ideal gas $h_\beta = c_{p\beta} T_\beta$, when $c_{p\beta}$ is constant and $T = T_\beta$, Eq.(8a) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_\beta = \rho_\beta h_\beta \mathbf{V}_{\beta t} = -\kappa_\beta \nabla T \tag{9}$$

where κ_β and $\alpha_\beta = \kappa_\beta / (\rho_\beta c_{p\beta})$ are the thermal conductivity and diffusivity. Similarly, (8b) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [5]

$$\boldsymbol{\tau}_{ij\beta} = \rho_\beta \mathbf{v}_{j\beta} \mathbf{V}_{ij\beta h} = -\mu_\beta \partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_i \tag{10}$$

Substitutions from (7a)-(7c) into (4)-(6), neglecting cross-diffusion terms and assuming constant transport coefficients with $Sc_\beta = Pr_\beta = 1$, result in [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{11}$$

$$h_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial h_\beta}{\partial t} + \mathbf{w}_\beta . \nabla h_\beta - \alpha_\beta \nabla^2 h_\beta \right] = 0 \tag{12}$$

$$\mathbf{v}_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta \right] = 0 \tag{13}$$

In the first and second parts of Eqs.(12)-(13), the *gravitational* versus the *inertial* contributions to the change in energy and momentum density are apparent. Substitutions from (11) into (12)-(13) result in the invariant forms of conservation equations [6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{14}$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta . \nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -h_\beta \Omega_\beta / (\rho_\beta c_{p\beta}) \tag{15}$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\mathbf{v}_\beta \Omega_\beta / \rho_\beta \tag{16}$$

An important feature of the modified equation of motion (16), that is similar to the *Smoluchowski* equation [7], is that it involves a convective velocity $\mathbf{w}_\beta$ that is different from the local fluid velocity $\mathbf{v}_\beta$. Because the convective velocity $\mathbf{w}_\beta$ is not *locally-defined* it cannot occur in *differential form* within the conservation equations [5]. This is because one cannot differentiate a function that is not locally, i.e. differentially, defined. To determine $\mathbf{w}_\beta$, one needs to go to the next higher scale (β+1) where $\mathbf{w}_\beta = \mathbf{v}_{\beta+1}$ becomes a local velocity. However, at this new scale one encounters yet another convective velocity $\mathbf{w}_{\beta+1}$ which is not known, requiring consideration of the higher scale (β+2). This unending chain constitutes the *closure problem* of the statistical theory of turbulence discussed earlier [5].

3 A Modified Theory of Laminar Flow Near a Rotating Disk

As examples of exact solutions of the modified equation of motion (16), the classical *Blasius* problem [2] of laminar flow over a flat plate [8], laminar boundary layer flow adjacent to an axisymmetric stagnation-point [9], laminar free convection on vertical hot plate [10], and laminar axi-symmetric and two-dimensional jets [11] have been investigated. In the present study, the modified equation of motion is solved for the classical problem of laminar flow near a rotating disk investigated by *von Kármán* [2, 12], *Cochran* [13], and *Theodorsen* and *Regier* [14]. The schematic diagram of the flow field and the axi-symmetric cylindrical coordinates is shown in Fig.1.

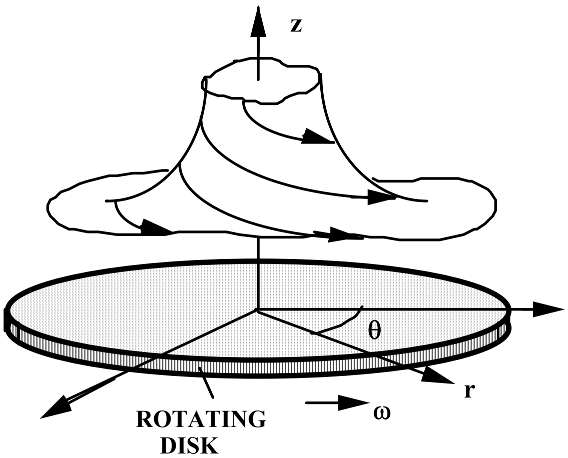

Fig.1 Laminar flow near a rotating disk.

The local axial, azimuthal, and radial velocities $(\mathbf{v}'_z, \mathbf{v}'_\theta, \mathbf{v}'_r)$ along the corresponding coordinates $(\mathbf{z}', \theta, \mathbf{r}')$ are made dimensionless in the forms

$$(\mathbf{w}_z, \mathbf{w}_r, \mathbf{v}_z, \mathbf{v}_\theta, \mathbf{v}_r) = (\mathbf{w}'_z, \mathbf{w}'_r, \mathbf{v}'_z, \mathbf{v}'_\theta, \mathbf{v}'_r)/\sqrt{\nu\omega}$$

$$r = \frac{r'}{\sqrt{\nu/\omega}} \quad , \qquad \varsigma = \frac{z'}{\sqrt{\nu/\omega}}$$

where ω is the known angular frequency of the disk and ν is the kinematic viscosity.

The steady dimensionless forms of the modified equation of motion (16) and the continuity equation (4) in axi-symmetric cylindrical coordinate for an incompressible fluid, under weak swirl, small azimuthal convective velocity $w'_\theta = 0$, and in the absence of chemical reactions $\Omega = 0$ reduce to

$$\mathbf{w}_r \frac{\partial \mathbf{v}_z}{\partial r} + \mathbf{w}_z \frac{\partial \mathbf{v}_z}{\partial \varsigma} = \frac{\partial^2 \mathbf{v}_z}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{v}_z}{\partial r} + \frac{\partial^2 \mathbf{v}_z}{\partial \varsigma^2} \tag{17}$$

$$\mathbf{w}_r \frac{\partial \mathbf{v}_r}{\partial r} + \mathbf{w}_z \frac{\partial \mathbf{v}_r}{\partial \varsigma} = \frac{\partial^2 \mathbf{v}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{v}_r}{\partial r} - \frac{\mathbf{v}_r}{r^2} + \frac{\partial^2 \mathbf{v}_r}{\partial \varsigma^2} \tag{18}$$

$$\mathbf{w}_r \frac{\partial \mathbf{v}_\theta}{\partial r} + \mathbf{w}_z \frac{\partial \mathbf{v}_\theta}{\partial \varsigma} = \frac{\partial^2 \mathbf{v}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{v}_\theta}{\partial r} - \frac{\mathbf{v}_\theta}{r^2} + \frac{\partial^2 \mathbf{v}_\theta}{\partial \varsigma^2} \tag{19}$$

$$\frac{\partial \mathbf{v}_r}{\partial r} + \frac{\mathbf{v}_r}{r} + \frac{\partial \mathbf{v}_z}{\partial \varsigma} = 0 \tag{20}$$

that are subject to the boundary conditions

$$\zeta = 0 \qquad v_r = v_z = v_\theta - r = 0 \tag{21}$$

$$\varsigma \to \infty \qquad v_r = v_\theta = 0 \tag{22}$$

Following *von Kármán* [2, 13], one assumes the similarity solutions in the form

$$v_r = r\,F(\zeta) \tag{23}$$

$$v_\theta = r\,G(\zeta) \tag{24}$$

$$v_z = H(\zeta) \tag{25}$$

To solve the system (17)-(22), it is first noted that the rotation of the disk results in an axial flow towards the disk to replace the fluid being thrown radially outward by the centrifugal forces. Therefore, the entire flow field will be divided into two distinguishable zones. (1) The outer zone with flow that is somewhat similar to the inviscid part of the classical stagnation point flow. (2) The relatively thinner inner zone that has a flow field somewhat similar to that within viscous boundary layer as schematically shown in Fig.2. The flow fields in these separate zones are similar to those discussed elsewhere in connection to stagnation-point flow in finite jets [9]. In the following the solution of the flow fields in the outer and the inner zones are discussed separately.

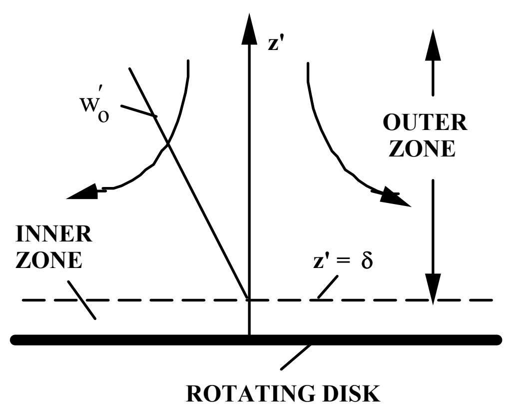

Fig.2 Schematic diagram of the inner and outer zones and outer convective velocity profile for flow near a rotating disk.

For the classical stagnation-point flow [2, 7], the radial and axial convective velocities w'_r and w'_z are [2, 7]

$$w'_r = \Gamma r' \quad , \quad w'_z = -2\Gamma(z' - \delta'/2) \tag{26}$$

where Γ is the velocity gradient and

$$\delta' = \sqrt{\nu/\Gamma} \tag{27}$$

is the viscous hydrodynamic length scale. The dimensionless form of (26) will be

$$w_r = \chi \quad , \quad w_z = -2(\xi - 1/2) \tag{28}$$

where $w = w'/\sqrt{\nu\Gamma}$ and $(\chi, \xi) = (r', z')/\sqrt{\nu/\Gamma}$. According to (26), the axial position where the convective velocity vanishes $w'_z = 0$ is displaced by $\delta'/2$ because of the presence of viscous boundary layer in accordance with the classical theory [2].

For the rotating disk flow, on the other hand, the viscous hydrodynamic length scale δ is defined as

$$\delta = \sqrt{\nu/\omega} \tag{29}$$

Hence, the angular frequency ω of the rotating disk flow in (27) will play the same role as the strain rate Γ of the forced-convection stagnation-point flow in (26) such that $\omega \Leftrightarrow \Gamma$. However, because of the passive nature of the flow in the outer region of the rotating disk, suction induced rather than forced-convection, the velocity gradient in the outer zone (Fig.2) will be smaller than that of the standard stagnation-point flow in (26). Therefore, the convective velocity for the outer region of the rotating disk flow will be taken as

$$w'_{ro} = \omega r'/4 \quad , \quad w'_{zo} = -\omega(z' - \delta)/2 \tag{30}$$

corresponding to a factor of 4 reduction in strain rates $\Gamma = \omega/4$. The reduction of the strain rate in (30) is like stretching of the axial coordinate, and hence the thickness of the entire viscous flow field, by a factor of 4. According to (30), the position of the stagnation plane where the outer convective velocity vanishes $w'_{zo} = 0$ is displaced by the thickness δ as shown in Fig.2. The dimensionless form of the convective velocity in the outer zone is obtained from (30) as

$$w_{ro} = r/4 \quad , \quad w_{zo} = -(\varsigma - 1)/2 \tag{31}$$

The flow field in the outer zone is now investigated by substitutions from (23), (25), and (31) into (17), (19), and (20) to obtain

$$H''_o + \frac{1}{2}(\varsigma - 1)H'_o = 0 \tag{32}$$

$$G''_o + \frac{1}{2}(\varsigma - 1)G'_o = 0 \tag{33}$$

$$2F_o + H'_o = 0 \tag{34}$$

subject to the boundary conditions

$$\begin{aligned} &\zeta = 1/4 && H_o = 0 \\ &\zeta = 1 && F'_o = G_o - a_o = 0 \\ &\varsigma \to \infty && F_o = G_o = 0 \end{aligned} \tag{35}$$

where subscript (o) refers to the outer zone (Fig.2). The constant a_o will be determined from matching to the inner solution to be discussed in the following. Also, the boundary condition at ζ = 1/4 may be anticipated on the basis of the fact that the outer convective velocity w'_{zo} also vanishes at this point.

The solution of (32) and (35) is

$$H_o = -\frac{\sqrt{\pi}}{2A}\int_{1/4}^{\varsigma} \exp[-(\varsigma/2 - 1/2)^2]d\varsigma \tag{36}$$

that by (25) gives the axial velocity in the outer zone

$$v_{zo} = -\frac{\sqrt{\pi}}{2A}\int_{1/4}^{\varsigma} \exp[-(\varsigma/2 - 1/2)^2]d\varsigma \tag{37}$$

where

$$A = \int_{1/4}^{\infty} \exp[-(\varsigma/2 - 1/2)^2]d\varsigma \approx 2.4887 \tag{38}$$

The factor $(-\sqrt{\pi}/2)$ comes from matching with the error-function type solutions of the inner zone to be discussed in the following. The solution (37) gives the axial velocity in the far field away from the disk

$$\varsigma \to \infty \qquad v_{zo} = -\frac{\sqrt{\pi}}{2} = -0.886 \tag{39}$$

that is in accordance with the exact numerical calculations of *Cochran* [2, 13]. The identification of the numerically determined value of the limit –0.886

[13] with $(-\sqrt{\pi}/2)$ is a mathematical evidence that error-function type solutions are indeed intrinsic to this problem. On the other hand, the behavior of the outer axial velocity (37) close to the edge of the "boundary layer" becomes

$$\varsigma \to 1 \qquad \frac{dv_{zo}}{d\varsigma} = -\frac{\sqrt{\pi}}{2A} \tag{40}$$

The slope in (40) will be matched with that within the inner zone (58) to be discussed later.

The radial velocity in the outer zone is considered next by substituting (36) into the continuity equation (34) to obtain

$$F_o = \frac{\sqrt{\pi}}{4A}\exp[-(\varsigma/2-1/2)^2] \tag{41}$$

that by (23) gives

$$v_{ro} = \frac{r\sqrt{\pi}}{4A}\exp[-(\varsigma/2-1/2)^2] \tag{42}$$

The limits of the above solution in the far field and near the edge of the inner zone are

$$\varsigma \to \infty \qquad v_{ro}(\infty) = 0 \tag{43}$$

$$\varsigma \to 1 \qquad v_{ro}(0) = \frac{r\sqrt{\pi}}{4A} \tag{44}$$

that satisfy (35). The radial velocity for the inner zone (56) to be discussed in the following must match the result (44).

Next, the azimuthal velocity is obtained from solution of (33) and (35) as

$$G_o = a_o[1-\mathrm{erf}(\varsigma/2-1/2)] \tag{45}$$

that by (24) gives

$$v_{\theta o} = r a_o[1-\mathrm{erf}(\varsigma/2-1/2)] \tag{46}$$

The constant a_o is determined from matching of the value and the slope of v_θ at the boundary between the inner and the outer zones $\zeta = 1$, expressed by (61)-(62) in the following, and found to be

$$a_o \simeq 0.639 \tag{47}$$

The convective velocity within the inner zone will be similar to the stagnation-point flow given in (28). However, the factor of 1/4 reduction in strain rate introduced in (29) is now removed leading to

$$w'_{ri} = \omega r'_i \quad , \quad w'_{zi} = -2\omega\varsigma'_i \tag{48a}$$

However, the increase of the strain rate by a factor of four results in reduction of the thickness of the viscous layer (29) by a factor of 2 such that

$$\delta_i = \delta/2 \quad , \quad \varsigma_i = 2\varsigma \tag{48b}$$

Substituting from (23)-(25), and (48a,b) into (18), and (20) and invoking the "boundary layer" assumption

$$\frac{\partial v_r}{\partial r} \ll \frac{\partial v_r}{\partial \varsigma} \tag{49}$$

result in equations

$$F_i'' + 8\varsigma F_i' = 0 \tag{50}$$

$$2F_i + H_i' = 0 \tag{51}$$

that are subject to the boundary conditions

$$\zeta = 0 \qquad F_i = H_i = 0 \tag{52}$$

$$\varsigma \to 1 \qquad F_i - \frac{\sqrt{\pi}}{4A} = H_i' + \frac{\sqrt{\pi}}{2A} = 0 \tag{53}$$

The result (53) is obtained by the requirement of matching with the outer solutions in (44).

The solution of (50), and (52)-(53) is

$$F_i = \frac{\sqrt{\pi}}{4A}\mathrm{erf}(2\varsigma) \tag{54}$$

that by (23) gives the radial velocity in the inner zone as

$$v_{ri} = \frac{r\sqrt{\pi}}{4A}\mathrm{erf}(2\varsigma) \tag{55}$$

As one approaches the edge of the inner zone, $\varsigma_i \to \infty$, the radial velocity (55) becomes

$$\varsigma \to 1 \qquad v_{ri}(\infty) = \frac{r\sqrt{\pi}}{4A} \tag{56}$$

that matches the outer solution (44).

Next, the solution of (52) and (53) after substitution from (54) is obtained as

$$H_i = -\frac{\sqrt{\pi}}{2A}\int_0^{\varsigma} \mathrm{erf}(2\varsigma)d\varsigma \qquad (57)$$

that by (25) gives the axial velocity in the inner zone

$$v_{zi} = -\frac{\sqrt{\pi}}{2A}\int_0^{\varsigma} \mathrm{erf}(2\varsigma)d\varsigma \qquad (58)$$

The gradient of (58) at the edge of the inner zone matches the slope of the outer solution in (40) as required.

Finally, for the azimuthal velocity within the thin inner zone we assume a linear profile

$$G_i = 1 - a_i\varsigma \qquad (59)$$

that by (24) leads to

$$v_{\theta i} = r(1 - a_i\varsigma) \qquad (60)$$

The constant a_i is determined from matching of the value and the slope of v_θ at $\zeta = 1$

$$[v_{\theta o}]_{\varsigma=1} = [v_{\theta i}]_{\varsigma=1} \qquad (61)$$

$$\left[\frac{dv_{\theta o}}{d\varsigma}\right]_{\varsigma=1} = \left[\frac{dv_{\theta i}}{d\varsigma}\right]_{\varsigma=1} \qquad (62)$$

as

$$a_i = 0.361 \qquad (63)$$

The constants a_o in (46) was also determined from the matching conditions (61)-(62).

The calculated velocity distributions for the outer and inner zones from the solutions (37), (42), (46), (55), (58), and (60) using the *Mathematica* [15] are shown in Fig.3 and are in excellent agreement with Fig.5.12 of *Schlichting* [2] that is based on the exact numerical calculations of *Cochran* [13]. This close agreement may be considered as evidence that the modified theory presented herein has indeed captured the essential elements of this complex flow field.

The results (59) and (63) are now used to predict the moment experienced by a rotating disk of radius R wetted on both sides. This is given in terms of the dimensionless moment coefficient

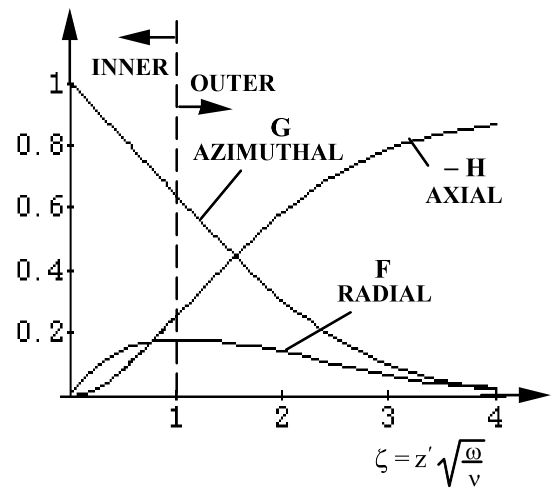

Fig.3 Calculated velocity distribution in inner and outer zones for flow near a rotating disk.

$$C_M = \frac{2M}{\frac{1}{2}\rho\omega^2 R^5} \qquad (64)$$

by the expression [2]

$$C_M = -\frac{2\pi G'(0)\nu^{1/2}}{R\omega^{1/2}} \qquad (65)$$

From (59) and (63) one obtains

$$G'(0) = -0.361 \qquad (66)$$

and its substitution in (65) results in

$$C_M = \frac{2.27}{\sqrt{R}} \qquad (67)$$

where the *Reynolds* number is defined as

$$R = \frac{R^2\omega}{\nu} \qquad (68)$$

The result (67) is to be compared with the classical finding [2]

$$C_M = \frac{3.87}{\sqrt{R}} \qquad (69)$$

that is based on the exact numerical calculations of *Cochran* [13] and is in good agreement with the experimental observations of *Theodorsen* and *Regier* [2, 14].

The quantity of fluid which is pumped outwards as a result of centrifugal forces from one side of the rotating disk of radius R is [2]

$$Q = 2\pi R \int_0^\infty v'_r dz' \tag{70}$$

After substitution from (42), (55), and (68) one obtains from (70)

$$Q = \frac{\pi^{3/2}\omega}{2AR^{1/2}} \left\{ \int_0^1 \mathrm{erf}(2\varsigma)d\varsigma + \int_1^\infty \exp[-(\varsigma/2 - 1/2)^2]d\varsigma \right\} \tag{71}$$

that is expressed as

$$Q = 0.886\pi\omega R^3 \sqrt{R} \left[\frac{A_1 + A_2}{A} \right] \tag{72}$$

where A_1 and A_2 are constants respectively representing the integrals on the right-hand-side of (70)

$$A_1 = \int_0^1 \mathrm{erf}(2\varsigma)d\varsigma = 0.7184 \tag{73}$$

$$A_2 = \int_1^\infty \exp[-(\varsigma/2 - 1/2)^2]d\varsigma = 1.7725 \tag{74}$$

The ratio involving the three integrals A_1, A_2, and A = 2.4887 from (38) is very close to unity

$$\frac{A_1 + A_2}{A} = 1.00088 \tag{75}$$

such that (72) becomes almost exactly the classical result of *Schlichting* [2]

$$Q = 0.886\pi\omega R^3 \sqrt{R} \tag{76}$$

By continuity equation, the quantity of the fluid flowing towards the disk in the axial direction is also equal to (76).

4 Concluding Remarks

Following the classical theory of *von Kármán,* the solution of the modified equation of motion for the classical problem of laminar flow near a rotating disk was determined. The predicted velocity profiles were found to be in excellent agreement with the exact numerical solutions. The quantity of fluid being pumped by a rotating disk of radius was found to be identical to that predicted by the classical theory. Also, the predicted moment on a rotating disk wetted on both sides was shown to be in reasonable agreement with the experimental observations and the classical findings based on numerical calculations.

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