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Viscoelastic Bilateral Contact Problem Involving Coulomb Friction Law *

ANNA OCHAL
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Nawojki 11, 30 072 Krakow
POLAND

Abstract: In this paper we study an evolution problem which describes the dynamic bilateral contact of viscoelastic body and foundation. The contact is modeled by a friction multivalued subdifferential boundary condition which involves the Coulomb law of fiction. We prove the existence and uniqueness of weak solutions to the hyperbolic variatioal inequality by using a surjectivity result for pseudomonotone operators and a fixed point argument.

Key–Words: Contact problem, variational inequality, subdifferential, Coulomb law, friction, hyperbolic, viscoelasticity.

1 Introduction

Mathematical theory of contact mechanics is a growing field in engineering and scientific computing. We deal with a model for a mechanical problem describing bilateral frictional contact between a viscoelastic body and a rigid foundation. The model consists of a hyperbolic system of equations of motion, representing momentum conservation, considered in a bounded domain subjected to mixed boundary conditions. Our main interest lies in the boundary conditions on the contact surface. The bilateral contact condition describes the situation when contact between the body and the foundation is maintained at all times. This is the case in many machines and in moving parts and components of mechanical equipment such as the contact between the piston rings and the engine block in a car and the frictional contact of the wheels with the rail when a train is braking. Mathematically there is no separation (no gap) between the body and the foundation, and the normal component of the displacement on the contact surface vanishes. We model the friction with a multivalued subdifferential boundary condition which incorporates the Coulomb law of friction.

The aim of the paper is proving the existence and uniqueness of a solution under the hypothesis that the friction coefficient is sufficiently small. It is solved by exploiting the surjectivity result for multivalued pseudomonotone operators and a fixed point argument.

The paper is organized as follows. In Section 2 we recall some notation and present some auxiliary material. In Section 3 we state the mechanical problem and describe the classical model for the process. We also derive the variational inequality formulation of the model and state the hypotheses. The statement of the main existence and uniqueness result is given in Section 4.

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2 Preliminaries and notation

We denote by $\mathcal{S}_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, or equivalently, the space $\mathbb{R}_s^{d\times d}$ of symmetric matrices of order d. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathcal{S}_d$ by

$$u \cdot v = u_i v_i, \ \|v\| = (v \cdot v)^{1/2}, \quad u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \ \|\tau\|_{\mathcal{S}_d} = (\tau : \tau)^{1/2}, \quad \sigma, \tau \in \mathcal{S}_d.$$

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The summation convention over repeated indices is used.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let n denote the outward unit normal vector to Γ. The assumption that Γ is Lipschitz ensures that n is defined a.e. on Γ. We use the following spaces

$$H = L^2(\Omega;\mathbb{R}^d), \quad \mathcal{H} = L^2(\Omega;\mathcal{S}_d),$$
$$H_1 = \{u \in H : \varepsilon(u) \in \mathcal{H}\} = H^1(\Omega;\mathbb{R}^d),$$
$$\mathcal{H}_1 = \{\tau \in \mathcal{H} : \operatorname{div}\tau \in H\},$$

where $\varepsilon\colon H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathcal{S}_d)$ and $\operatorname{div}\colon \mathcal{H}_1 \to L^2(\Omega;\mathbb{R}^d)$ denote the deformation and the divergence operators, respectively, given by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{div}\sigma = \{\sigma_{ij,j}\}$$

and the index following a comma indicates a partial derivative. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$\langle u,v\rangle_H = \int_\Omega u\cdot v\,dx, \qquad \langle \sigma,\tau\rangle_{\mathcal{H}} = \int_\Omega \sigma:\tau\,dx,$$
$$\langle u,v\rangle_{H_1} = \langle u,v\rangle_H + \langle \varepsilon(u),\varepsilon(v)\rangle_{\mathcal{H}},$$
$$\langle \sigma,\tau\rangle_{\mathcal{H}_1} = \langle \sigma,\tau\rangle_{\mathcal{H}} + \langle \operatorname{div}\sigma,\operatorname{div}\tau\rangle_H.$$

The associated norms in H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

For every $v \in H_1$ we denote by v its trace γv on Γ, where $\gamma\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ is the trace map. Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_N and v_T the usual normal and the tangential components of v on the boundary Γ $v_N = v\cdot n$, $\quad v_T = v - v_N n$. Similarly, for a regular (say C^1) tensor field $\sigma\colon \Omega \to \mathcal{S}_d$, we define its normal and tangential components by $\sigma_N = (\sigma n)\cdot n$, $\quad \sigma_T = \sigma n - \sigma_N n$.

The following surjectivity result (see [2]) for L-pseudomonotone operators will be used in our existence theorem.

Proposition 1 *If Y is a reflexive, strictly convex Banach space, $L\colon D(L) \subset Y \to Y^*$ is a linear densely defined maximal monotone operator and $T\colon Y \to 2^{Y^*}\setminus\{\emptyset\}$ is bounded coercive and pseudomonotone with respect to $D(L)$, then $L+T$ is surjective.*

3 Problem formulation

In this section we describe the classical model and then we give its variational formulation.

We consider a deformable viscoelastic body which occupies the reference configuration $\Omega \subset \mathbb{R}^d$, $d = 2,3$. We suppose that Ω is a bounded domain with Lipschitz boundary Γ and Γ is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C such that $meas(\Gamma_D) > 0$. The body is held fixed on Γ_D, so the displacement field vanishes there and we use the homogeneous Dirichlet condition on Γ_D. Volume forces of density f_1 act in Ω and the surface tractions of density f_2 are applied on Γ_N so we use the Neumann condition on Γ_N. The body may come in contact with a foundation over the potential contact surface Γ_C.

We denote by $u(x,t) = \{u_i(x,t)\}$ the displacement vector for $(x,t) \in Q = \Omega\times(0,T)$ with $0 < T < +\infty$, by $\sigma = \{\sigma_{ij}\}$ the stress tensor and by $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ the linearized (small) strain tensor. We suppose the Kelvin-Voigt viscoelastic constitutive relation

$$\sigma(u,u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)),$$

where $\mathcal{C}$ and $\mathcal{G}$ are given nonlinear and linear constitutive functions, respectively. We remark that in linear viscoelasticity the above law takes of the form

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}(u') + g_{ijkl}\varepsilon_{kl}(u),$$

where $\mathcal{C} = \{c_{ijkl}\}$ and $\mathcal{G} = \{g_{ijkl}\}$ are the viscosity and elasticity tensors, respectively.

The classical model for dynamic bilateral contact with friction is as follows: **(P)** find a displacement $u\colon Q \to \mathbb{R}^d$ and a stress field $\sigma\colon Q \to \mathcal{S}_d$ such that

$$\left\{\begin{array}{ll}
u'' - \operatorname{div}\sigma(u,u') = f_1 & \text{in } Q\\
\sigma(u,u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)) & \text{in } Q\\
u = 0 & \text{on } \Gamma_D\times(0,T)\\
\sigma n = f_2 & \text{on } \Gamma_N\times(0,T)\\
u_N = 0, & \text{on } \Gamma_C\times(0,T)\\
-\sigma_T \in \mu\, p(|R\sigma_N|)\partial\|u_T'\|_{\mathbb{R}^d} & \text{on } \Gamma_C\times(0,T)\\
u(0) = u_0, \quad u'(0) = u_1 & \text{in } \Omega.
\end{array}\right.$$

Here, for the sake of simplicity, the material density is assumed constant and set equal to one. We set the problem (P) in a variational form.

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To this end we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1 \,:\, v = 0 \ \text{ on } \Gamma_D,\, v_N = 0 \ \text{ on } \Gamma_C\}\,.$$

This is a Hilbert space with the inner product and the corresponding norm given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\ \|v\| = \|\varepsilon(v)\|_{\mathcal{H}},\ \ u, v \in V.$$

From the Korn inequality $\|v\|_{H_1} \le c\|\varepsilon(v)\|_{\mathcal{H}}$ for $v \in V$ with $c > 0$, it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|$ are the equivalent norms on V. Identifying H with its dual, we obtain an evolution triple of spaces $V \subset H \subset V^*$ (cf. e.g. [10]) with dense, continuous and compact embeddings. We denote by $\langle \cdot, \cdot\rangle$ the duality of V and its dual V^*, by $\|\cdot\|_{V^*}$ the norm in V^*. We have $\langle u, v\rangle = \langle u, v\rangle_H$ for all $u \in H$ and $v \in V$.

In what follows we need the spaces $\mathcal{V} = L^2(0,T;V)$, $\widehat{\mathcal{H}} = L^2(0,T;H)$ and $\mathcal{W} = \{w \in \mathcal{V} \,:\, w' \in \mathcal{V}^*\}$, where the time derivative involved in the definition of $\mathcal{W}$ is understood in the sense of vector valued distributions. Endowed with the norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*}$, the space $\mathcal{W}$ becomes a separable reflexive Banach space. We also have $\mathcal{W} \subset \mathcal{V} \subset \widehat{\mathcal{H}} \subset \mathcal{V}^*$. The duality for the pair $(\mathcal{V}, \mathcal{V}^*)$ is denoted by $\langle\langle z, w\rangle\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle z(s), w(s)\rangle\, ds$ for $z \in \mathcal{V}^*$, $w \in \mathcal{V}$. It is well known (cf. [10]) that the embeddings $\mathcal{W} \subset C(0,T;H)$ and $\{w \in \mathcal{V} \,:\, w' \in \mathcal{W}\} \subset C(0,T;V)$ are continuous.

We suppose that the coefficient of friction μ, the force and traction densities f_1, f_2, the initial displacement and velocity u_0 and u_1 and the regularization operator R satisfy the following conditions, respectively.

$\underline{H(\mu)}$: $\ \mu \in L^\infty(\Gamma_C \times (0,T)), \mu \ge 0$ a.e. (x,t);

$\underline{H_0}$: $\ u_0 \in V$, $u_1 \in H$, $f_1 \in L^2(0,T;H)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$;

$\underline{H(R)}$: $\ R \in \mathcal{L}(H^{-1/2}(\Gamma); L^2(\Gamma))$.

The assumptions on the friction function p (cf. (8.5.8) in [7]) and the contact (superpotential) function j are as follows.

$\underline{H(p)}$: $\ p\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(1) $p(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$;

(2) $|p(x,r_1) - p(x,r_2)| \le L_p|r_1 - r_2|$ for all r_1, $r_2 \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $L_p > 0$;

(3) $p(\cdot, 0) \in L^2(\Gamma_C)$.

The viscosity and elasticity operators satisfy the following conditions.

$\underline{H(\mathcal{C})}$: $\ \mathcal{C}\colon Q \times \mathcal{S}_d \to \mathcal{S}_d$ satisfies the properties

(1) $\mathcal{C}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon \in \mathcal{S}_d$;

(2) $\mathcal{C}(\cdot,\cdot,0) \in L^2(Q, \mathcal{S}_d)$;

(3) $\|\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)\|_{\mathcal{S}_d} \le L_{\mathcal{C}}\|\varepsilon_1 - \varepsilon_2\|_{\mathcal{S}_d}$ for all ε_1, $\varepsilon_2 \in \mathcal{S}_d$, a.e. $(x,t) \in Q$ with $L_{\mathcal{C}} > 0$;

(4) $(\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge 0$ for all ε_1, $\varepsilon_2 \in \mathcal{S}_d$, a.e. $(x,t) \in Q$;

(5) $\mathcal{C}(x,t,\varepsilon) : \varepsilon \ge \alpha\|\varepsilon\|^2_{\mathcal{S}_d} - \alpha_1(x,t)\|\varepsilon\|_{\mathcal{S}_d}$ for all $\varepsilon \in \mathcal{S}_d$, a.e. $(x,t) \in Q$ with $\alpha > 0$, $\alpha_1 \in L^2(Q)$, $\alpha_1 \ge 0$.

Remark 2 *If the conditions $H(\mathcal{C})(2)$ and (3) hold, then $\|\mathcal{C}(x,t,\varepsilon)\|_{\mathcal{S}_d} \le L_{\mathcal{C}}\|\varepsilon\|_{\mathcal{S}_d} + b(x,t)$ for all $\varepsilon \in \mathcal{S}_d$, a.e. $(x,t) \in Q$, where $b(x,t) = \|\mathcal{C}(x,t,0)\|_{\mathcal{S}_d}$, $b \in L^2(Q)$, $b \ge 0$.*

In Section 4, in the second part of the proof of Theorem 5 we need the following hypothesis:

$\underline{H(\mathcal{C})_1}$: $\ \mathcal{C}\colon Q \times \mathcal{S}_d \to \mathcal{S}_d$ satisfies $H(\mathcal{C})(1)$, (2), (3) and the strong monotonicity condition $(\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge m\|\varepsilon_1 - \varepsilon_2\|^2_{\mathcal{S}_d}$ for all ε_1, $\varepsilon_2 \in \mathcal{S}_d$, a.e. $(x,t) \in Q$ with $m > 0$;

It easy to see that if the operator $\mathcal{C}$ satisfies $H(\mathcal{C})_1$, then it satisfies $H(\mathcal{C})$.

$\underline{H(\mathcal{G})}$: $\ \mathcal{G}\colon \Omega \times \mathcal{S}_d \to \mathcal{S}_d$ is of the form $\mathcal{G}(x,\varepsilon) = \mathbb{E}(x)\varepsilon$ (the Hooke law) with a symmetric and nonnegative elasticity tensor $\mathbb{E}$, i.e. $\mathbb{E} = \{G_{ijkl}\}$, $i,j,k,l = 1,\ldots,d$ with $G_{ijkl} \in L^\infty(\Omega)$, $G_{ijkl} = G_{jikl} = G_{lkij}$ and $G_{ijkl}(x)\chi_{ij}\chi_{kl} \ge 0$ for all symmetric tensors $\chi = \{\chi_{ij}\}$ and for a.e. $x \in \Omega$.

Next, for f_1 and f_2 satisfying the regularity in H_0, we define $f \in \mathcal{V}^*$ by

$$\langle f(t), v\rangle = (f_1(t), v)_H + (f_2(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)}.$$

Example 3 *Since the subdifferential of $\|\cdot\|_{\mathbb{R}^d}$ is the unit vector in the direction of ξ when $\xi \ne 0$ and it is the unit ball $B_1 = \{\xi \in \mathbb{R}^d \,:\, \|\xi\|_{\mathbb{R}^d} \le 1\}$, that is, $\partial(\|\xi\|_{\mathbb{R}^d})$ equals $\xi/\|\xi\|$ if $\xi \ne 0$ and equals B_1 if $\xi = 0$, we easily observe that*

$\|\partial(\|\xi\|_{\mathbb{R}^d})\|_{\mathbb{R}^d} \le 1$ *for all* $\xi \in \mathbb{R}^d$. *Then the contact boundary condition*

$$-\sigma_T(x,t) \in \mu(x,t)p(x,|R\sigma_N(x,t)|)\,\partial\,\|u_T'(x,t)\|_{\mathbb{R}^d} \quad (1)$$

on $\Gamma_C \times (0,T)$ *is equivalent to*

$$\begin{array}{l} \|\sigma_T\| \le \mu p(|R\sigma_N|) \text{ with} \\ \quad \|\sigma_T\| < \mu p(|R\sigma_N|) \;\Rightarrow\; u_T' = 0, \\ \quad \|\sigma_T\| = \mu p(|R\sigma_N|) \;\Rightarrow\; \exists\,\lambda \ge 0 : \\ \qquad \sigma_T = -\lambda u_T' \text{ on } \Gamma_C \times (0,T). \end{array} \quad (2)$$

In the case when p *is a known function which is independent of* $|R\sigma_N|$, *i.e.* $p(x,r) = h(x)$ *with* $h \in L^\infty(\Gamma_C)$, $h \ge 0$, p *satisfies* $H(p)$ *and the conditions (2) become the Tresca friction law (cf. Section 2.6 of [7] for a detailed discussion). If* $p(x,r) = |r|$, *then* $H(p)$ *holds and (2) reduces to the usual regularized Coulomb friction boundary condition*

$$\begin{array}{l} \|\sigma_T\| \le \mu |R\sigma_N| \text{ with} \\ \quad \|\sigma_T\| < \mu |R\sigma_N| \;\Rightarrow\; u_T' = 0, \\ \quad \|\sigma_T\| = \mu |R\sigma_N| \;\Rightarrow\; \exists\,\lambda \ge 0 : \\ \qquad \sigma_T = -\lambda u_T' \quad \text{on } \Gamma_C \times (0,T), \end{array}$$

which was extensively used in the literature (cf. e.g. [3, 7, 1, 6]). If $p(x,r) = |r|(1-\delta|r|)_+$ *with* $(\cdot)_+ = \max\{\cdot,0\}$, *where* δ *is a small positive coefficient related to the wear and hardness of the surface, then* $H(p)_1$ *holds and we obtain a modification to the Coulomb law of friction. Such a modification, called the SJK model, consists of the factor* $(1-\delta|\cdot|)_+$ *and was derived in [9] from the thermodynamical considerations. It leads to the condition*

$$\begin{array}{l} \|\sigma_T\| \le \mu |R\sigma_N|(1-\delta|R\sigma_N|)_+ \text{ with} \\ \quad \|\sigma_T\| < \mu |R\sigma_N|(1-\delta|R\sigma_N|)_+ \;\Rightarrow\; u_T' = 0, \\ \quad \|\sigma_T\| = \mu |R\sigma_N|(1-\delta|R\sigma_N|)_+ \;\Rightarrow\; \exists\,\lambda \ge 0 : \\ \qquad \sigma_T = -\lambda u_T' \quad \text{on } \Gamma_C \times (0,T). \end{array}$$

For the discussion of the SJK generalization of the Coulomb law, we refer to [9,6,7,8].

In order to obtain the variational formulation of the problem (P), we use the dynamic equations of motion in (P), multiply them by $v - u'(t)$ with $v \in V$, apply the Green formula (assuming the regularity of the functions involved) and take into account the boundary conditions on Γ. Introducing the contact functional $J\colon (0,T) \times \mathcal{H} \times V \to \mathbb{R}$ given by

$$J(t,g,z) = \int_{\Gamma_C} \mu(x,t)p(x,|R\sigma_N(x)|)\|z_T\|_{\mathbb{R}^d}\,d\Gamma, \quad (3)$$

we obtain the following variational formulation of (P): find a displacement field $u\colon (0,T) \to V$ such that

$$\left\{ \begin{array}{l} \langle u''(t), v - u'(t)\rangle + \\ \quad +(\sigma(t), \varepsilon(v) - \varepsilon(u'(t)))_{\mathcal{H}} + \\ \quad +J(t,\sigma(t),v) - J(t,\sigma(t),u'(t)) \ge \\ \quad \ge \langle f(t), v - u'(t)\rangle \quad \text{for } v \in V, \text{ a.e. } t \\ \sigma(t) = \mathcal{C}(\varepsilon(u'(t))) + \mathcal{G}(\varepsilon(u(t))) \quad \text{a.e. } t \\ u(0) = u_0, \quad u'(0) = u_1. \end{array} \right. \quad (4)$$

4 Main result

The goal of this section is to state the main existence and uniqueness result for the variational inequality (4). We also provide the properties of the data involved in this problem.

Definition 4 *A function* $u \in \mathcal{V}$ *is said to be weak solution of the problem* (P) *if and only if* $u' \in \mathcal{W}$ *and (4) holds.*

Theorem 5 *If the hypotheses* $H(\mu)$, (H_0'), $H(R)$, $H(p)$, $H(\mathcal{G})$, $H(\mathcal{C})_1$ *hold, then there is the unique weak solution to the problem* (P). *Moreover, the stress field satisfies* $\sigma \in L^2(0,T;\mathcal{H})$ *and* $\operatorname{div}\sigma \in \mathcal{V}^*$.

The proof of this theorem will be carried out in two main steps. In the first step we study the problem (4) when the stress field σ on the contact boundary Γ_C is supposed to be known. The existence and uniqueness of solutions to this auxiliary problem is obtained by employing the surjectivity result for multivalued L-pseudomonotone operators (cf. Proposition 1). In the second step of the proof we use the Banach fixed point theorem and obtain existence and uniqueness result for (4).

In the first step of the proof of Theorem 5, for every fixed $g \in L^2(0,T;\mathcal{H})$, we consider the following problem: (P_g) find $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ such that

$$\left\{ \begin{array}{l} \langle u''(t), v - u'(t)\rangle + (\sigma(t), \varepsilon(v) - \varepsilon(u'(t)))_{\mathcal{H}} + \\ \quad +J(t,g(t),v) - J(t,g(t),u'(t)) \ge \\ \quad \ge \langle f(t), v - u'(t)\rangle \quad \text{for } v \in V \text{ a.e. } t \\ \sigma(t) = \mathcal{C}(\varepsilon(u'(t))) + \mathcal{G}(\varepsilon(u(t))) \quad \text{a.e. } t \\ u(0) = u_0, \quad u'(0) = u_1. \end{array} \right.$$

Since in this problem, the stress on the contact boundary is known, by inserting $\sigma(t)$ into the inequality, we have

$$\langle u''(t) + A(t, u'(t)) + Bu(t) - f(t), v - u'(t)\rangle +$$
$$+ J(t, g(t), v) - J(t, g(t), u'(t)) \geq 0$$

for all $v \in V$ and a.e. $t \in (0, T)$, where the operators $A\colon (0,T)\times V \to V^*$ and $B\colon V \to V^*$ are defined by

$$\langle A(t,u), v\rangle = (\mathcal{C}(x,t,\varepsilon(u)), \varepsilon(v))_{\mathcal{H}}, \quad (5)$$

$$\langle Bu, v\rangle = (\mathcal{G}(x,\varepsilon(u)), \varepsilon(v))_{\mathcal{H}} \quad (6)$$

for u, $v \in V$ and $t \in (0,T)$. Hence, by the definition of the subdifferential, we obtain the following equivalent form of the problem (P_g): find $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ such that

$$(I_g)\quad \begin{cases} f(t) \in u''(t) + A(t, u'(t)) + Bu(t) + \\ +\partial J(t, g(t), u'(t)) \ \text{ a.e. } t \in (0,T) \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

Here the symbol ∂J denotes the subdifferential of J with respect to the third variable.

Thus, in order to establish the existence and uniqueness to (P_g), it is enough to study (I_g).

4.1 Auxiliary results

We establish the properties of the operators A and B defined by (5) and (6), respectively, and the contact functional given by (3).

Lemma 6 *Under the hypothesis $H(\mathcal{C})$, the operator A given by (5) satisfies*

$\underline{H(A)}$: $A\colon (0,T)\times V \to V^*$ *is such that*

(1) $A(\cdot, v)$ *is measurable on* $(0,T)$ *for all* $v \in V$;
(2) $A(\cdot, 0) \in \mathcal{V}^*$;
(3) $\|A(t,u_1) - A(t,u_2)\|_{V^*} \leq L_{\mathcal{C}}\|u_1 - u_2\|$ *for all* u_1, $u_2 \in V$, *a.e.* $t \in (0,T)$ *with* $L_{\mathcal{C}} > 0$;
(4) $A(t,\cdot)$ *is monotone;*
(5) $\langle A(t,v), v\rangle \geq \alpha\|v\|^2 - a(t)\|v\|$ *for* $v \in V$ *a.e.* t, *where* $\alpha > 0$, $a \geq 0$, $a \in L^2(0,T)$.

Under the hypothesis $H(\mathcal{C})_1$, the operator A satisfies

$\underline{H(A)_1}$: $A\colon (0,T)\times V \to V^*$ *satisfies* $H(A)(1)$, (2), (3) *and the strong monotonicity condition* $\langle A(t,u_1) - A(t,u_2), u_1 - u_2\rangle \geq m\|u_1 - u_2\|^2$ *for all* u_1, $u_2 \in V$, *a.e.* t *with* $m > 0$.

Lemma 7 *Under the assumption $H(\mathcal{G})$, the operator $B\colon V \to V^*$ defined by (6) satisfies*

$\underline{H(B)}$: $B\colon V \to V^*$ *is a bounded, linear, monotone and symmetric operator, (i.e.* $B \in \mathcal{L}(V,V^*)$, $\langle Bv, v\rangle \geq 0$ *for all* $v \in V$, $\langle Bv, w\rangle = \langle Bw, v\rangle$ *for all* v, $w \in V$*).*

We also remark that if H_0 is satisfied, then (H_0') holds, where

$\underline{(H_0')}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

Lemma 8 *Assume that $H(\mu)$ and $H(p)$ hold, then the functional J defined by (3) satisfies*

$\underline{H(J)}$: $J\colon (0,T)\times\mathcal{H}\times V \to \mathbb{R}$ *is such that*

(1) $J(\cdot, g, z)$ *is measurable on* $(0,T)$ *for all* $g \in \mathcal{H}$, $z \in V$;
(2) $J(t, g, \cdot)$ *is well defined and convex for* $t \in (0,T)$, $g \in \mathcal{H}$;
(3) $\|\partial J(t,g,z)\|_{Z^*} \leq c\|\mu\|$ *for* $(t,g,z) \in (0,T)\times\mathcal{H}\times V$, *where* $c = 2\|\bar{\gamma}\|\,\|p_g\|_{L^2(\Gamma_C)}$, $p_g(x) = p(x, |Rg_N(x)|)$, $\|\bar{\gamma}\| = \|\bar{\gamma}\|_{\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^d))}$ *and* $\|\mu\| = \|\mu\|_{L^\infty(\Gamma_C\times(0,T))}$.

For proofs of these lemmata see [4].

In order to establish the existence and uniqueness of solution to the evolution inclusion (I_g) we begin the study of (I_g) with the a priori estimate for the solutions.

Lemma 9 *Let $g \in L^2(0,T;\mathcal{H})$ be fixed, $H(A)$, $H(B)$, (H_0') hold and let u be a solution to (I_g). If $H(J)$ holds or $H(J)_1$ is satisfied with $\|\mu\|$ sufficiently small, then*

$$\begin{aligned} &\|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \leq \\ &\leq C\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*}) \end{aligned} \quad (7)$$

with a positive constant C.

Theorem 10 *Let $g \in L^2(0,T;\mathcal{H})$ and assume the hypotheses $H(A)$, $H(B)$ and (H_0'). If $H(J)$ holds, then the problem (I_g) has the unique solution.*

The main idea of the proof is based on [5], in particular on the surjectivity result for multivalued pseudomonotone operators.

The goal of this part of section is to apply the Banach fixed point theorem to the problem (I_g) and deduce the existence and uniqueness of solutions to (4). The main additional hypothesis of this section is the strong monotonicity of the operator $A(t, \cdot)$.

From the first part of the proof, we know that under the hypotheses of Theorem 10, for every $g \in L^2(0,T;\mathcal{H})$, there exists the unique $u = u_g$ solution to (I_g) such that $u_g \in \mathcal{V}$ and $u'_g \in \mathcal{W}$. Equivalently, we have that for every $g \in L^2(0,T;\mathcal{H})$, there is the unique solution u_g of the problem (P_g) with the above mentioned regularity. We take $\sigma_g(t) = \mathcal{C}\varepsilon(u'_g(t)) + \mathcal{G}\varepsilon(u_g(t))$ for a.e $t \in (0,T)$ and consider the operator $\Lambda \colon L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined by

$$\Lambda g = \sigma_g \quad \text{for } g \in L^2(0,T;\mathcal{H}).$$

We have the following

Theorem 11 *Under the hypotheses $H(\mu)$, (H_0), $H(R)$, $H(p)$, $H(\mathcal{G})$, $H(\mathcal{C})_1$, if $\|\mu\|_{L^\infty(\Gamma_C \times (0,T))}$ is sufficiently small, then the operator Λ has a unique fixed point $g^* \in L^2(0,T;\mathcal{H})$.*

Proof of Theorem 5. From the definition of the operator Λ and Theorem 11, if $\|\mu\|$ is sufficiently small, we deduce that the solution u_{g^*} of the problem (P_{g^*}) is a solution of the variational inequality (4). The uniqueness of the solution to (4) is a consequence of the uniqueness of the solution of (P_{g^*}) and the uniqueness of the fixed point of Λ. Moreover, we have following regularity of the solution and the corresponding stress tensor $u \in W^1(0,T;V) \cap C(0,T;V)$, $u' \in C(0,T;H)$, $u'' \in L^2(0,T;V^*)$ and $\sigma \in L^2(0,T;\mathcal{H})$ with div $\sigma \in L^2(0,T;V^*)$. $\square$

5 Conclusions

In the recent years a considerable progress has been achieved in the modeling and analysis of mechanical processes involved in contact between deformable bodies. The first attempts to deal with the dynamic contact problem with friction was carried out by G. Duvaut and J.-L. Lions in 1972.

The present paper presents a new technics of the proof and can be extended to more general case. It is possible to consider the contact boundary condition with a locally Lipschitz superpotential j instead of $\|u'_T\|_{\mathbb{R}^d}$.

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