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Axisymmetric Flow of a Generalized Newtonian Fluid in a Straight Pipe Using a Director Theory Approach

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Abstract: The aim of this paper is to analyze the axisymmetric unsteady flow of an incompressible generalized Newtonian fluid in a straight rigid and impermeable tube with circular cross-section of constant radius. To study this problem, we use an approach based on the Cosserat theory (also called director theory) related to fluid dynamics which reduces the exact three-dimensional equations to a system depending only on time and on a single spatial variable. From this system we obtain for a flow without swirling motion the relationship between mean pressure gradient and volume flow rate over a finite section of the pipe for the specific case of the power law viscosity function. Moreover, we compare the 3D exact solution for steady volume flow rate with the corresponding solution obtained by the Cosserat theory using nine directors.

Key–Words: Cosserat theory, nine directors, steady solution, axisymmetric motion, volume flow rate, power law viscosity function.

1 Introduction

Generalized Newtonian fluids are inelastic non-Newtonian fluids for which the viscosity is shear-dependent and that can be written as a function of the modulus of the symmetric velocity gradient. If the shear viscosity function increases with shear rate, the corresponding fluids are shear-thickening (or dilatant) while fluids with viscosity decreasing monotonically with shear rate are fluids termed shear-thinning (or pseudoplastic). Examples of non-Newtonian fluids abound in industry and nature, and include fluid suspensions, emulsions, polymeric fluids, magma, food products or biological fluids such as blood which is a complex fluid with shear-thinning behavior (see e.g. Chien et al. [4], [5]). The development and study of mathematical models for non-Newtonian fluids is a very rich field of research with many fascinating problems. We refer to the monographs [1], [22], [21] for relevant issues related to non-Newtonian fluids behavior and modeling.

In this paper we introduce a 1D model for generalized Newtonian flows in an axisymmetric pipe, based on the nine-director approach developed by Caulk and Naghdi [3]. This theory includes an additional structure of directors (deformable vectors) assigned to each point on a space curve (Cosserat curve). It is a one-dimensional theory since a 3D system of equations is replaced by a system of equations depending on time and on a single spatial variable. The idea of using directors in continuum mechanics goes back to Duhen [8] who regards a body as a collection of points together with directions associated to them. Theories based on such a model of an oriented medium were further developed by Cosserat brothers [6], [7] and have been also used by several authors in studies of rods, plates and shells (see e.g. Ericksen and Truesdell [9], Truesdell and Toupin [19], Toupin [20], Ericksen [10], Green et al. [14], [13] and Naghdi [16]). An analogous hierarchial theory for unsteady and steady flows has been developed more recently by Caulk and Naghdi [3] in straight pipes of circular cross-section and by Green and Naghdi [15] in channels. The same theory was applied to unsteady viscous fluid flow in curved pipes of circular and elliptic cross-section by Green et al. [11], [12]. Recently, a director theory approach for modeling blood flow in the arterial system, as an alternative to the classical 1D models, has been introduced by Robertson and Sequeira [18].

The relevance of using a theory of directed curves is not in regarding it as an approximation to three-dimensional equations, but rather in their use as independent theories to predict some of the main properties of the three-dimensional problems. Advantages of the director theory include: (i) the theory incorporates all components of the linear momentum; (ii) it is a hierarchical theory, making it possible to increase the accuracy of the model; (iii) there is no need for closure approximations; (iv) invariance under superposed rigid body motions is satisfied at each order and (v) the wall shear stress enters directly in the formulation as a dependent variable.

In this paper we are interested in studying the initial boundary value problem for an incompressible homogeneous power law fluid model in a straight circular rigid and impermeable tube with constant radius where the fluid velocity field, given by the director theory, can be approximated by the following finite series:[1]

$$\boldsymbol{v}^* = \boldsymbol{v} + \sum_{N=1}^{k} x_{\alpha_1} \dots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \dots \alpha_N}, \tag{1}$$

with

$$\boldsymbol{v} = v_i(z,t)\boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \dots \alpha_N} = W^i_{\alpha_1 \dots \alpha_N}(z,t)\boldsymbol{e}_i. \tag{2}$$

Here, $\boldsymbol{v}$ represents the velocity along the axis of symmetry z at time t, $x_{\alpha_1} \dots x_{\alpha_N}$ are the polynomial weighting functions with order[2] k, the vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are the director velocities which are completely symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. When we use the director theory, the 3D system of equations governing the fluid motion is replaced by a system which depends only on a single spatial and time variables, as previously mentioned. From this new system, we obtain the unsteady relationship between mean pressure gradient and volume flow rate, and the correspondent equation for the wall shear stress.

The goal of this paper is to develop a nine-director theory ($k = 3$ in equation (1)) for the steady flow of a power law fluid in a straight pipe with constant radius to compare the corresponding volume flow rate with the 3D exact solution given in [1].

[1] Throughout the paper, Latin indices subscript take the values $1, 2, 3$, Greek indices subscript $1, 2$, and the usual summation convention is employed over a repeated index.

[2] The number k identifies the order of hierarchical theory and is related to the number of directors.

2 Governing Equations

We consider a homogeneous fluid moving within a circular straight and impermeable tube, the domain Ω (see Fig.1) subset of the three-dimensional space $\mathbb{R}^3$. Its boundary $\partial\Omega$ is composed by different parts, namely the proximal cross-section Γ_1 (upstream part of the tube), the distal cross-section Γ_2 (downstream district of the tube) and the lateral wall of the tube, denoted by Γ_w.

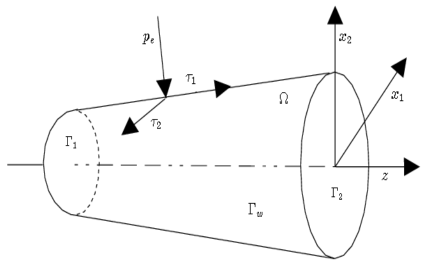

Figure 1: Fluid domain Ω with the components of the surface traction vector τ_1, τ_2 and p_e.

Let x_i $(i = 1, 2, 3)$ be the rectangular Cartesian coordinates and for convenience set $x_3 = z$. Consider the axisymmetric motion of an incompressible fluid without body forces, inside a surface of revolution, about the z axis and let $\phi(z,t)$ denote the instantaneous radius of that surface at z and time t. The three-dimensional equations governing the fluid motion are given in $\Omega' = \Omega \times (0,T)$ by[3]

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{v}^*}{\partial t} + v^*_{,i}\boldsymbol{v}^*_i\Big) = t_{i,i}, & \\ & in \;\; \Omega', \\ v^*_{i,i} = 0, & \\ t_i = -p^*\boldsymbol{e}_i + \sigma_{ij}\boldsymbol{e}_j, \; \boldsymbol{t} = \vartheta^*_i t_i, & \end{cases} \tag{3}$$

with the initial condition

$$\boldsymbol{v}^*(x,0) = \boldsymbol{v}_0(x) \;\; \text{in} \;\; \Omega, \tag{4}$$

and the boundary condition

$$\boldsymbol{v}^*(x,t) = 0 \;\; \text{on} \;\; \Gamma_w \times (0,T), \tag{5}$$

where $\boldsymbol{v}^* = v^*_i\boldsymbol{e}_i$ is the velocity field and ρ is the constant fluid density. Equation $(3)_1$ represents the balance of linear momentum and $(3)_2$ is the incompressibility condition. In equation $(3)_3$, p^*

[3] Here and in the sequel we use the notation $v^*_{i,j} = \partial v^*_i/\partial x_j$ and $v^*_{,i}\boldsymbol{v}^*_i = v^*_i \partial \boldsymbol{v}^*/\partial x_i$ adopted in Naghdi et al. [3], [11], [12].

is the pressure and σ_{ij} are the components of the extra stress tensor. For a generalized Newtonian fluid the components of the extra stress tensor are given by

$$\sigma_{ij} = \mu(|\dot{\gamma}|)\big(v^*_{i,j} + v^*_{j,i}\big), \quad i,j = 1,2,3$$

where $\dot{\gamma}$ is the shear rate and

$$\mu(|\dot{\gamma}|) : \mathbb{R}^+ \to \mathbb{R}^+$$

is the shear rate dependent viscosity function.

In the case of a power law fluid model the viscosity function is given by

$$\mu(|\dot{\gamma}|) = k|\dot{\gamma}|^{n-1} \tag{6}$$

where the parameters k and n are the consistency and the flow index (positive constants), respectively. If $n = 1$ in (6), the viscosity is a constant $\mu = k$ and the fluid is Newtonian. If $n < 1$ then

$$\lim_{|\dot{\gamma}|\to\infty} \mu(|\dot{\gamma}|) = 0, \ \lim_{|\dot{\gamma}|\to 0} \mu(|\dot{\gamma}|) = \infty,$$

and we have a shear-thinning fluid. For $n > 1$ we get

$$\lim_{|\dot{\gamma}|\to\infty} \mu(|\dot{\gamma}|) = \infty, \ \lim_{|\dot{\gamma}|\to 0} \mu(|\dot{\gamma}|) = 0,$$

and the fluid is shear-thickening. This theoretical model has limited applications to real fluids due to the unboundedness of the viscosity asymptotic limits.

In $(3)_3$, $\boldsymbol{t}$ denotes the stress vector on the surface whose outward unit normal is $\vartheta^* = \vartheta^*_i \boldsymbol{e}_i$, and t_i are the components of $\boldsymbol{t}$. The initial velocity field $\boldsymbol{v}_0$ is assumed to be known.

The lateral surface Γ_w of the axisymmetric domain is defined by

$$\phi^2 = x_\alpha x_\alpha, \tag{7}$$

and the components of the outward unit normal to this surface are

$$\vartheta^*_\alpha = \frac{x_\alpha}{\phi\big(1+\phi_z^2\big)^{1/2}}, \quad \vartheta^*_3 = -\frac{\phi_z}{\big(1+\phi_z^2\big)^{1/2}}, \tag{8}$$

where a subscript variable denotes partial differentiation. Since equation (7) defines a material surface, the velocity field must satisfy the condition

$$\phi\phi_t + \phi\phi_z v^*_3 - x_\alpha v^*_\alpha = 0 \tag{9}$$

at the boundary (7).

Let us consider $S(z,t)$ as a generic axial section of the domain at time t defined by the spatial variable z and bounded by the circle defined in (7) and let $A(z,t)$ be the area of this section $S(z,t)$. The volume flow rate Q is defined by

$$Q(z,t) = \int_{S(z,t)} v^*_3(x_1,x_2,z,t)\,da, \tag{10}$$

and the average pressure $\bar{p}$ is defined by

$$\bar{p}(z,t) = \frac{1}{A(z,t)} \int_{S(z,t)} p^*(x_1,x_2,z,t)\,da. \tag{11}$$

In what follows, this general framework will be applied to the specific case of the nine-director theory in a rigid pipe.

2.1 Cosserat Theory with Nine Directors

Starting with representation (1) it follows from [3], that the approximation of the three-dimensional velocity field $\boldsymbol{v}^* = v^*_i(x_1,x_2,z,t)\boldsymbol{e}_i$ using nine directors, is given by

$$\begin{aligned} \boldsymbol{v}^* &= \Big[\, x_1(\xi + \sigma(x_1^2+x_2^2)) - x_2(\omega + \eta(x_1^2+x_2^2)) \,\Big]\boldsymbol{e}_1 \\ &+ \Big[\, x_1(\omega + \eta(x_1^2+x_2^2)) + x_2(\xi + \sigma(x_1^2+x_2^2)) \,\Big]\boldsymbol{e}_2 \\ &+ \Big[\, v_3 + \gamma(x_1^2+x_2^2) \,\Big]\boldsymbol{e}_3 \end{aligned} \tag{12}$$

where $\xi, \omega, \gamma, \sigma, \eta$ are scalar functions of the spatial variable z and time t. The physical significance of these scalar functions in (12) is the following: γ is related to tranverse shearing motion, ω and η are related to rotational motion (also called swirling motion) about $\boldsymbol{e}_3$, while ξ and σ are related to transverse elongation.

Now, using the boundary condition (5), the velocity field (12) on the surface (7) is given by

$$\xi + \phi^2\sigma = 0, \ \ \omega + \phi^2\eta = 0, \ \ v_3 + \phi^2\gamma = 0. \tag{13}$$

The incompressibility condition $(3)_2$ applied to the velocity field (12), can be written as

$$(v_3)_z + 2\xi + x_\alpha x_\alpha\big(\gamma_z + 4\sigma\big) = 0. \tag{14}$$

For equation (14) to hold at every point in the fluid, the velocity coefficients must satisfy the conditions

$$(v_3)_z + 2\xi = 0, \ \ \gamma_z + 4\sigma = 0. \tag{15}$$

Taking into account $(13)_{1,3}$, these separate conditions (15), reduce to

$$(v_3)_z + 2\xi = 0, \ \ \big(\phi^2 v_3\big)_z = 0. \tag{16}$$

Moreover, replacing the velocity field (12) in condition (9) defined at the boundary (7), we get

$$\phi_t + \big(v_3 + \phi^2\gamma\big)\phi_z - \big(\xi + \phi^2\sigma\big)\phi = 0. \qquad (17)$$

Let us now consider flow in a rigid tube, i.e.

$$\phi = \phi(z), \qquad (18)$$

without swirling motion ($\omega = \eta = 0$). From (18) and (13) we verify that the kinematic condition (17) is satisfied identically. Conditions (10), (12), $(13)_3$ and $(16)_2$ imply that the volume flow rate Q is a function of time t, given by

$$Q(t) = \frac{\pi}{2}\,\phi^2(z)v_3(z,t). \qquad (19)$$

Then, for a flow in a rigid tube without rotation with volume flow rate (19) and verifying conditions $(13)_{1,3}$ and $(16)_1$, the velocity field (12) becomes

$$\begin{aligned} \boldsymbol{v}^* &= \Big[x_1\big(1 - \frac{x_1^2+x_2^2}{\phi^2}\big)\frac{2\phi_z Q}{\pi\phi^3}\Big]\boldsymbol{e}_1 \\ &+ \Big[x_2\big(1 - \frac{x_1^2+x_2^2}{\phi^2}\big)\frac{2\phi_z Q}{\pi\phi^3}\Big]\boldsymbol{e}_2 \\ &+ \Big[\frac{2Q}{\pi\phi^2}\big(1 - \frac{x_1^2+x_2^2}{\phi^2}\big)\Big]\boldsymbol{e}_3, \end{aligned} \qquad (20)$$

and the inicial condition (4) is satisfied, when we consider $Q(0) = ct$.

Moreover, the stress vector on the lateral surface Γ_w is given by

$$\begin{aligned} \boldsymbol{t}_w &= \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_1\phi_z - p_e x_1 \\ &- \tau_2 x_2(1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_1 \\ &+ \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_2\phi_z - p_e x_2 \\ &+ \tau_2 x_1(1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_2 \\ &+ \Big[\frac{1}{(1+\phi_z^2)^{1/2}}\Big(\tau_1 + p_e\phi_z\Big)\Big]\boldsymbol{e}_3. \end{aligned} \qquad (21)$$

Instead of satisfying the momentum equation $(3)_1$ pointwise in the fluid, we impose the following integral conditions

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)\Big]da = 0, \qquad (22)$$

$$\int_{S(z,t)} \Big[t_{i,i} - \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)\Big]x_{\alpha_1}\dots x_{\alpha_N}da = 0, \qquad (23)$$

where $N = 1, 2, \dots, k$.

Using the divergence theorem and integration by parts, equations (22) – (23) for nine directors, can be reduced to the four vector equations ($k = 3$):

$$\frac{\partial \boldsymbol{n}}{\partial z} + \boldsymbol{f} = \boldsymbol{a}, \qquad (24)$$

$$\frac{\partial \boldsymbol{m}^{\alpha_1\dots\alpha_N}}{\partial z} + \boldsymbol{l}^{\alpha_1\dots\alpha_N} = \boldsymbol{k}^{\alpha_1\dots\alpha_N} + \boldsymbol{b}^{\alpha_1\dots\alpha_N}, \qquad (25)$$

where $\boldsymbol{n}$, $\boldsymbol{k}^{\alpha_1\dots\alpha_N}$, $\boldsymbol{m}^{\alpha_1\dots\alpha_N}$ are resultant forces defined by

$$\boldsymbol{n} = \int_S \boldsymbol{t}_3 da, \quad \boldsymbol{k}^\alpha = \int_S \boldsymbol{t}_\alpha da, \qquad (26)$$

$$\boldsymbol{k}^{\alpha\beta} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta + \boldsymbol{t}_\beta x_\alpha\Big)da, \qquad (27)$$

$$\boldsymbol{k}^{\alpha\beta\gamma} = \int_S \Big(\boldsymbol{t}_\alpha x_\beta x_\gamma + \boldsymbol{t}_\beta x_\alpha x_\gamma + \boldsymbol{t}_\gamma x_\alpha x_\beta\Big)da, \qquad (28)$$

$$\boldsymbol{m}^{\alpha_1\dots\alpha_N} = \int_S \boldsymbol{t}_3 x_{\alpha_1}\dots x_{\alpha_N}da. \qquad (29)$$

The quantities $\boldsymbol{a}$ and $\boldsymbol{b}^{\alpha_1\dots\alpha_N}$ are inertia terms defined by

$$\boldsymbol{a} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)da, \qquad (30)$$

$$\boldsymbol{b}^{\alpha_1\dots\alpha_N} = \int_S \rho\Big(\frac{\partial \boldsymbol{v}^*}{\partial t} + \boldsymbol{v}^*_{,i}v^*_i\Big)x_{\alpha_1}\dots x_{\alpha_N}da, \qquad (31)$$

and $\boldsymbol{f}$, $\boldsymbol{l}^{\alpha_1\dots\alpha_N}$, which arise due to surface traction on the lateral boundary, are defined by

$$\boldsymbol{f} = \int_{\partial S}\Big(1+\phi_z^2\Big)^{1/2}\boldsymbol{t}_w ds, \qquad (32)$$

$$\boldsymbol{l}^{\alpha_1\dots\alpha_N} = \int_{\partial S}\Big(1+\phi_z^2\Big)^{1/2}\boldsymbol{t}_w x_{\alpha_1}\dots x_{\alpha_N}ds. \qquad (33)$$

These quantities will be used to calculate the equation for the average pressure as a function of the volume flow rate, using the director approach.

3 Results

In this section we compare the 3D exact solution for steady volume flow rate (see [1]) with the corresponding solution obtained by the Cosserat theory using nine directors in the case of a straight circular rigid and impermeable tube with constant radius, i.e. $\phi = ct$, for a general flow index n.

Replacing (26) – (33), obtained for the nine-director model, into equations (24) – (25) we get the unsteady relationship[4]

$$\bar{p}_z(z,t) = -\frac{4\rho\dot{Q}(t)}{3\pi\phi^2} - \frac{4k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n+1}} \tag{34}$$

and the correspondent wall shear stress

$$\tau_1 = -\frac{\rho\dot{Q}(t)}{6\pi\phi^2} - \frac{k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n}}.$$

Integrating equation (34), over a finite section of the tube, between z_1 and position z_2 ($z_1 < z_2$), we get the mean pressure gradient

$$G(t) = \frac{4\rho\dot{Q}(t)}{3\pi\phi^2} + \frac{4k\big(2^{\frac{5n+1}{2}}\big)Q^n(t)}{(n+3)\pi^n\phi^{3n+1}} \tag{35}$$

where

$$G(t) = \frac{\bar{p}(z_1,t) - \bar{p}(z_2,t)}{z_2 - z_1}.$$

From (35), the volume flow rate in the steady case is given by

$$Q_{9directors} = \frac{(n+3)^{1/n}\pi\phi^3}{2^{\frac{5n+3}{2n}}}\Big(\frac{\phi G}{2k}\Big)^{1/n}. \tag{36}$$

In order to evaluate the flow predictions of the nine-director theory developed here, we next consider the exact three-dimensional volume flow rate of an axisymmetric steady flow through a straight tube with circular cross-section of constant radius ϕ and length $z_2 - z_1$, given by (see [1])

$$Q_{3D} = \frac{\pi\phi^3}{(1/n)+3}\Big(\frac{\phi G}{2k}\Big)^{1/n}. \tag{37}$$

Shown in Fig.2 is the normalized nine-director solution (36) by the three-dimensional solution (37), given by

$$\frac{Q_{9directors}}{Q_{3D}} = \frac{(3n+1)(n+3)^{1/n}}{n\big(2^{\frac{5n+3}{2n}}\big)}, \tag{38}$$

versus the flow index n.

[4]We use the notation $\dot{Q}$ for time differentiation.

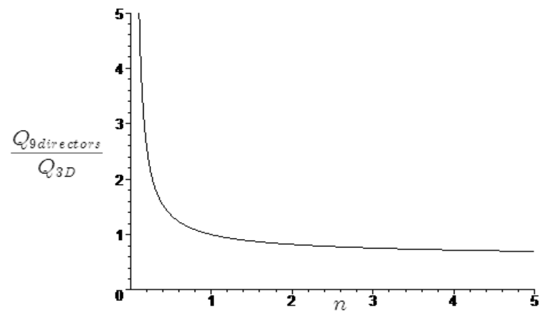

Figure 2: Normalized volume flow rate (38) as a function of the flow index n for straight circular tube with constant radius.

Finally, Table 1 shows the error obtained when solution (36) is compared with the 3D exact solution (37). For both shear-thinning and shear-thickening fluids we observe a good quantitative agreement, when the index flow n is close to one.

	shear-thinning			shear-thickening		
n	0.7	0.8	0.9	1.1	1.2	1.3
error	0.4%	0.2%	0.1%	0.1%	0.2%	0.3%

Table 1: The error of the nine-director aproximation related with the index flow n.

4 Conclusion

The predictive capability of a nine-director theory applied to study the axsymmetric unsteady flow behavior of a power law fluid in a straight pipe with uniform circular cross-section has been evaluated by comparing its solution with the 3D exact solution for steady flows. A good match of the results for a range of power-law indices close to one is in agreement with the established theory for incompressible Newtonian fluids in straight pipes ([3], [18]). This theory has strong limitations for sufficiently low and/or high index flow n. The case of a tube with non constant radius is more difficult to handle, specially for a shear thinning viscosity, due to singularities appearing in some of the integral equations. One of the important extensions of this work is the application of the Cosserat theory to blood flows in both rigid and flexible walled straight and curved vessels, and in vessels with branches or bifurcations. More detailed discussion of these issues can be found in [2].

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