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Spline approximation of fuzzy functions

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Abstract: The usefulness of fuzzy input fuzzy output functions and their interpolation/approximation by different operators in fuzzy control motivate their deeper theoretical study. We obtain some properties of fuzzy B-spline series such as variation and uncertainty diminishing property. We also propose spline approximation of fuzzy input fuzzy output functions.

Key–Words: Fuzzy numbers, Fuzzy splines, Approximation, B-splines

1 Introduction

The idea of interpolation is present in fuzzy logic from the very beginning. For example in [19] it is proposed the problem of interpolating some fuzzy data. Since then many results in this sense are obtained (see e.g. [8], [14]). Also, the idea of exploiting approximation capabilities of fuzzy systems in practice is present in the literature implicitly (not always explicitly) from the very beginning of fuzzy control (the output of a fuzzy controller can be regarded as a function that approximates some target function).

An important approach to fuzzy control is the interpolative control. As controllers of this type we mention the controllers with conditionally firing rules ([16]) and the Kóczy-Hirota, [9] interpolators based on Shepard-type operators.

In many recent papers universal approximation properties of fuzzy systems, fuzzy neural networks, fuzzy polynomials are studied (see e.g. [10], [7], [3], [12]).

Splines are widely applied in many different fields of the classical mathematics. Fuzzy spline-type interpolation or approximation is a relatively unexploited field both from theoretical and practical point of view. Fuzzy spline interpolation of crisp input fuzzy number output functions was introduced in [8], natural and complete spline approximation were studied in [1] and [2]. Also, fuzzy B-spline series were introduced in [4] and they were successfully applied to digital terrain modeling. Recently a theoretical study of the approximation by fuzzy B-spline series was given in [5]. Here, error estimates are obtained for approximation of fuzzy-number-valued functions by fuzzy B-spline series.

All the papers on fuzzy splines mentioned above have crisp input (only the values of the function are fuzzy). So naturally raises the question if we could extend the fuzzy splines to the case of fuzzy inputs. We study this question in the present paper. The most natural way to extend the fuzzy B-splines to the case of fuzzy input fuzzy output splines is to use Zadeh's extension principle.

After a preliminary section, in Section 3, we present some results on crisp input fuzzy output splines such as variation and uncertainty diminishing property, then in Section 4 we introduce and study fuzzy input fuzzy output splines. In Section 5, we present some conclusions and proposals of further research topics.

2 Preliminaries

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers, i.e. fuzzy subsets of the real axis $\mathbb{R}$ satisfying the following properties:

(i) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal i.e. $\exists x_u \in \mathbb{R}$ with $u(x_u) = 1$;

(ii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$);

(iii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semi-continuous on $\mathbb{R}$;

(iv) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$, because any real number $x_0 \in \mathbb{R}$, can be described as the fuzzy number whose value is 1 for $x = x_0$ and 0 otherwise.

For $0 < \alpha \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$ let $[u]^\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Then it is well known that for each $\alpha \in [0,1]$, $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ is a bounded closed interval ($\underline{u}^\alpha, \overline{u}^\alpha$ denote the endpoints of the $\alpha-$level set). For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we have the sum $u \oplus v$ and the product $\lambda \cdot u$ defined by $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda \cdot u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

A fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ is said to be positive if $\underline{u}^1 \geq 0$, strict positive if $\underline{u}^1 > 0$, negative if $\overline{u}^1 \leq 0$ and strict negative if $\overline{u}^1 < 0$. We say that u and v have the same sign if they are both positive or both negative. If u is positive (negative) then $\ominus u = (-1) \cdot u$ is negative (positive).

A special class of fuzzy numbers is the class of triangular fuzzy numbers. Given $a \leq b \leq c$, $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}^\alpha = a + (b-a)\alpha$ and $\overline{u}^\alpha = c - (c-b)\alpha$, for all $\alpha \in [0,1]$. Then $\underline{u}^0 = a$, $\underline{u}^1 = \overline{u}^1 = b$ and $\overline{u}^0 = c$.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$ by

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}^\alpha - \underline{v}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right|\right\}.$$

The following properties are known:

$D(u \oplus w, v \oplus w) = D(u, v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$

$D(k \cdot u, k \cdot v) = |k| D(u, v)$, $\forall u, v \in \mathbb{R}_{\mathcal{F}}$, $\forall k \in \mathbb{R}$;

$D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Also, the following is known.

Theorem 1 *(i) If we denote $\tilde{0} = \chi_{\{0\}}$ then $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is neutral element with respect to $\oplus$, i.e. $u \oplus \tilde{0} = \tilde{0} \oplus u = u$, for all $u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) With respect to $\tilde{0}$, none of $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has opposite in $\mathbb{R}_{\mathcal{F}}$ (with respect to $\oplus$).

(iii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have

$$(a + b) \cdot u = a \cdot u \oplus b \cdot u .$$

For general $a, b \in \mathbb{R}$, the above property does not hold.

(iv) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have

$$\lambda \cdot (u \oplus v) = \lambda \cdot u \oplus \lambda \cdot v .$$

(v) For any $\lambda, \mu \in \mathbb{R}$ and any $u \in \mathbb{R}_{\mathcal{F}}$, we have

$$\lambda \cdot (\mu \cdot u) = (\lambda \cdot \mu) \cdot u .$$

(vi) If we denote $\|u\|_{\mathcal{F}} = D\left(u, \tilde{0}\right)$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, then $\|\cdot\|_{\mathcal{F}}$ has the properties of an usual norm on $\mathbb{R}_{\mathcal{F}}$, i.e. $\|u\|_{\mathcal{F}} = 0$ iff. $u = \tilde{0}$, $\|\lambda \cdot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}}$ and $\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}$, $\left| \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \right| \leq D(u, v)$.

The uniform distance between fuzzy-number-valued functions is defined by

$$D(f, g) = \sup\{D(f(x), g(x) | x \in [a, b]\}$$

for $f, g : [a, b] \to \mathbb{R}_{\mathcal{F}}$.

For $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. The function $\omega(f, \cdot) : \mathbb{R}_+ \to \mathbb{R}$

$$\omega(f, \delta) = \sup\{D(f(x), f(y))$$

$$|x, y \in [a, b], |x - y| \leq \delta\}$$

is called the modulus of continuity of the fuzzy-number-valued function f.

The fuzzy splines as introduced by [8], are given below. Let S_l be the family of all splines of order l with the knots t_i, $i = 0, 1, ..., n$. Then $fs(x) = \sum_{i=0}^n s_i(x) \cdot u_i$, where $s_i \in S_l$ is the crisp spline interpolating the data (x_i, f_j), $j = 0, 1, ..., n$, where $f_j = 1$ if $i = j$ and 0 otherwise, and $u_i \in \mathbb{R}_{\mathcal{F}}$ are fuzzy constants.

3 Approximation by Fuzzy B-Spline series

Firstly, let us recall the definitions of the crisp B-splines. Let $t_0 \leq t_1 \leq .... \leq t_r$ be points in $\mathbb{R}$, with $t_r \neq t_0$. The B-spline M is given by

$$M(x) = M(x; t_0, ..., t_r) = r[t_0, ..., t_r](\cdot - x)_+^{r-1}, \quad (1)$$

where $[t_0, ..., t_r]f$ denotes the divided difference of f (see e.g. [6, p. 137]).

The B-spline N is defined by

$$N(x; t_0, ..., t_r) = \frac{1}{r}(t_r - t_0) M(x; t_0, ..., t_r). \quad (2)$$

The fuzzy B-spline series are defined as follows (see [4]).

Let $A = [a, b]$ or $A = \mathbb{R}$. Let $T = (t_i)$ be a sequence of points in A called basic knots satisfying $t_i \leq t_{i+1}$ and $t_i < t_{i+r}$, for any $t_i \in A$, $i = 0, ..., n$ if $A = [a, b]$ and $i \in \mathbb{Z}$ if $A = \mathbb{R}$. If $A = [a, b]$ we need some auxiliary knots $t_{-r+1} \leq ... \leq t_0 = a$ and $b = t_{n+1} \leq ... \leq t_{n+r}$. To a given sequence of knots corresponds a sequence of crisp B-splines $N_j(x) = N(x; t_j, ..., t_{j+r})$, for $j \in \Lambda$, where $\Lambda = \mathbb{Z}$ if $A = \mathbb{R}$

and $\Lambda = \{-r+1, ..., n\}$ if $A = [a, b]$. A fuzzy B-spline series on A ($A = \mathbb{R}$ or $A = [a, b]$) having knots in $T = (t_i)$, $i \in \Lambda$ is a function $S : A \to \mathbb{R}_{\mathcal{F}}$, of the form

$$S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j,$$

where $c_j \in \mathbb{R}_{\mathcal{F}}$.

It is easy to see that the function S is continuous in the Hausdorff distance. In what follows we present variation and uncertainty diminishing property.

Let us recall two useful properties of crisp B-splines (see e.g. [6]).

$$N_j(x) \geq 0 \text{ for } x \in [t_j, t_{j+r}]$$

$$\text{and } N(x) = 0 \text{ for } x \notin [t_j, t_{j+r}] \tag{3}$$

The following identity holds

$$\sum_{j\in\Lambda} N_j(x) = 1. \tag{4}$$

We say that a fuzzy-number-valued function changes sign in $[x_0, x_1]$ if $f(x_0)$ is negative (positive) and $f(x_1)$ is positive (negative).

Theorem 2 *A fuzzy B-spline series* $S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j$ *has the variation diminishing property, i.e.* S *changes its sign at most as many times as the sequence* $(c_j)_{j\in\Lambda}$ *changes it's sign.*

Proof. We observe that since $N_j(x) \geq 0$ for any $x \in A$ and $j \in \Lambda$, we have

$$\underline{S(x)}^1 = \sum_{j\in\Lambda} N_j(x)\underline{c_j}^1$$

and

$$\overline{S(x)}^1 = \sum_{j\in\Lambda} N_j(x)\overline{c_j}^1,$$

where $\underline{c_j}^1$ and $\overline{c_j}^1$ are the endpoints of the 1-level set of the fuzzy number $c_j \in \mathbb{R}_{\mathcal{F}}$. By variation diminishing property of crisp splines it is easy to see that $\underline{S(x)}^1$ changes its sign at most as many times as the sequence $\underline{c_j}^1$changes sign. Analogously, $\overline{S(x)}^1$ changes its sign as many times as $\overline{c_j}^1$ does. Now the conclusion of the theorem is immediate if we observe that $(c_j)_{j\in\Lambda}$ changes its sign if and only if $\underline{c_j}^1$and $\overline{c_j}^1$ change their signs. ■

Next we prove the uncertainty diminishing property. We denote by $len(u)$ the length of the support of the fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$, $i.e.\ len(u) = \overline{u}^0 - \underline{u}^0$. It is easy to see that $len(u)$ interpreted from a possibilistic point of view can be regarded as the uncertainty of the fuzzy number u.

Theorem 3 *A fuzzy B-spline series* $S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j$ *has the uncertainty diminishing property, i.e. the length of the support of* $S(x)$ *does not exceed the maximum length of the support of the fuzzy numbers* c_j*. (the length of the support of a fuzzy number can be interpreted as the uncertainty on it).*

Proof. As in the proof of the previous theorem we have

$$\underline{S(x)}^0 = \sum_{j\in\Lambda} N_j(x)\underline{c_j}^0,$$

$$\overline{S(x)}^0 = \sum_{j\in\Lambda} N_j(x)\overline{c_j}^0$$

and we get

$$\begin{aligned} len(S(x)) &= \overline{S(x)}^0 - \underline{S(x)}^0 \\ &= \sum_{j\in\Lambda} N_j(x) \cdot len(c_j) \\ &\leq \max_{j\in\Lambda} len(c_j). \end{aligned}$$

■

Remark 4 *We observe that the splines defined by [8] cannot be written as fuzzy B-spline series. Indeed, by Curry-Schoenberg theorem (see e.g. [6]), the crisp B-splines are a basis for the Schoenberg space of all splines. Let* s_i *be the splines in Definition 2. Then*

$$s_i(x) = \sum_{j\in\Lambda} N_j(x) d_{ij},$$

with $d_{ij} \in \mathbb{R}$*. Let* fs *be as in Definition 2. Then*

$$fs(x) = \sum_{i=0}^{n} s_i(x) \cdot u_i = \sum_{i=0}^{n} \left(\sum_{j\in\Lambda} N_j(x) d_{ij} \right) \cdot u_i,$$

where $u_i \in \mathbb{R}_{\mathcal{F}}$*,* $i = 0, ..., n$*. By Theorem 1, (iii) the two sums cannot be interchanged, because the splines* s_i *(and so also the coefficients* d_{ij}*) change their sign at each knot. Changing the order of the sums is possible if all* d_{ij} *have the same sign for* $j \in \Lambda$*. The same remark is true for fuzzy splines defined in [1] and [2].*

Remark 5 *Let us observe that the fuzzy B-spline series* $S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j$ *can be easily computed by using fuzzy arithmetic (i.e. addition of fuzzy numbers and multiplication of a fuzzy number by a crisp real).*

The approximation of a continuous fuzzy-number-valued function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ by some fuzzy B-spline series $S : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is proposed in [4], [11] from a practical point of view, and it was successfully applied to digital terrain modeling. The theoretical study of approximation properties is given in [5]. Let us recall some definitions and properties that will be useful in what follows. Let $f : [0,1] \to \mathbb{R}_{\mathcal{F}}$ be the target function. We consider the sequence of knots $0 < t_1 \leq .... \leq t_n < 1$, and auxiliary knots $t_{-r+1} \leq .... \leq t_0 = 0$, $1 = t_{n+1} \leq .... \leq t_{n+r}$. Let $\xi_j \in [0,1] \cap \operatorname{supp} N_j$, $j = -r+1, ..., n$ (where $\operatorname{supp} N_j = \{x \in [0,1] : N_j(x) \neq 0\}$). Then we consider the fuzzy B-spline series

$$S(f,x) = \sum_{j=-r+1}^{n} N_j(x) \cdot f(\xi_j). \qquad (5)$$

Let also $\delta = \max_{0 \leq j \leq n}(t_{j+1} - t_j)$.

We observe that everything is also valid for $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. Approximation properties of fuzzy B-spline series are given in the following theorem.

Theorem 6 *For $f : [0,1] \to \mathbb{R}_{\mathcal{F}}$ continuous we have:*

$$D(f(x), S(f,x)) \leq r\omega(f,\delta),$$

where $\omega(f,\delta)$ is the modulus of continuity of the function f.

Better estimates can be obtained for fuzzy splines of Schoenberg type (for crisp Schoenberg splines see e.g. [17], [15]). Let the knots and auxiliary knots given as above, and $\xi_j = \frac{t_{j+1}+...+t_{j+r-1}}{r-1}$, $j = -r+1, ..., n$. We define the fuzzy spline of Schoenberg type

$$S(f,x) = \sum_{j=-r+1}^{n} N_j(x) \cdot f(\xi_j).$$

If we have given fuzzy data $(\xi_j, f(\xi_j))$, $j = -r+1, ..., n$, ξ_j being the nodes, then the sequence of knots t_j, $j = -r+1, ..., n$ considered for the fuzzy Schoenberg splines is as in the crisp case a linear functional of the nodes (see [17], [15]). If there are no basic knots in the interior of the interval $[0,1]$ then the fuzzy Schoenberg spline reduces to the fuzzy Bernstein polynomial similar to the crisp case, so the results of this paper extend the results in [12]. As in [15] we obtain the error bound in approximation by fuzzy splines of Schoenberg type:

Theorem 7 *Concerning the error in approximation by fuzzy Schoenberg splines we have:*

$$D(f(x), S(f,x)) \leq (1 + h(r,\delta))\, \omega(f,\delta), \qquad (6)$$

where

$$h(r,\delta) = \min\left\{ \frac{1}{\sqrt{2r-2}}, \sqrt{\frac{r}{12}\delta} \right\}. \qquad (7)$$

Particularly simple error bounds can be obtained for fuzzy splines of Schoenberg type with equally spaced knots and for dyadic fuzzy splines of Schoenberg type.

Corollary 8 *For fuzzy splines of Schoenberg type with equally spaced knots we have*

$$D(f(x), S(f,x)) \leq \left(1 + h\left(r, \frac{1}{n}\right)\right) \omega\left(f, \frac{1}{n}\right),$$

where $h(r,\delta)$ as in (7).

Corollary 9 *For dyadic fuzzy splines of Schoenberg type (i.e. having the knots $t_j = j \cdot 2^{-n}$, $j = 1, ..., 2^n - 1$) we have*

$$D(f(x), S(f,x)) \leq \left(1 + h\left(r, \frac{1}{2^n}\right)\right) \omega\left(f, \frac{1}{2^n}\right),$$

where $h(r,\delta)$ as in (7).

4 Fuzzy input fuzzy output spline approximation

The main problem of fuzzy control is that given a fuzzy rule base i.e. fuzzy IF-THEN rules:

$$\text{IF } \xi_j \text{ THEN } f(\xi_j)$$

find an output $f(x)$ for any input x. The fuzzy sets ξ_j are called the typical inputs, while $f(\xi_j)$ are called the typical outputs. Shepard-type operators are used in the case of Kóczy-Hirota interpolators. In this section we propose B-spline approximation as solution of the same problem. The wide applicability of fuzzy controllers motivate this study.

In order to define the fuzzy input fuzzy output B-spine series we will use Zadeh's extension principle ([19]) and the Stacking Theorem.

Theorem 10 *If $u \in \mathbb{R}_{\mathcal{F}}$ then*

(i) $[u]^\alpha$ is a closed and bounded interval for any $\alpha \in [0,1]$;

(ii) $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ implies $[u]^{\alpha_2} \subseteq [u]^{\alpha_1}$;

(iii) for any $(\alpha_n)_{n\in\mathbf{N}} \subset [0,1]$ converging increasingly to $\alpha \in [0,1]$, $\bigcap_{n\in\mathbf{N}} [u]^{\alpha_n} = [u]^\alpha$.

Conversely, if $\{M_\alpha; \alpha \in [0,1]\}$ fulfills $(i)-(iii)$ there exists an unique $u \in \mathbb{R}_{\mathcal{F}}$ such that $[u]^\alpha = M_\alpha$ for $\alpha \in (0,1]$ and $[u]^0 \subseteq M_0$.

Let K be a compact subset of $\mathbb{R}_{\mathcal{F}}$ such that any $u \in K$ has its support in the $[0,1]$ interval (we say K is lying in the $[0,1]$ interval. Let $f : K \to \mathbb{R}_{\mathcal{F}}$ be a continuous target function. We consider the sequence of knots $0 < t_1 \leq .... \leq t_n < 1$, and auxiliary knots $t_{-r+1} \leq .... \leq t_0 = 0, 1 = t_{n+1} \leq .... \leq t_{n+r}$. Let $N_j(x)$ be the crisp B-splines defined as in eq. (2). Let ξ_j be the typical inputs and $f(\xi_j)$ be the typical outputs. Let us suppose that $\xi_j \in K \cap \operatorname{supp} N_j$, $j = -r+1, ..., n$ (where $\operatorname{supp} N_j = \{x \in [0,1] : N_j(x) \neq 0\}$). Then we consider the fuzzy input fuzzy output B-spline series $S : K \to \mathbb{R}_{\mathcal{F}}$ defined levelwise

$$[S(f,u)]^\alpha = \sum_{j=-r+1}^{n} N_j([u]^\alpha) \cdot [f(\xi_j)]^\alpha, \alpha \in [0,1]. \tag{8}$$

Let also $\delta = \max_{0 \leq j \leq n}(t_{j+1} - t_j)$.

We observe that everything is also valid for K compact subset of the fuzzy reals lying in an interval $[a,b]$.

First let us prove that the function in (8) is correctly defined.

Theorem 11 *The (crisp) intervals $[S(f,u)]^r$, $r \in [0,1]$ given by eq. (8) define a fuzzy number for any $u \in K$, i.e., $S(f,u)$ is a function having inputs and outputs fuzzy numbers.*

Proof. Indeed, since N_j are continuous and since $f(\xi_j) \in \mathbb{R}_{\mathcal{F}}$, then obviously $N_j([u]^\alpha)$ and $[f(\xi_j)]^\alpha$ are closed, bounded intervals. This implies that $[S(f,u)]^\alpha$ is a closed bounded interval, i.e. condition (i) of Theorem 10 is satisfied.

Since $f(\xi_j)$ are supposed to be fuzzy numbers, they satisfy condition (ii) of Theorem 10. Also, since $[u]^{\alpha_1} \subset [u]^{\alpha_2}$ for $\alpha_1 \geq \alpha_2$ and since N_j are continuous we get $N_j([u]^{\alpha_1}) \subset N_j([u]^{\alpha_2})$ and we get $N_j([u]^{\alpha_1}) \cdot [f(\xi_j)]^{\alpha_1} \subset N_j([u]^{\alpha_2}) \cdot [f(\xi_j)]^{\alpha_2}$ for any $j \in \{-r+1, ..., n\}$. By the properties of the usual interval arithmetic it follows that condition (ii) in Theorem 10 is fulfilled.

Since N_j and f are continuous and since N_j is positive we obtain the endpoints of the r-level set $N_j([u]^\alpha) \cdot [f(\xi_j)]^\alpha$ are $N_j(\underline{u}^\alpha) \cdot \underline{f(\xi_j)}^\alpha$ and $N_j(\overline{u}^\alpha) \cdot \overline{f(\xi_j)}^\alpha$, respectively. Now, for any $(\alpha_n)_{n \in \mathbf{N}} \subset [0,1]$ converging increasingly to $\alpha \in [0,1]$ we have $N_j(\underline{u}^{\alpha_n}) \cdot \underline{f(\xi_j)}^{\alpha_n} \to N_j(\underline{u}^\alpha) \cdot \underline{f(\xi_j)}^\alpha$ and $N_j(\overline{u}^{\alpha_n}) \cdot \overline{f(\xi_j)}^{\alpha_n} \to N_j(\overline{u}^\alpha) \cdot \overline{f(\xi_j)}^\alpha$ and codition (iii) of Theorem 10 is satisfied.

Finally, by Theorem 10 we get the required conclusion. ∎

Continuity of the function $S(f,u)$ is proved in the next theorem.

Theorem 12 *The function $S : K \to \mathbb{R}_{\mathcal{F}}$ is continuous on $\mathbb{R}_{\mathcal{F}}$ equipped with the Hausdorff distance.*

Proof. Let $u, v \in K$. By direct computation we have

$$D(S(f,u), S(f,v)) =$$

$$= \sup_{\alpha \in [0,1]} \max \left\{ \left| \underline{S(u)}^\alpha - \underline{S(v)}^\alpha \right|, \left| \overline{S(u)}^\alpha - \overline{S(v)}^\alpha \right| \right\}$$

By the properties and the definition of the Hasdorff distance D we have:

$$D(S(f,u), S(f,v))$$
$$\leq \sum_{j=-r+1}^{n} \sup_{\alpha \in [0,1]} \max\{|N_j(\underline{u}^\alpha) - N_j(\underline{v}^\alpha)| \cdot |\underline{f(\xi_j)}^\alpha|,$$
$$, |N_j(\overline{u}^\alpha) - N_j(\overline{v}^\alpha)| \cdot |\overline{f(\xi_j)}^\alpha|\}.$$

It is easy to check that the above relation leads to the continuity of $S(f,u)$. ∎

5 Conclusions and Further Research

Fuzzy B-spline series are generalizations of the spline for approximation of functions having crisp inputs and fuzzy output. In this paper we have studied these functions. We have obtained variation and uncertainty diminishing property of fuzzy B-spline series. Their approximation properties together with considerations regarding possible applications to fuzzy control motivate the generalization of the splines to case when both the inputs and the outputs are fuzzy. For this case we have introduced by using Zadeh's extension principle fuzzy input fuzzy output splines and we proved that the definition is correct and that it gives a functions with inputs and outputs in the space of fuzzy numbers.

For further research we propose the study of approximation properties of these functions, together with the study of a fuzzy control algorithm based on the above defined splines.

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