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On local canonical form of boundary tangency manifolds for 2-dimensional gradient-like Morse-Smale controlled systems

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Abstract: - Boundary tangency manifolds (BTMs) of a nonlinear state equation are for constructing the level surface of an unknown control Lyapunov function generalized to gradient-like Morse-Smale controlled systems. We differentiate a Morse-Smale function repeatedly along the 'state equation'by regarding the input as an independent variable. The BTMs are the nested sequence of submanifolds obtained from these Lie-derivatives. In this paper, we show that there is a certain kind of local canonical form of BTMs on the neighborhood of a noteworthy point. If the BTMs are in local canonical form then the trajectories of the closed-loop system under a constant input has only the points of external tangency on its boundary of the defining set of the BTMs.

Key-Words: - global asymptotic stabilization, gradient-like Morse-Smale controlled system, Conley index

1 Introduction

A gradient-like Morse-Smale controlled (GLMSC) system [6] is a closed-loop system that provides a gradient-like Morse-Smale (GLMS) flow. We are developing the theory of the GLMSC systems [2, 3, 4, 5, 7, 8, 9, 10, 11, 12]. It is aimed to solve the global asymptotic stabilization problem for general nonlinear state equations (may not necessarily be feedback linearizable).

For the global asymptotic stabilization problem of a given nonlinear system, the key idea of permitting singular points to be contained in the controlled system has been already presented in the beginning of 1970's [14, 18]. However, in order to carry out the idea it has been necessary to wait for the development in the theory of Conley index [1, 17, 20]. The Conley index theory, which is a topological theory of dynamical systems, offers us various tools for analyzing the global topological structures of dynamical systems. Almost all these results were shown in the early 1990's. Our GLMSC system can be easily handled as a system that can realize the above-mentioned idea, and is based on the Conley index theory.

Boundary tangency manifolds (BTMs) of a nonlinear state equation are for constructing the level surface of a control Lyapunov function generalized to GLMSC systems in the viewpoint of differential topology [19].

We differentiate a Morse-Smale function repeatedly along that section of the vector bundle on the configuration space (see Section 2 for detail) which is defined by a given state equation. Briefly speaking, the boundary tangency manifolds of a state equation are obtained from the nested sequence of subsets of common zeros of these Lie-derivatives.

In this paper, we show the existence of local canonical form of the BTMs for a given nonlinear state equation that satisfies a suitable assumption. If the BTMs are in local canonical form then trajectories of the closed-loop system under a constant input have only the points of external tangency on its boundary of the defining set of the BTMs. Local canonical form of BTMs is closely associated with the topology of the given state equation. If we specify a topological structure to the GLMSC system, then we get significant information for constructing an arbitrary level surface of the generalized control Lyapunov function from local canonical form of BTMs.

The level surface of our generalized control Lyapunov function has a self-intersection at each saddle point of GLMSC systems. To analyze the level surface with self-intersections, we need the discussion for expressing the compact attractor of GLMSC system as a CW-complex [13]. Thus, this subject is not mentioned here. Further, in this paper, we restrict our discussion to nonlinear state equations with two-state variables and one-input.

This paper is an expansion of our paper [7]. In [7], we discussed the same notion for linear state equations.

2 Basic definitions and results

In this section, we outline the basic definitions and results for GLMSC systems from our previous papers [2, 3, 4, 5, 8, 9].

Let us consider a class C^3-nonlinear state equation with $x = (x^1, x^2) \in \mathbb{X}$ and $u \in \mathbb{U}$:

$$\dot{x} = f(x,u), \tag{1}$$

where the state space $\mathbb{X}$ is a manifold that is homeomorphic to $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$ ($\mathbb{R}$ is the real line and $\mathbb{S}^1$ is the unit circle), and the input space $\mathbb{U}$ is $\mathbb{R}$.

Let

$$\pi_u : \mathcal{Q}_0 := \mathbb{X} \times \mathbb{U} \to \mathbb{X}, \tag{2}$$

be a trivial fibered manifold over the state space $\mathbb{X}$ with the total manifold $\mathbb{X} \times \mathbb{U}$, the base manifold $\mathbb{X}$, the typical fiber $\mathbb{U}$ and the projection π_u as a surjective submersion. We call it the configuration space of state feedback control systems. We often represent the fibered manifold (2) by the total manifold $\mathcal{Q}_0$. Each point $q := (x,u) \in \mathcal{Q}_0$ is called a configuration of the control system.

Let $\mathcal{E}$ be a fibered manifold or a vector bundle. $\Gamma(\mathcal{E})$ denotes the set of all sections of $\mathcal{E}$. The map in the right hand side of a state feedback $u = k(x)$ defines a section $k \in \Gamma(\mathcal{Q}_0)$. The image $\mathcal{M}_u(k) := k(\mathbb{X})$ is a 2-dimensional submanifold of $\mathcal{Q}_0$. We call it an input manifold:

$$\mathcal{M}_u(k) := \{q \in \mathcal{Q}_0 \mid u - k(x) = 0\}. \tag{3}$$

Let $T\mathbb{X}$ be the tangent bundle over $\mathbb{X}$. Pulling back $T\mathbb{X}$ over $\mathcal{Q}_0$ by π_u, we construct a 2-dimensional vector bundle, denoted by

$$\tau : (\pi_u)^{-1}(T\mathbb{X}) =: E \to \mathcal{Q}_0. \tag{4}$$

The map f of the state equation (1) defines a section of E. We will also refer to $f \in \Gamma(E)$ as the state equation.

The restriction of $f \in \Gamma(E)$ to an input manifold $\mathcal{M}_u(k) \subset \mathcal{Q}_0$,

$$\sigma := f|\mathcal{M}_u(k), \tag{5}$$

is called a controlled system. The σ can be identified with the vector field on $\mathcal{M}_u(k)$. It is locally given by

$$\sigma(q) = f(q)|_{q=(x,k(x))}. \tag{6}$$

On the other hand, a vector filed $\hat{f} \in \mathfrak{X}(\mathbb{X})$ on $\mathbb{X}$ that makes commutative the following diagram:

$$\begin{array}{ccc} \overset{(x,u;v)}{E} & \xrightarrow{\pi_{u*}} & \overset{(x;v)}{T\mathbb{X}} \\ f\uparrow & & \uparrow\hat{f} \\ \underset{(x,u)}{\mathcal{Q}_0} & \xleftarrow{k} & \underset{(x)}{\mathbb{X}} \end{array} \tag{7}$$

is the coordinate expression of the controlled system $\sigma \in \mathfrak{X}(\mathcal{M}_u(k))$ on $\mathbb{X}$, and it is locally given by

$$\hat{f}(x) = f(x,k(x)). \tag{8}$$

We also call it a controlled system.

We denote especially the zeros of the state equation $f \in \Gamma(E)$ by $\ker(f)$:

$$\ker(f) := f^{-1}(0) = \{q \in \mathcal{Q}_0 \mid f(q) = 0\}. \tag{9}$$

A constant solution (a singular point) of a controlled system is the intersection of the input manifold $\mathcal{M}_u(k)$ and $\ker(f)$. Let $J(f)(q)$ be the Jacobian matrix of f at $q \in \mathcal{Q}_0$, and we denote its rank by $\mathrm{rank}_q(f)$. For $a = 0,1,2$, we define

$$K_a^2(f) := \{q \in \mathcal{Q}_0 \mid \dim \mathbb{R}^2 - \mathrm{rank}_q(f) = a\}.$$

$K_0^2(f)$ is the set of regular points of f. The intersection of $K_0^2(f)$ and $\ker(f)$:

$$\mathcal{N}_x(f) := \ker(f) \cap K_0^2(f) \tag{10}$$

is an one-dimensional submanifold of $\mathcal{Q}_0$. We call it the null manifold of the state equation (1). In general, $\ker(f) \setminus \mathcal{N}_x(f) \neq \varnothing$. We can classify the complement in some categories [8, 9] by using $K_1^2(f)$ and $K_2^2(f)$, which are the sets of critical points. Nevertheless, the null manifold plays an essential role in our application [3, 4, 6].

Let $\bar{q} \in \mathcal{M}_u(k)$ be a singular point of a controlled system $\sigma \in \mathfrak{X}(\mathcal{M}_u(k))$. If the derivative of $D\sigma(\bar{q})$ does not have $0 \in \mathbb{C}$ as an eigenvalue, then we say that $\bar{q}$ is simple. If $D\sigma(\bar{q})$ have no eigenvalue on the imaginary axis, then we say that $\bar{q}$ is hyperbolic. The following theorem is a basic result for us:

Theorem 1 ([3, 5]) *A constant solution $\bar{q} = (\bar{x}, \bar{u}) \in \mathcal{M}_u(k) \cap \ker(f) \subset \mathcal{Q}_0$ of a controlled system is a simple singular point if and only if $\bar{q}$ is a transversal intersection between $\mathcal{N}_x(f)$ and $\mathcal{M}_u(k)$ in $\mathcal{Q}_0$. If $\bar{q}$ is a hyperbolic point then it is a transversal intersection between $\mathcal{N}_x(f)$ and $\mathcal{M}_u(k)$ in $\mathcal{Q}_0$.* ∎

Let E^0 be the zero section of the vector bundle E. Then, $\ker(f) = \mathcal{N}_x(f)$ if and only if $f(\mathcal{Q}_0) \pitchfork E^0$ in E. Since $f \in \Gamma(E)$ is a continuous map, we have $\mathrm{cl}(\mathcal{N}_x(f)) \subset \ker(f)$. The state equation (1) will said to be simple if $\ker(f) = \mathcal{N}_x(f)$, and almost simple if $\ker(f) = \mathrm{cl}(\mathcal{N}_x(f))$. Henceforth, we suppose that the state equation is almost simple.

In general, $\mathcal{N}_x(f)$ has connected components $\mathcal{N}_x^1, \mathcal{N}_x^2, \ldots$. The union of some connected components

$$\mathcal{N}_{x:\lambda} := \mathcal{N}_x^{k_1} \cup \mathcal{N}_x^{k_2} \cup \cdots \cup \mathcal{N}_x^{k_L} \tag{11}$$

will said to be an unit component or λ-component of $\mathcal{N}_x(f)$ with the index $\lambda := (k_1, k_2, \cdots, k_L)$, if it has the following properties:

(u1) $\mathrm{cl}(\mathcal{N}_{x:\lambda})$ is the closure of a C^3 immersion of a connected 1-dimensional manifold $\mathcal{K}_\lambda$.

(u2) $\mathrm{cl}(\mathcal{N}_{x:\lambda})$ is maximal as a subset of $\ker(f)$ satisfying the condition (u1).

Let $\mathcal{N}_{x:\lambda}$ be an unit component of $\mathcal{N}_x(f)$, and

$$(x^1, x^2, u) = (\phi^1(\beta), \phi^2(\beta), \phi^3(\beta)),\ \beta \in \mathcal{K}_\lambda \quad (12)$$

be a parametric representation of $\mathrm{cl}(\mathcal{N}_{x:\lambda})$. Suppose that $\mathcal{M}_u(k)$ and $\mathcal{N}_x(f)$ intersect transversally at a point $\bar{q} \in \mathcal{M}_u(k) \cap \mathcal{N}_{x:\lambda} \subset \mathcal{Q}_0$ in $\mathcal{Q}_0$ and the point $\bar{q}$ is on an connected component $\mathcal{N}^k_{x:\lambda}$ of $\mathcal{N}_{x:\lambda}$. Then, we define the local intersection number of $\mathcal{M}_u(k)$ with $\mathcal{N}_{x:\lambda}$ at $\bar{q}$ by

$$(\mathcal{M}_u(k) \circ \mathcal{N}_{x:\lambda})_{\bar{q}} := \mathrm{sign}(\mathrm{Det}(G(\bar{q}))), \quad (13)$$

where

$$G(\bar{q}) := \begin{bmatrix} 1 & 0 & \partial_1 k(\bar{q}(\alpha)) \\ 0 & 1 & \partial_2 k(\bar{q}(\alpha)) \\ d_\beta \phi^1(\bar{q}) & d_\beta \phi^2(\bar{q}) & d_\beta \phi^3(\bar{q}) \end{bmatrix}.$$

Consider the matrix:

$$F(\bar{q}) := \begin{bmatrix} \partial_1 f^1(\bar{q}) & \partial_2 f^1(\bar{q}) & \partial_u f^1(\bar{q}) \\ \partial_1 f^2(\bar{q}) & \partial_2 f^2(\bar{q}) & \partial_u f^2(\bar{q}) \\ -\partial_1 k(\bar{q}) & -\partial_2 k(\bar{q}) & 1 \end{bmatrix},$$

obtained from $f(q) = 0$ and $u - k(x) = 0$. There exists an unique $\mu_k \in \{1, -1\}$ for each connected component $\mathcal{N}^k_{x:\lambda}$ such that

$$\mathrm{sign}(\mathrm{Det}(F(\bar{q}))) = \mu_k\, \mathrm{sign}(\mathrm{Det}(G(\bar{q}))) \quad (14)$$

at any transversal intersection $\bar{q} \in \mathcal{M}_u(k) \cap \mathcal{N}^k_{x:\lambda}$.

The Morse index of a hyperbolic point is the dimension of the unstable manifold. A hyperbolic point is denoted by Σ^+ if the Morse index is an even number, and is denoted by Σ^- if that is an odd number. Then, we have the following theorem on the relation between a local intersection number of $\mathcal{M}_u(k)$ with $\mathcal{N}_{x:\lambda}$ and the parity of the Morse index:

Theorem 2 ([5, 9]) *Let $\bar{q} \in \mathcal{N}^k_{x:\lambda} \cap \mathcal{M}_u$ be a hyperbolic point and $(\mathcal{M}_u \circ \mathcal{N}_{x:\lambda})_{\bar{q}} \in \{1, -1\}$ be the local intersection number. Then there exists an unique isomorphism*

$$\theta(\mathcal{N}^k_{x:\lambda}) : \{1, -1\} \longrightarrow \{\Sigma^+, \Sigma^-\} \quad (15)$$

for each connected component $\mathcal{N}^k_{x:\lambda}$, and the isomorphism does not depend on the position of $\bar{q}$ on $\mathcal{N}^k_{x:\lambda}$.

To probe the theorem, we define

$$\theta(\mathcal{N}_{x:\lambda})((\mathcal{M}_u(k) \circ \mathcal{N}_{x:\lambda})_{\bar{q}}) := \begin{cases} \Sigma^+ & \text{if} \quad \mu_k \cdot (\mathcal{M}_u(k) \circ \mathcal{N}_{x:\lambda})_{\bar{q}} = 1, \\ \Sigma^- & \text{if} \quad \mu_k \cdot (\mathcal{M}_u(k) \circ \mathcal{N}_{x:\lambda})_{\bar{q}} = -1, \end{cases}$$

by μ_k in (14).

3 Boundary tangency manifolds

In this section, we summarize the basic definitions and results for boundary tangency manifolds from our previous papers [10, 11].

Let $\xi \in \mathfrak{X}(\mathbb{X})$ be a vector field on a manifold $\mathbb{X}$. A flow on $\varphi : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is the set of all solution trajectories of a ξ. We say that $\bar{x}$ is a constant solution of the flow if $\varphi(\mathbb{R}, \bar{x}) = \bar{x}$ at a point $\bar{x} \in \mathbb{X}$. If there exists a function that strictly decrease along the solutions except on the constant solutions, i.e. Lyapunov function in a generalized sense, then the flow is called gradient-like.

Let φ be a gradient-like flow on $\mathbb{X}$. We say that the flow φ is gradient-like Morse-Smale (GLMS)

- if the constant solutions of φ are a finite number of hyperbolic points only, and
- if the unstable manifold $W^u(y)$ and the stable manifold $W^s(z)$ intersect transversally in $\mathbb{X}$ for any hyperbolic points $y, z \in \mathbb{X}$ [22].

Let V be a function on $\mathbb{X}$. If the negative gradient flow of $\dot{x} = -\mathrm{grad}(V)$ is GLMS, then the V is called a Morse-Smale function on $\mathbb{X}$. In general, the Lyapunov function of GLMS flow is obtained from a Morse-Smale function, and that itself is a Morse-Smale function.

Let $M \subset \mathbb{X}$ be a level set of a Morse-Smale function on the state space $\mathbb{X}$. Suppose that M is a connected submanifold with a boundary ∂M of class C^3. Let us denote restrictions of the fibered manifold $\mathcal{Q}_0$ to M and ∂M by

$$\pi|M : \mathbb{M} := M \times \mathbb{U} \to M, \quad (16)$$
$$\pi|\partial M : \partial\mathbb{M} := \partial M \times \mathbb{U} \to \partial M \quad (17)$$

respectively. We will represent these fibered manifolds by their total manifolds $\mathbb{M}$ and $\partial\mathbb{M}$.

The Poincaré-Hopf index formula for vector fields on a manifold with boundary [16] has been extended by C. C. Pugh [19] to the case that the vector field does not have uniform direction at each point on the boundary. Moreover, this result has been extended by C. McCord [15] to flows that include isolated invariant sets. The notion of boundary tangency manifolds of control systems is a generalization of the results of C. C. Pugh and C. McCord for control systems.

Let $T\mathcal{Q}_0$ be the tangent bundle and $v \in \Gamma(E)$ be a section. Then we define a map $v_* : \mathcal{Q}_0 \to T\mathcal{Q}_0$ by

$$v_*(x, u) = (x, u; v(x), 0) \in T\mathcal{Q}_0. \quad (18)$$

Now we define subsets of $\partial\mathbb{M}$:

$$\mathcal{B}^1_M := \{q \in \mathcal{B}^0_M |\ f_*(q) \in T_q(\mathcal{B}^0_M)\},$$
$$\mathcal{B}^2_M := \{q \in \mathcal{B}^1_M |\ f_*(q) \in T_q(\mathcal{B}^1_M)\},$$

where $\mathcal{B}_M^0 := \partial\mathbb{M}$, and we call them the first and second boundary tangency sets on the defining set M respectively.

We denote the intersection of $\mathcal{N}_x(f)$ and $\partial\mathbb{M}$ by

$$\mathcal{N}_x(f|\partial M) := \mathcal{N}_x(f) \cap \partial\mathbb{M}. \tag{19}$$

Since $f_*(\bar{q}) = 0$ at each point $\bar{q} \in \mathcal{N}_x(f|\partial M)$, we get $\bar{q} \in \mathcal{B}_M^1$ and $\bar{q} \in \mathcal{B}_M^2$. For $j = 1,2$, we define the jth open boundary tangency set by

$$\hat{\mathcal{B}}_M^j := \mathcal{B}_M^j \setminus \mathcal{N}_x(f|\partial M). \tag{20}$$

If they satisfy the transversality condition:

$$f_* \pitchfork_{\hat{\mathcal{B}}_M^j} T\mathcal{B}_M^j \text{ in } T\mathcal{B}_M^{j-1}|\mathcal{B}_M^j,$$

then we call $\hat{\mathcal{B}}_M^j$ the jth boundary tangency manifold (BTM), and $\{\hat{\mathcal{B}}_M^1, \hat{\mathcal{B}}_M^2\}$ the boundary tangency manifolds (BTMs). If we need to specify the state equation, we write such as $\hat{\mathcal{B}}_M^1(f)$ for $\hat{\mathcal{B}}_M^1$.

Let us define $\mathcal{R}_-^0 := \mathbb{M}$ and $\mathcal{R}_-^1 := \{q \in \partial\mathcal{R}_-^0 |\ f_*(q)$ points outward from $\mathcal{R}_-^0\}$, where $\partial\mathcal{R}_-^0 = \mathcal{B}_M^0$, and we call $\mathcal{R}_-^1$ the first exit region. Further, $\mathcal{R}_+^1 := \{q \in \partial\mathcal{R}_-^0 |\ f_*(q)$ points inward toward $\mathcal{R}_-^0\}$ is called the first entrance region. We will write such as $\mathcal{R}_-^1(f;M)$ for $\mathcal{R}_-^1$ if we need.

A subset $\mu(\partial M) \subset \mathcal{Q}_0$ is called a dissipative boundary of the control system if there exist a defining set $M \subset \mathbb{X}$ of BTMs and a section $\mu \in \Gamma(\partial\mathbb{M})$ such that

$$\mu(\partial M) \subset \mathcal{R}_+^1(f). \tag{21}$$

For an input manifold $\mathcal{M}_u(k)$ including a dissipative boundary, the subset $\mu(\partial M) \subset \mathcal{M}_u(k)$ is a level surface of a Lyapunov function defined locally on the neighbourhood of $\mu(\partial M)$. This result does not depend on the gradient and the curvature of $\mathcal{M}_u(k)$ at each $\bar{q} \in \mu(\partial M)$.

Using the above definitions, the problem to construct a level surface of a generalized control Lyapunov function is equivalent to the problem how to find a defining set M of BTMs that allow a dissipative boundary of the control system. We will follow C. C. Pugh for considering the problem, and let us define the second exit and entrance sets.

The second exit set $\mathcal{R}_-^2$ is a set of $q \in \partial(\bar{\mathcal{R}}_-^1)$ such that $f_*(q)$ points outward from $\mathcal{R}_-^1$ in the way similar to the $\mathcal{R}_-^1$, and the second entrance set $\mathcal{R}_+^2$ is a set of $q \in \partial(\bar{\mathcal{R}}_-^1)$ such that $f_*(q)$ points inward toward $\mathcal{R}_-^1$, where $\bar{R} = \mathrm{cl}(R)$.

If $\hat{\mathcal{B}}_M^1$ is the first BTM then we have

$$\partial\bar{\mathcal{R}}_-^1 = \partial\bar{\mathcal{R}}_+^1 = \mathcal{B}_M^1.$$

Further, if $\hat{\mathcal{B}}_M^2$ is the second BTM then we have

$$\mathcal{B}_M^1 = \mathcal{R}_-^2 \sqcup \mathcal{B}_M^2 \sqcup \mathcal{R}_+^2, \tag{22}$$

where $\sqcup$ means disjoint union. Let $\mathcal{M}_u^c := \mathcal{M}_u(\bar{k})$ be an input manifold such that $u = \bar{k}(x) =$ constant. It is easy to show that the intersection point of $\mathcal{M}_u^c$ and $\hat{\mathcal{B}}_M^1$ belongs to $\hat{\mathcal{B}}_M^2$ if and only if the point is a tangency point of them.

Let us suppose that $\hat{\mathcal{B}}_M^2$ is the BTM, and consider the geometrical meaning of $\mathcal{R}_-^2$ and $\mathcal{R}_+^2$. Fix orientations of $\mathcal{Q}_0$ and $\mathcal{M}_u(k)$. A point $\bar{q} \in \mathcal{M}_u(k) \cap \hat{\mathcal{B}}_M^1$ be the transversal intersection in $\mathcal{Q}_0$, and $\mathcal{M}_u^c$ be an input manifold with a constant input such that $\bar{q} \in \mathcal{M}_u^c$. Then, we can define the local intersection numbers $(\mathcal{M}_u \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$ and $(\mathcal{M}_u^c \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$ at the point $\bar{q}$ if $\mathcal{M}_u^c \pitchfork_{\bar{q}} \hat{\mathcal{B}}_M^1$ in $\mathcal{Q}_0$. Thus, we define another local intersection number $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}}$ by the following way: $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = 1$ if $(\mathcal{M}_u \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$ agrees with $(\mathcal{M}_u^c \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$, and $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = -1$ if $(\mathcal{M}_u \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$ disagrees with $(\mathcal{M}_u^c \circ \hat{\mathcal{B}}_M^1)_{\bar{q}}$. Then, we have the following result:

Theorem 3 ([5, 11]) *Suppose that $\mathcal{M}_u \pitchfork_{\bar{q}} \hat{\mathcal{B}}_M^1$ in $\mathcal{Q}_0$ at $\bar{q} \in \mathcal{M}_u \cap \hat{\mathcal{B}}_M^1$. If $\bar{q} \in \mathcal{R}_-^2$ and $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = 1$ then the controlled trajectory has $\bar{q} = (\bar{x}, k(\bar{x})) \in \mathcal{M}_u$ as a point of internal tangency, and if $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = -1$ then that has $\bar{q}$ as a point of external tangency. On the other hand, if $\bar{q} \in \mathcal{R}_+^2$ and $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = 1$ then the controlled trajectory has $\bar{q}$ as a point of external tangency, and if $(\mathcal{M}_u \bullet \hat{\mathcal{B}}_M^1)_{\bar{q}} = -1$ then that has $\bar{q}$ as a point of internal tangency.*

4 Local canonical form

In this section, we define local canonical form of BTMs, and state the result of the existence of canonical form.

If we use an input transformation:

$$u = \omega(x, \tilde{u}) \tag{23}$$

for a given state equation (1), then we denote the new state equation by

$$\dot{x} = \tilde{f}(x, \tilde{u}) := f(x, \omega(x, \tilde{u})). \tag{24}$$

We denote the inverse map of (23) by $\tilde{u} = \omega_x^{-1}(u)$ for a fixed $x \in \mathbb{X}$.

Let $M \subset \mathbb{X}$ be a defining set of BTMs, and

$$\mathcal{N}_x(f;\partial M) := \mathcal{N}_x(f) \cap \partial\mathbb{M}$$

be the intersection of $\mathcal{N}_x(f)$ and $\partial\mathbb{M}$. In general, $\mathcal{N}_x(f;\partial M)$ has several connected components.

If $\partial\mathbb{M}$ and $\mathcal{N}_x(f)$ intersect transversally then each connected component is a discrete point (it is always true for $\dim\mathbb{U} = 1$). We choose such a M and denote one of them by $q_0 = (x_0, u_0) \in \mathcal{N}_x(f; \partial M)$.

Definition 1 *For a given state equation (1), if there exist an input transformation (23) and a defining set $M \subset \mathbb{X}$ of a BTM $\hat{\mathcal{B}}^1_M$ such that the $\hat{\mathcal{B}}^1_M$ satisfies*

$$\hat{\mathcal{B}}^1_M(\tilde{f}) \cap W_{\tilde{q}_0} \subset \mathcal{R}^2_+(\tilde{f}, M) \tag{25}$$

on an open neighborhood $W_{\tilde{q}_0} \subset \partial\mathbb{M}$ of $\tilde{q}_0 := (x_0, \tilde{u}_0) = (x_0, \omega^{-1}_{x_0}(u_0)) \in \mathcal{N}_x(\tilde{f}; \partial M)$, then we say that the BTMs $\{\hat{\mathcal{B}}^1_M(\tilde{f}), \hat{\mathcal{B}}^2_M(\tilde{f})\}$ is in local canonical form at the intersection $\tilde{q}_0$.

The following result is a direct consequence of Theorem 3:

Theorem 4 *Suppose that $\{\hat{\mathcal{B}}^1_M(\tilde{f}), \hat{\mathcal{B}}^2_M(\tilde{f})\}$ is in local canonical form at a intersection $\tilde{q}_0 \in \mathcal{N}_x(\tilde{f}; M)$. Let $\mathcal{M}^c_{\tilde{u}}$ be a constant input manifold that intersects with $\hat{\mathcal{B}}^1_M(\tilde{f})$ at points that are sufficiently close to $\tilde{q}_0$. Then the controlled trajectory has each point $\tilde{q} \in \hat{\mathcal{B}}^1_M(\tilde{f}) \cap \mathcal{M}^c_{\tilde{u}}$ as a point of external tangency.*

For simplicity, we use $\partial_u := \partial/\partial u$ and $\partial_x := \partial/\partial x$. The main result is as follows:

Theorem 5 *If $\mathrm{Rank}(\partial_u f(q_0)) \neq 0$ at a point $q_0 = (x_0, u_0) \in \mathcal{N}_x(f)$ and the linear part $(A, b) := (\partial_x f(q_0), \partial_u f(q_0))$ of the state equation (1) have no uncontrollable eigenvalues on the imaginary axis, then there exists an input transformation (23), and the corresponding state equation (24) has a BTMs $\hat{\mathcal{B}}^1_M(\tilde{f})$ with local canonical form at $\tilde{q}_0$.*

proof *The proof is easy but lengthy, and we only give sketch it.*

Let $\rho_0(x) = 0$ be the implicit expression of ∂M. The gradient vector $\partial_x\rho_0(x)$ points outward from M. Let us define a vector z and a matrix Q by

$$z := \begin{bmatrix} \partial_1\rho_0(x_0) \\ \partial_2\rho_0(x_0) \end{bmatrix}, Q := \begin{bmatrix} \partial_{11}\rho_0(x_0) & \partial_{12}\rho_0(x_0) \\ \partial_{21}\rho_0(x_0) & \partial_{22}\rho_0(x_0) \end{bmatrix},$$

where $\partial_i = \partial/\partial x^i$ and $\partial_{ij} = \partial/\partial x^i \partial x^j$. Let γ_1 and γ_2 be the eigenvalues of Q. By using an input transformation (23), the eigenvalues of $F := \partial_x \tilde{f}(\tilde{q}_0)$ are specified for 0 and $\lambda \neq 0$. If the linear part (A, b) has an uncontrollable eigenvalue, then we take it to be λ. Choose $\partial_x\rho_0(x_0)$ such that $F^T z = 0$. Let $\tilde{q}_0 = (x_0, \tilde{u}_0) \in \mathcal{N}_x(\tilde{f})$ be the corresponding coordinates of $q_0 \in \mathcal{N}_x(f)$. We set $\rho_1(x, \tilde{u}) := \tilde{f}^i(x, \tilde{u})\partial_i\rho_0(x)$ and $\rho_2(x, \tilde{u}) := \tilde{f}^i(x, \tilde{u})\partial_i\rho_1(x, \tilde{u})$, where $\tilde{f}^i\partial_i\rho$ is written in summation convention, then we have $\mathcal{B}^1_M(f) = \rho_0^{-1}(0) \cap \rho_1^{-1}(0)$ and $\mathcal{B}^2_M(f) = \rho_0^{-1}(0) \cap \rho_1^{-1}(0) \cap \rho_2^{-1}(0)$. Let $\zeta = (\zeta^1, \zeta^2) := x - x_0$ and $\eta := \tilde{u} - \tilde{u}_0$ be first order of a small parameter ε, and let us expand $\rho_2(x, \tilde{u}) = \rho_2(x_0 + \zeta, \tilde{u}_0 + \eta)$ up to second order of ε. Substituting $\rho_2(x_0, \tilde{u}_0) = 0$ in that, we rewrite the remaining terms by

$$\bar{\rho}_2(\zeta, \eta) := \begin{bmatrix} \zeta^T & \eta \end{bmatrix} \begin{bmatrix} \Psi_1 & \psi_3 \\ \psi_3^T & \psi_2 \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix}.$$

Denote the symmetric matrix in the right-hand side by Ψ. Each element of Ψ includes a term depending on the second partial derivatives of $\tilde{f}$. It can be vanished by the choice of the second derivatives of (23). The Ψ after vanishing the terms has an eigenvalue of 0, and the eigenvector indicate the direction of $\mathcal{N}_x(\tilde{f})$. After some calculation, we can lead an inequality of γ_1 and γ_2 for the condition that the remaining eigenvalue of Ψ becomes positive regardless of the sign of λ. This means that

$$\bar{\rho}_2(\zeta, \eta) > 0 \tag{26}$$

for any non-zero (ζ, η). On the other hand, (26) is the condition that $\tilde{q} \in \mathcal{R}^2_+(\tilde{f}; M)$ for $\tilde{q} \in \hat{\mathcal{B}}^1_M$. Thus, we have proved the theorem. □

Let us illustrate the meaning of local canonical form of BTMs by a simple example. Consider the following linear state equation:

$$\dot{x}^1 = x^2 + u =: f^1(q),\ \dot{x}^2 = -x^1 + 2x^2 =: f^2(q).$$

The null manifold of the equation, which is homeomorphic to $\mathbb{R}$, is represented as

$$\mathcal{N}_x(f) = \{q \in \mathcal{Q}_0 \,|\, (x^1, x^2, u) = \alpha(2, 1, -1), \alpha \in \mathbb{R}\}.$$

We take a ellipse:

$$\rho_0(x) = (x^1)^2 + x^1x^2 + (x^2)^2 - 1 < 0$$

as the defining set of the BTMs. (See Figure 1 (a).) The horizontal axis of Figure 1 is angle θ in the polar coordinate representation of $\rho_0(x) = 0$. The first entrance region $\mathcal{R}^1_+(f)$ (the gray colored portions in Figure 1) has 2-connected components, thus there is no dissipative boundary with respect to the defining set. After a coordinate change $u = 19x^1 - 10x^2 + \tilde{u}$, we take an another ellipse

$$\frac{15}{128}(x^1)^2 - \frac{35}{32}x^1x^2 + \frac{135}{32}(x^2)^2 - 1 < 0$$

as a defining set of the BTMs. (See Figure 1 (b).) If we take the λ appeared in the proof of Theorem 5 as negative, then the first entrance region $\mathcal{R}^1_+$ becomes connected as shown in the figure (b). On the other hand, if we take the λ as positive then $\mathcal{R}^1_-$ becomes connected. In Figure 1, A, A', C and C' are the intersection points of $\mathcal{N}_x$ and $\partial\mathbb{M}$. The first BTM $\hat{\mathcal{B}}^1_M$ is the boundary line between $\mathcal{R}^1_+$ and $\mathcal{R}^1_-$ (uncolored portions) except these points. In Figure 1 (b), the BTMs are locally canonicalized at C and C', thus there appears no second BTMs. In the nonlinear case, we can suggest that this fact leads to global information of $\hat{\mathcal{B}}^1_M$ from local canonical form of BTMs by using the Conley index theory.

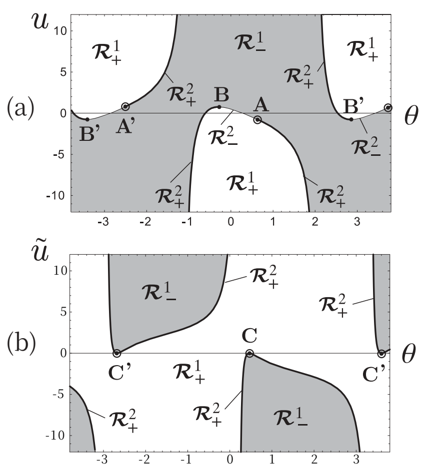

Figure 1: Examples of BTMs

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