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On the Approximation Properties of *q*-Laguerre type Modification of Meyer König and Zeller Operators

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Abstract: - In the present paper, we introduce a Laguerre type positive linear operators based on the *q*-integers including the *q*-Meyer König and Zeller operators defined by Doğru and Duman in [7]. Then we obtain some results about Korovkin type approximation properties and rates of convergence for this generalization.

Key-Words: - Positive linear operators, *q*-Meyer König and Zeller operators, *q*- Laguerre polynomials, modulus of continuity.

1 Introduction

The following operators were introduced by Meyer-König and Zeller [11]:

$$M_n^*(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)m_{n,k}(x) \quad (0\le x<1) \quad (1.1)$$

where

$$m_{n,k}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}.$$

To obtain the monotonicity properties of the operators (1.1), Cheney and Sharma [4] were introduced the following operators:

$$M_n(f;x)=\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)m_{n,k}(x) \quad (0\le x<1). \quad (1.2)$$

The operators (1.2) are also called as Bernstein power series in the literature.

A generalization of the Meyer-König and Zeller operators has been given by Doğru in [5]. Then a Stancu type generalization of the operators in [5] is defined by Agratini in [1].

Recently, in [2], Altin, Doğru and Taşdelen studied on some approximation properties of a generalization of Meyer-König and Zeller operators by generating functions.

The *q*-type generalization in approximation theory were introduced by Phillips [14] for the classical Bernstein polynomials in 1996. This generalization is obtained by replacing the general binomial expansion with *q*-binomial expansion. The rate of convergence and Voronovskaja type asymptotic estimate are obtained by Phillips and colleagues for this *q*-generalization of Bernstein polynomials. The different convergence properties of this generalization has been obtained by Goodman, Oruç and Phillips [8].

In this point, recalling some definitions about *q*-integers will be suitable:

For any fixed real number $q>0$, we denote *q*-integers by [r] where

$$[r]=\begin{cases}(1-q^r)/(1-q) & ;\text{ if } q\neq 1\\ r & ;\text{ if } q=1\end{cases}. \quad (1.3)$$

Also, *q*-binomial coefficients are defined by

$$\begin{bmatrix}n\\r\end{bmatrix}=\frac{[n]!}{[r]![n-r]!},\ r=0,1,\ldots,n,$$

where

$$[r]!=\begin{cases}[r][r-1]\ldots[1] & ;\text{ if } r=1,2,\ldots\\ 1 & ;\text{ if } r=0\end{cases}$$

and $n,r\in N_0$.

It is clear that when $q=1$, the *q*-binomial coefficients reduce to ordinary binomial coefficients.

In [15], Triff defined the Meyer-König and Zeller operators based on the *q*-integers as follows:

$$M_{n,q}(f;x)=u_{n,q}(x)\sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right)\begin{bmatrix}n+k\\k\end{bmatrix}x^k \quad (1.4)$$

for $0 \le x \le a < 1$ where

$$u_{n,q}(x) = \prod_{s=0}^{n}(1 - xq^s).$$

But, unfortunately, it is not possible to obtain the explicit formulae for the second moment of $M_{n,q}(f;x)$. Therefore, in [7] following generalization of the q-Meyer König and Zeller operators is introduced by Doğru and Duman:

$$M_n(f;q;x) = u_{n,q}(x)\sum_{k=0}^{\infty} f\left(\frac{q^n[k]}{[k+n]}\right)\begin{bmatrix} n+k \\ k \end{bmatrix} x^k \quad (1.5)$$

for $0 \le x \le a < 1$.

The A-statistical approximation properties of $M_n(f;q;x)$ are investigated in [7]. Moreover, in [7], the rates of A-statistical approximation of $M_n(f;q;x)$ to $f(x)$ are estimated by using the modulus of continuity, Peetre K-functionals and Lipschitz type maximal functions.

In this study, the q-Laguerre type positive linear operators including the operators $M_n(f;q;x)$ are defined and their Korovkin type approximation properties and rates of convergence are investigated.

2 Construction of *q*-Laguerre Type Operators

In [4], Cheney and Sharma also introduced the following operators:

$$P_n(f;x,t) = (1-x)^{n+1}\exp\left(\frac{tx}{1-x}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)L_k^{(n)}(t)x^k$$

for $0 \le x < 1, -\infty < t \le 0$ where $L_k^{(n)}(t)$ denotes the Laguerre polynomials and investigated the approximation properties of these operators. Because of $L_k^{(n)}(0) = \binom{n+k}{k}$, $M_n(f;x)$ is the special case of the operators $P_n(f;x)$.

In this part, we will define a modification of the operators $P_n(f;x)$ based on the q-integers.

The q-Laguerre polynomials have the explicit expression (see [9, p.29], [10, p.57] and [12, p.21])

$$L_n^{(\alpha)}(t;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k\, q^{\binom{k}{2}}(1-q)^k(q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k(q;q)_k} \quad (2.1)$$

where

$$(x;q)_n = \begin{cases} 1 & \text{if } n=0 \\ (1-x)(1-xq)\dots(1-xq^{n-1}) & \text{if } n \in N. \end{cases}$$

Moak [12] gave the following recurrence relation and generating function for the q-Laguerre polynomials

$$tL_{k-1}^{(\alpha+1)}(t;q) = [k+\alpha]q^{-\alpha-k}L_{k-1}^{(\alpha)}(t;q) - [k]q^{-\alpha-k}L_k^{(\alpha)}(t;q) \quad (2.2)$$

$$F_\alpha(x,t) = \frac{(xq^{\alpha+1};q)_\infty}{(x;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{m^2+\alpha m}[-(1-q)xt]^m}{(q;q)_m(xq^{\alpha+1};q)_m} = \sum_{k=0}^{\infty}x^kL_k^{(\alpha)}(t;q) \quad (2.3)$$

where $\operatorname{Re}\alpha > -1,\ k = 1,2,...$ and

$$(x;q)_\infty = \prod_{s=0}^{\infty}(1-xq^s).$$

We consider the sequence of linear positive operators

$$V_n(f;q;x,t) = \frac{1}{F_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{q^n[k]}{[k+n]}\right)L_k^{(n)}(t;q)x^k \quad (2.4)$$

where $x \in [0,1], t \in (-\infty,0], q \in (0,1]$ and $\{F_n(x,t)\}_{n\in N}$ is the generating functions for the q-Laguerre polynomials which was given in (2.3).

If we replace $f\left(\frac{[k]}{[k+n]}\right)$ by $f\left(\frac{q^n[k]}{[k+n]}\right)$ in (2.4), then these operators turns to q-Laguerre type generalization of Trif's operators which was investigated by Özarslan in [13].

Notice that, since

$$L_k^{(n)}(0;q) = \frac{(nq;q)_k}{(q;q)_k} = \begin{bmatrix} n+k \\ k \end{bmatrix}$$

and

$$\frac{1}{F_n(x,0)} = \frac{(x;q)_\infty}{(xq^{n+1};q)_\infty} = \prod_{s=0}^{n}(1-xq^s) = u_{n,q}(x)$$

then the operators in (1.5) is the special case of $V_n(f;q;x,t)$ for $t = 0$. Also note that, we have

$$V_n(f;1;x,t) = P_n(f;x,t)$$

To obtain the approximation properties of the operators $V_n(f;q;x,t)$, we need the following lemmas.

Lemma 2.1. We have

$$F_{n+1}(x,t) \le \frac{F_n(x,t)}{(1-aq^{n+1})}.$$

Proof. Since $q \le 1$ and $(xq^{n+m+1};q)_m \ge (1-aq^{n+1})$ we get

$$F_{n+1}(x,t) = \frac{(xq^{n+2};q)_\infty}{(x;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2+nm+m}[-(1-q)xt]^m}{(q;q)_m (xq^{n+2};q)_m}$$

$$= \frac{(xq^{n+1};q)_\infty}{(x;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2+nm+m}[-(1-q)xt]^m}{(q;q)_m (xq^{n+1};q)_m (1-xq^{n+m+1})}$$

$$\le \frac{(xq^{n+1};q)_\infty}{(1-aq^{n+1})(x;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2+nm}[-(1-q)xt]^m}{(q;q)_m (xq^{n+1};q)_m}$$

$$= \frac{F_n(x,t)}{(1-aq^{n+1})}.$$

Lemma 2.2. For all $n \in N$, $x \in [0,a] (0 < a < 1)$, we have

$$\left|V_n(s;q;x,t) - x\right| \le \frac{|t| x}{[n](1-xq^{n+1})} + (q^n - 1)x.$$

Proof. Using (2.2) in (2.4), we have

$$V_n(s;q;x,t) = \frac{q^n}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{[k]}{[n+k]} L_k^{(n)}(t;q) x^k$$

$$= \frac{q^n x}{F_n(x,t)} \sum_{k=1}^{\infty} \left\{ L_{k-1}^{(n)}(t;q) - \frac{q^{k+n} t}{[n+k]} L_{k-1}^{(n+1)}(t;q) \right\} x^{k-1}. \quad (2.5)$$

Since

$$\frac{-x}{F_n(x,t)} \sum_{k=0}^{\infty} \frac{q^{k+n+1} t}{[n+k+1]} L_k^{(n+1)}(t;q) x^k \ge 0$$

then

$$V_n(s;q;x,t) - x \ge (q^n - 1)x. \quad (2.6)$$

Taking into consideration $[n] \le [n+k] (n,k \in N_0)$ and $0 < q \le 1$ in (2.5) and using Lemma 2.1, we get

$$V_n(s;q;x,t) \le x - \frac{tx}{[n]} \frac{F_{n+1}(x,t)}{F_n(x,t)} \le x - \frac{tx}{[n](1-aq^{n+1})}. \quad (2.7)$$

From (2.6) and (2.7) the proof is completed.

Lemma 2.3. For all $n \in N$, $x \in [0,a] (0 < a < 1)$, we have

$$\left|V_n(s^2;q;x,t) - x^2\right| \le \frac{|t| x(1+x)}{[n](1-xq^{n+1})} + \frac{x}{[n]}. \quad (2.8)$$

Proof. From the definition of the operator one can write

$$V_n(s^2;q;x,t) = \frac{q^{2n}}{F_n(x,t)} \sum_{k=1}^{\infty} \left(\frac{[k]}{[n+k]} \right)^2 L_k^{(n)}(t;q) x^k. \quad (2.9)$$

Using the recurrence formula (2.2) twice and the fact that

$$[k] = [k-1] + q^{k-1},$$

we can prove that

$$\left(\frac{[k]}{[n+k]} \right)^2 L_k^{(n)}(t;q) = \frac{[k-1]}{[n+k-1]} L_{k-2}^{(n)}(t;q) - \frac{q^{n+k-1} t}{[n+k]} L_{k-2}^{(n+1)}(t;q) + \frac{q^{k-1}}{[n+k]} L_{k-1}^{(n)}(t;q) - \frac{q^{n+k}[k] t}{([n+k])^2} L_{k-1}^{(n+1)}(t;q).$$

So,

$$V_n(s^2;q;x,t) - x^2 \le \left(\frac{q^{2n}}{F_n(x,t)} \sum_{k=2}^{\infty} \frac{[k-1]}{[n+k-1]} L_{k-2}^{(n)}(t;q) x^k - x^2 \right) + \left| \frac{q^{2n} t}{F_n(x,t)} \sum_{k=2}^{\infty} \frac{q^{n+k-1}}{[n+k]} L_{k-2}^{(n+1)}(t;q) x^k \right| + \left| \frac{q^{2n}}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{q^{k-1}}{[n+k]} L_{k-1}^{(n)}(t;q) x^k \right| + \left| \frac{q^{2n} t}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{q^{n+k}[k]}{([n+k])^2} L_{k-1}^{(n+1)}(t;q) x^k \right|. \quad (2.10)$$

Thus the right member of (2.10) splits naturally into four parts, which we analysis separately below. Since $0 < q < 1$ and $\{[k],[n]\} \le [k+n]$, it is obvious that

$$\frac{q^{n+k}[k]}{([n+k])^2} \le \frac{1}{[n]}.$$

We get, using Lemma 2.1,

$$\left| \frac{q^{2n} t}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{q^{n+k}[k]}{([n+k])^2} L_{k-1}^{(n+1)}(t;q) x^k \right| \le \frac{|t| x}{[n](1-xq^{n+1})} \quad (2.11)$$

and

$$\frac{q^{2n}}{F_n(x,t)} \sum_{k=1}^{\infty} \frac{q^{k-1}}{[n+k]} L_{k-1}^{(n)}(t;q) x^k \le \frac{x}{[n]}. \quad (2.12)$$

In a similar manner,

$$\left| \frac{q^{2n} t}{F_n(x,t)} \sum_{k=2}^{\infty} \frac{q^{n+k-1}}{[n+k]} L_{k-2}^{(n+1)}(t;q) x^k \right| \le \frac{x^2 t}{[n](1-xq^{n+1})}. \quad (2.13)$$

Finally, since $[k-1] \le [n+k-1]$, we can write

$$\frac{q^{2n}}{F_n(x,t)} \sum_{k=2}^{\infty} \frac{[k-1]}{[n+k-1]} L_{k-2}^{(n)}(t;q) x^k - x^2 \le 0 \cdot \quad (2.14)$$

On the other hand, using the expression

$$s^2 - x^2 = (s-x)^2 + 2xs - 2x^2,$$

we may write

$$V_n(s^2;q;x,t) - x^2 = V_n((s-x)^2;q;x,t) + 2xV_n((s-x);q;x,t).$$

Thus from (2.11), (2.12), (2.13) and (2.14) we have (2.8) immediately.
In the proof of these lemmas, we used the similar technique given by Doğru in [5] (see also [6]).

3 Rate of Convergence

In this section, we will compute the rate of convergence of $V_n(f;q;x,t)$ to $f(x)$ by means of the classical modulus of continuity.
Let $f \in C[a,b]$. The modulus of continuity of f denoted by $\omega(f;\delta)$, is defined as

$$\omega(f;\delta) = \sup_{\substack{|s-x|\le\delta \\ s,x\in[a,b]}} |f(s) - f(x)|.$$

It is also well known that for any $\delta \ge 0$

$$|f(s) - f(x)| \le \omega(f,\delta)\left(\frac{|s-x|}{\delta} + 1\right). \qquad (3.1)$$

Notice that, we will use the notation $\|f\|$ instead of $\|f\|_{C[0,a]}$ for abbreviation.

Details for modulus of continuities and smoothness can be found in [3].

Theorem 3.1. For all $f \in C[0,a]$, we have

$$\|V_n(f;q,t) - f\| \le 2\omega(f;\delta_n)$$

where

$$\delta_n = \left(\frac{|t|(2a+a^2)}{[n](1-aq^{n+1})} + \frac{a}{[n]} + (q^n - 1)a\right)^{\frac{1}{2}}.$$

Proof. Let $f \in C[0,a]$. By linearity and monotonicity of $V_n(f;q;x,t)$ to according to $f(x)$ and using (3.1), we obtain

$$|V_n(f;q;x,t) - f(x)| \le V_n(|f(s) - f(x)|;q;x,t)$$

$$\le \omega(f;\delta_n)V_n\left(1 + \frac{|s-x|}{\delta_n};q;x,t\right)$$

$$= \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\frac{1}{F_n(x,t)}\sum_{k=1}^{\infty}\left|\frac{q^n[k]}{[k+n]} - x\right|L_k^{(n)}(t;q)x^k\right].$$

By the Cauchy - Schwarz inequality we have

$$|V_n(f;q;x,t) - f(x)| \le \omega(f;\delta_n)\left\{1 + \frac{1}{\delta_n}(V_n((s-x)^2;q;x,t))^{\frac{1}{2}}\right\}.$$

This implies that

$$\|V_n(f;q;.,t) - f(.)\| \le \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n} \times \sup_{x\in[0,a]}(V_n((s-x)^2;q;x,t))^{\frac{1}{2}}\right]. \qquad (3.2)$$

For each $x \in [0,a]$, one can write

$$V_n((s-x)^2;q;x,t) \le \left|V_n(s^2;q;x,t) - x^2\right| + 2x|V_n(s;q;x,t) - x|.$$

So, by Lemma 2.2 and Lemma 2.3 we get

$$\sup_{x\in[0,a]} V_n((s-x)^2;q;x,t) \le \left\|V_n(s^2;q;x,t) - x^2\right\| + 2a\|V_n(s;q;x,t) - x\| \le \frac{|t|a(2+a)}{[n](1-aq^{n+1})} + \frac{a}{[n]} + (q^n-1)a \qquad (3.3)$$

and combining (3.2) with (3.3), the proof is completed.

Remark 3.2. Since $\delta_n \to 0$ as $n \to \infty$, under the assumption $\frac{t}{[n]} \to 0$, we obtain a rate of convergence for $V_n(f;q;x,t)$ by Theorem 3.1.

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