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The Kantorovich Form of Ibragimov-Gadjiev Operators

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Abstract: - *T*he paper is devoted to study on sequence of operators representing on integral form in Kantorovich sense of Ibragimov-Gadjiev Operators. Approximation properties of these operators are established for integrable functions.

Key-Words: - Positive linear operators, Kontorovich type operators,

1 Introduction

In [7], Ibragimov-Gadjiev gave a general sequence of linear positive operators defined by

$$M_n(f;x)=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)m_{n,\nu}(x), \qquad (I)$$

where

$$m_{n,\nu}(x)=\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\frac{\left[-\alpha_n\psi_n(0)\right]^\nu}{\nu!}.$$

It is known that Ibragimov-Gadjiev operators include some well-known classical linear positive operators such as Bernstein, Bernstein-Chlodowsky, Szász-Mirakyan and V. A Baskakov operators.

1. Choosing $K_n(x,t,u)=\left[1-\frac{ux}{1+t}\right]^n, \alpha_n=n$ and $\psi_n(0)=\frac{1}{n}$, we have Bernstein operators.

2. Choosing $K_n(x,t,u)=\left[1-\frac{ux}{1+t}\right]^n, \alpha_n=n$ and $\psi_n(0)=\frac{1}{nb_n}\left(\lim_{n\to\infty}b_n=\infty,\lim_{n\to\infty}\frac{b_n}{n}=0\right)$, we have Bernstein-Chlodowsky operators.

3. Choosing $K_n(x,t,u)=e^{-n(t+ux)}, \alpha_n=n$ and $\psi_n(0)=\frac{1}{n}$, we have Szász-Mirakyan operators.

4. Choosing $K_n(x,t,u)=K_n(t+ux), \alpha_n=n$ and $\psi_n(0)=\frac{1}{n}$, we have Baskakov operators.

Some new properties of the operators (I) and some generalizations investigated in [3, 5, 10].

The aim of this paper is to give an integral generalization of M_n in Kantorovich sense. For a Bernstein type approximation process for integrable functions, as Kantorovich done, we show that the derivatives of the Ibragimov-Gadjiev operators converge to the derivative of the function. Then we define Kantorovich type generalization of M_n as taking instead of M_n of f derivative of M_n of $F(x)=\int_0^x f(t)dt$. For details see [8].

For the sake of simplicity we are going to choose $\alpha_n=n$ and $\psi_n(0)=\frac{1}{n}$ in (I) and consider following conditions:

Assume that a family of functions of three variables $K_n(x,t,u)$ $x,t\in I_n=[0,A]$ (for fixed A>0 or $I_n=[0,\infty)$ (that is to say A=∞),-∞<u<∞) satisfies the following conditions:

(a) $K_n(x,0,1)=1$ for $x\in I_n$

(b) $(-1)^\nu\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\ t=0}}\geq 0 \quad \nu=0,1,2,\ldots$

(c) $\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\ t=0}}=-nx\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{n+m}(x,t,u)\Big|_{\substack{u=1\\ t=0}}$

$(x\in I_n, n\in N, \nu=0,1,2)\, n+m\in N_0=N\cup\{0\}$.

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Supposed that the function $K_n(x,t,u)$, in addition to the condition a)-c), satisfies:

(d) $K_n(x,t,u)$ is continuously differentiable with respect to x for any fixed u and t on the interval I_n and

$$\frac{\partial}{\partial x}K_n(x,0,1) = -nK_{n+m}(x,0,1).$$

(e)

$$\frac{1+\nu m}{1+mx}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}} = n\frac{\partial^\nu}{\partial u^\nu}K_{n+m}(x,t,u)\Big|_{\substack{u=1\\t=0}}.$$

Thus, we consider the operator

$$L_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right)\frac{(-1)^\nu}{\nu!}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$

$$:= \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right)A_n(\nu;x) \quad . \tag{1}$$

2. Auxilary Results

In this section we shall give some properties of operators which we shall apply the proofs of the main theorems.

Lemma 2.1 The condition (d) is equivalent to the following equality

$$\frac{d}{dx}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}} = \frac{\nu}{x}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$

$$-n\frac{\partial^\nu}{\partial u^\nu}K_{n+m}(x,t,u)\Big|_{\substack{u=1\\t=0}} \tag{2}$$

Proof. By ν-multiple application of condition (c), we obtain

$$\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}} = (-1)^\nu n(n+m)...[n+(\nu-1)m]$$

$$\times x^\nu K_{n+\nu_m}(x,1,0) \tag{3}$$

Applying condition d) and (3) we get

$$(-1)^\nu \frac{d}{dx}\frac{\partial^\nu}{\partial u^\nu}K_n(x,0,1) = n(n+m)...[n+(\nu-1)m]$$

$$\times\left\{\nu x^{\nu-1}K_{n+\nu m}(x,0,1) + x^\nu(n+\nu m)K_{n+(\nu+1)m}(x,0,1)\right\}$$

Using again (3), we have desired result.

Lemma2.2 For the polynomial

$$T_{\tau,n}(x) = S_n\left((t-nx)^\tau\right)(x), \qquad \tau = 0,1,2,... \tag{4}$$

where

$$\mathrm{S_n}(x) = \sum_{\nu=0}^{\infty}\nu A_n(x,\nu) \tag{5}$$

and $x \in I_n$, we have

$$\left|T_{\tau,n}(x)\right| \le Cn^{\left\|\frac{\tau}{2}\right\|},$$

where C is a positive constant, n is a natural number and $\left\|\frac{\tau}{2}\right\|$ is greatest integer less than $\frac{\tau}{2}$.

Proof. Setting $\mathrm{S_n}\left(t(t-1)(t-2)...(t-(l-1))\right), l \ge 1$ the following elementary identities hold true by (c) and (5).

$$\mathrm{S_n}\left(t(t-1)(t-2)...(t-(l-1))\right)(x)$$

$$= \sum_{\nu=l}^{\infty}\nu(\nu-1)...(\nu-(l-1))A_n(x,\nu)$$

$$= \sum_{\nu=l}^{\infty}\frac{(-1)^{\nu-l}}{(\nu-l)!}n(n+m)...(n+(l-1)m)$$

$$\times x^l\frac{\partial^{\nu-l}}{\partial u^{\nu-l}}K_{n+lm}(x,t,u)\Big|_{\substack{u=1\\t=0}}$$

$$= n(n+m)...(n+(l-1)m)x^2\sum_{\nu=l}^{\infty}A_{n+lm}(x,\nu-l).$$

Since $\sum_{\nu=l}^{\infty}A_{n+lm}(x,\nu-l) = 1$ we have

$$\mathrm{S_n}\left(t(t-1)(t-2)...(t-(l-1))\right)$$

$$= n(n+m)...(n+(l-1)m)x^l$$

It is easily seen that by this equality

$$S_n(1)(x)=1$$
$$S_n(t)(x)=nx$$
$$S_n(t^2)(x)=nx\{x(n+m)+1\}$$
$$S_n(t^3)(x)=nx\{(n^2+m(3n+2m))x^2+3x(n+m)+1\}$$
$$S_n(t^4)=nx\{(n^3+6n^2m+11nm^2+6m^3)x^3 +6(n^2+3nm+2m^2)x^2+7(n+m)x+1\}.$$

$$T_{0,n}(x)=1$$
$$T_{1,n}(x)=0$$
$$T_{2,n}(x)=nx(1+mx)$$
$$T_{3,n}(x)=nx(mx(2mx+3)+1)$$
$$T_{4,n}(x)=n^2(11m^2x^4+6mx^3+3x^2-8m^2x^2) +n(6m^3x^4+12m^2x^3+7mx^2+x).$$

Note that by the equalities above, it can be said that $T_{0,n}$ and $T_{1,n}$ are polynomials in n of order $\|0\|$ and $\left\|\frac{1}{2}\right\|$, respectively. Moreover, $T_{2,n}$ and $T_{3,n}$ are polynomials of degree $\|1\|$ and $\left\|\frac{3}{2}\right\|$. Finally, $T_{4,n}$ is polynomial of degree $\left\|\frac{4}{2}\right\|$. This completes the proof of lemma.

Lemma2.3 For $x\in I_n$

$$\sum_{\left\{\nu:\left|\frac{\nu}{n}-x\right|\ge n^{\frac{1}{8}}\right\}}^{\infty} A_n(x,\nu)\le C\frac{1}{n\sqrt{n}}$$

where $A_n(x,\nu)$ defined as in (1) and C is a positive constant independent of n.

Proof: Since $\left(\frac{\nu}{n}-x\right)^4 n^{-\frac{1}{2}}\ge 1$ we have from Lemma 2.2

$$\sum_{\left\{\nu:\left|\frac{\nu}{n}-x\right|\ge n^{\frac{1}{8}}\right\}}^{\infty} A_n(x,\nu)=\sum_{\nu=0}^{\infty}\left(\frac{\nu}{n}-x\right)^4 n^{-\frac{1}{2}}A_n(x,\nu)$$

$$=\frac{\sqrt{n}}{n^4}T_{4,n}\le\frac{C}{n\sqrt{n}}.$$

Lemma2.4 If

$$\lim_{x\to A} x^{\nu}K_{n+m+1}(x,0,1)=0$$
$$(\nu=0,1,2...\quad and\quad n+m\in N)$$

then

$$\int_0^A x^{\nu}K_{n+(\nu+1)m+1}(x,0,1)dx=\frac{\nu!}{(n+1)(n+m+1)...(n+\nu m+1)}.$$

Proof: By $(\nu+1)-$multiple application of condition (d), we obtain

$$K_{n+(\nu+1)m+1}(x,0,1)=-\frac{1}{(n+\nu m+1)}\frac{\partial}{\partial x}K_{n+\nu m+1}(x,0,1)$$
$$=\frac{1}{(n+(\nu-1)m+1)(n+\nu m+1)}\frac{\partial^2}{\partial x^2}K_{n+(\nu-1)m+1}(x,0,1)$$
$$=\vdots$$
$$=\frac{(-1)^{\nu+1}}{(n+1)(n+m+1)\cdots(n+\nu m+1)}\frac{\partial^{\nu+1}}{\partial x^{\nu+1}}K_{n+1}(x,0,1)$$

3.The convergence results

Now we introduce the convergence issues of the derivative of $L_n(f)$.

Theorem 3.1. Let f is bounded function on I_n. If f has finite derivative $f'(x)$ at the point $x\in I_n$, then

$$\lim_{n\to\infty}(L_nf)'(x)=f'(x).$$

Proof: Let $x\neq 0$. Using (2) we get

$$(L_nf)'(x)=\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)\frac{(-1)^{\nu}}{\nu!}\frac{\partial}{\partial x}\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$
$$=\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)$$
$$\times\left\{\frac{\nu}{x}\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}-n\frac{\partial^{\nu}}{\partial u^{\nu}}K_{n+m}(x,t,u)\Big|_{\substack{u=1\\t=0}}\right\}$$

By condition (e)

$$(L_nf)'(x)=\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)\frac{(-1)^{\nu}}{\nu!}\frac{(\nu-nx)}{x(1+mx)}\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,0,1)$$
$$=\frac{1}{x(1+mx)}\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)(\nu-nx)A_n(x,\nu).$$

Since f' exists then by Lagrange's theorem we can write $f\left(\frac{\nu}{n}\right)=f(x)+\left(f'(x)+\alpha\left(\frac{\nu}{n}\right)\right)\left(\frac{\nu}{n}-x\right)$, where $\alpha(t)\to 0$ as $t\to x$.

Evidently we have

$$(L_n f)'(x)=\frac{1}{x(1+mx)}f(x)T_{1,n}(x)$$
$$+\frac{1}{nx(1+mx)}f'(x)T_{2,n}(x)+\tau_n$$

where

$$\tau_n=\frac{1}{nx(1+mx)}\sum_{\nu=0}^{\infty}\alpha\left(\frac{\nu}{n}\right)(\nu-nx)^2A_n(\nu,x).$$

By Lemma2.2, $T_1(x)=0$ and $T_2(x)=nx(1+mx)$ we get

$$(L_n f)'(x)=f'(x)+\tau_n. \tag{6}$$

Now we find upper bound of τ_n for a large n. Since $\alpha(t)\to 0$ as $t\to x$, for $\varepsilon>0$ there exists a number n such that for $|t-x|\le n^{-\frac{1}{8}}$, thus, we have

$$|\alpha_n(t)|<\frac{\varepsilon}{8}.$$

From above observation we can write

$$\left|\frac{1}{x(1+mx)}\sum_{\left\{\nu:\left|\frac{\nu}{n}-x\right|<n^{-\frac{1}{8}}\right\}}^{\infty}\alpha\left(\frac{\nu}{n}\right)(\nu-nx)^2A_n(x,\nu)\right|$$
$$\le\frac{\varepsilon}{2}\frac{1}{nx(1+mx)}T_2(x)=\frac{\varepsilon}{2}. \tag{7}$$

Since the function $\alpha\left(\frac{\nu}{n}\right)(\nu-nx)^2$ is bounded there exists an upper bound M such that by Lemma 2.3

$$\left|\frac{1}{x(1+mx)}\sum_{\left\{\nu:\left|\frac{\nu}{n}-x\right|\ge n^{-\frac{1}{8}}\right\}}^{\infty}\alpha\left(\frac{\nu}{n}\right)(\nu-nx)^2A_n(x,\nu)\right|$$
$$\le nM\sum_{\left\{\nu:\left|\frac{\nu}{n}-x\right|\ge n^{-\frac{1}{8}}\right\}}^{\infty}A_n(x,\nu)\le\frac{MC}{\sqrt{n}}.$$

Thus sufficiently large n one has $\frac{MC}{\sqrt{n}}<\frac{\varepsilon}{2}$. From this inequality and (7) the inequality $|\tau_n|<\varepsilon$ holds. In result, (6) implies

$$\lim_{n\to\infty}(L_n f)'(x)=f'(x).$$

For x=0 we consider the equality

$$(L_n f)'(x)=\frac{1}{x(1+mx)}\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n}\right)(\nu-nx)$$
$$\times\frac{(-1)^\nu}{\nu!}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$
$$=\frac{1}{x(1+mx)}\Big\{-f(0)nxK_n(x,0,1)$$
$$-f\left(\frac{1}{n}\right)(1-nx)\frac{\partial}{\partial u}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$
$$+\sum_{\nu=2}^{\infty}f\left(\frac{\nu}{n}\right)(\nu-nx)\frac{(-1)^\nu}{\nu!}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}\Big\}$$

Using (3) and condition (c) the above equality can be re-written in the form

$$(L_n f)'(x)=\frac{1}{x(1+mx)}\Big\{-f(0)nxK_n(x,0,1)$$
$$-nf\left(\frac{1}{n}\right)(1-nx)\frac{\partial}{\partial u}K_{n+m}(x,t,u)\Big|_{\substack{u=1\\t=0}}$$
$$+\sum_{\nu=2}^{\infty}f\left(\frac{\nu}{n}\right)n(n+m)\cdots\frac{x^{\nu-1}}{\nu!}K_{n+\nu m}(x,t,u)\Big|_{\substack{u=1\\t=0}}\Big\}$$

By taking x=0 we get

$$(L_n f)'(0)=n\left(f\left(\frac{1}{n}\right)-f(0)\right)$$

and

$$\lim_{n\to\infty}(L_n f)'(0)=f(0).$$

This completes the proof of Theorem 3.1.
For the function

$$F(x)=\int_0^x f(t)\,dt$$

we can write

$$L_n(F)(x)=\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu}{n}} f(t)dt\right\}\frac{(-1)^{\nu}}{\nu!}\frac{\partial^{\nu}}{\partial u^{\nu}}K_n(x,t,u)\Big|_{\substack{u=1\\t=0}}$$

ν -multiple usage of condition (c) yields

$$L_n(F)(x)=\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu}{n}} f(t)dt\right\}$$

$$\times\frac{x^{\nu}n\cdots(n+(\nu-1)m)}{\nu!}K_{n+\nu m}(x,0,1)$$

and writing $n+1$ instead of n we get

$$L_{n+1}(F)(x)=\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu}{n+1}} f(t)dt\right\}$$

$$\times\frac{x^{\nu}(n+1)\cdots(n+(\nu-1)m+1)}{\nu!}K_{n+\nu m+1}(x,0,1)$$

By condition (d)

$$\frac{d}{dx}L_{n+1}(F)(x)=\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu+1}{n+1}} f(t)dt\right\}K_{n+(\nu+1)m+1}(x,0,1)$$

$$\times\frac{x^{\nu}(n+1)\cdots(n+\nu m+1)}{\nu!}$$

$$-\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu}{n+1}} f(t)dt\right\}x^{\nu}\frac{\partial}{\partial x}K_{n+\nu m+1}(x,0,1)$$

$$\times\frac{(n+1)\cdots(n+(\nu-1)m+1)}{\nu!}$$

holds.

Representing $\frac{\partial}{\partial x}L_{n+1}(F)(x)$ with $A_n(f)(x)$ which is a linear positive operator such that it is a Ibragimov-Gadjiev operator in Kantorovich form.

$$A_n(f)(x)=\sum_{\nu=0}^{\infty}\left\{\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)dt\right\}x^{\nu}\frac{\partial}{\partial x}K_{n+(\nu+1)m+1}(x,0,1)$$

$$\times\frac{(n+1)\cdots(n+(\nu-1)m+1)}{\nu!}$$

or

$$A_n(f)(x)=$$

$$(n+1)\sum_{\nu=0}^{\infty}\left\{\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)dt\right\}\frac{(-1)^{\nu}}{\nu!}\frac{\partial^{\nu}}{\partial x^{\nu}}K_{n+m+1}(x,0,1) \quad (8)$$

We note that in [1, 2, 4, 9] similar generalizations were developed for different operators.

Let us denote by $L[I_n]$, the class of integrable function in I_n. The norm on I_n is defined by

$$\|f\|_L=\int_0^A|f(x)|dx.$$

As a consequence of Theorem 3.1, we give the following theorem.

Theorem 3.2 At any point $x\in I_n$

$$\lim_{n\to\infty}A_n(f)(x)=f(x),$$

where $f(x)$ is the derivative of its indefinite integral.

Theorem 3.3 If for each $\nu=0,1,2,\ldots$ with $n+m\in N$

$$\lim_{x\to A}x^{\nu}K_{n+m+1}(x,0,1)=0$$

and $f\in L[I_n]$ then

$$\lim_{n\to\infty}\|A_n(f)-f\|_L=0.$$

Proof : Firstly, we show that $A_n, n\in N$, is a operator from $L[I_n]$ to itself. By applying condition (c) $\nu-$times we obtain from Lemma 2.4

$$\int_0^A A_n(f)(x)dx$$

$$=\sum_{\nu=0}^{\infty}\left\{\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)dt\right\}\int_0^A\frac{(-1)^{\nu}}{\nu!}\frac{\partial^{\nu}}{\partial x^{\nu}}K_{n+m+1}(x,0,1)dx$$

$$=\sum_{\nu=0}^{\infty}\left\{\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} f(t)dt\right\}=\|f\|_L.$$

We choose $f_{\nu,n}(x)=[u_n]^{-1}\chi_u(x)$, where $u_n=\frac{r}{n+1}$ and χ_u is the characteristic function of the interval $[0,u_n]$.

Since the set of all step functions is dense in the space $L[0,\infty)$ to prove the theorem it is sufficient to prove only for the characteristic function χ_u.
It is obvious that

$$\int_0^R A_n\left(\chi_{u_n};x\right)dx=\sum_{\nu=0}^{r-1}\int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}}dx=\frac{r}{n+1} \quad . \qquad (9)$$

For a large n, we can write

$$\left\|\chi_{u_n}-A_n\left(\chi_{u_n}\right)\right\|_L=\int_0^R\left[\chi_{u_n}(x)-A_n\left(\chi_{u_n};x\right)\right]dx$$

$$=\int_0^{u_n}\left[1-A_n\left(\chi_{u_n};x\right)\right]dx+\int_{u_n}^R A_n\left(\chi_{u_n};x\right)dx$$

$$=\int_0^{u_n}\left[A_n(1;x)-A_n\left(\chi_{u_n};x\right)\right]dx+\int_0^R A_n\left(\chi_{u_n};x\right)dx$$

$$-\int_0^{u_n}A_n\left(\chi_{u_n};x\right)dx$$

$$=\int_0^{u_n}\left[2A_n\left(\overline{\chi_{u_n}};x\right)-A_n(1;x)\right]dx+\int_0^R A_n\left(\chi_{u_n};x\right)dx,$$

where $\overline{\chi_{u_n}}=1-\chi_{u_n}(x)$.
Also,

$$A_n\left(\overline{\chi_{u_n}};x\right)=\frac{d}{dx}\left(L_{n+1}\left(\int_0^x\overline{\chi_{u_n}}(t)\,dt\right);x\right)$$

$$=\frac{d}{dx}\left(\sum_{\nu=0}^{\infty}\left\{\int_0^{\frac{\nu}{n+1}}\overline{\chi_{u_n}}dt\right\}\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Bigg|_{\substack{u=1\\t=0}}\right)$$

$$=\frac{d}{dx}\left(\sum_{\nu=0}^{\infty}\left(\frac{\nu}{n+1}-\frac{r}{n+1}\right)\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Bigg|_{\substack{u=1\\t=0}}\right)$$

Thus

$$\int_0^{u_n}A_n\left(\overline{\chi_{u_n}};x\right)dx=\sum_{\nu=0}^{\infty}\left(\frac{\nu}{n+1}-\frac{r}{n+1}\right)\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Bigg|_{\substack{u=1\\t=0}}$$

$$=L_n(\varphi,u_n),$$

where

$$\varphi(x)=\begin{cases}0, x\le u_n\\ x-u_n, x>u_n\end{cases}.$$

Since $L_n(\varphi,u_n)-\varphi(x)\to 0\,(n\to\infty)$ then

$$\lim_{n\to\infty}\int_0^{u_n}A_n\left(\overline{\chi_{u_n}};x\right)dx=0.$$

From (9),(10) and (11), the proof is completed.

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