Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATICS, Tenerife, Spain, December 16-18, 2005 (pp334-338)

Solution of a Class of Riccati Equations

K. BUSAWON, P. JOHNSON
School of Computing, Engineering and Information Sciences
Northumbria University
Ellison Building, Newcastle-upon-Tyne, NE1 8ST
UNITED KINGDOM

Abstract: In this paper, we derive the analytical closed form solution of a class of Riccati equations. Since it is well known that to every Riccati equation there corresponds a linear homogeneous ordinary differential equation (ODE), the result obtained is subsequently employed to derive the analytical solution of the class of second order linear homogeneous ODEs corresponding to the class of Riccati equations considered. In addition, we present some examples in order to demonstrate the validity of the results obtained.

1 Introduction

A large class of dynamical systems appearing throughout the field of applied mathematics are described by ODEs of order 2 of the following form:

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = r(t) \tag{1}$$

where $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, and $p(t)$, $q(t)$ and $r(t)$ are real valued, scalar functions of the independent variable t. It is not obvious to perform a complete analysis of the equation (1) since there is no universal method for solving the linear inhomogeneous ODE of order 2. There do however exist well established methods for solving (1) in a number of special cases, one of which being when $p(t)$ and $q(t)$ are constant [1]. A survey of the various existing methods (analytical and otherwise) for solving the ODE (1) is given in many reference textbooks (see e.g. [1]-[10]).

On the other hand, it is well known that solving the ODE (1) in the homogeneous case when $r(t) = 0$, amounts to solving a Riccati equation (see e.g. [11]). More precisely, the following second order homogeneous ODE

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0 \tag{2}$$

can be transformed into the following Riccati equation

$$\dot{y}(t) + p(t)y(t) - y^2(t) = q(t) \tag{3}$$

by making the following change of variable:

$$y(t) = -\frac{\dot{x}(t)}{x(t)}. \tag{4}$$

In general, it is not at all a simple matter to solve (3) for arbitrary functions $p(t)$ and $q(t)$. This implies that the transformation (4) does not make it any easier to find the solution of (2). However, as Euler was able to demonstrate, when one particular solution of (3) is obtained, the general solution can be readily found [11]. The difficulty in using (3) to obtain the solution of (2) therefore resides in the problem of finding a particular solution of (3). It ought to be stressed that when $p(t)$ and $q(t)$ assume some special form, then the general solution of (3) can be found. For example, when $q(t) \equiv 0$, the above Riccati equation reduces to the Bernoulli differential equation and thus can be reduced to a first order linear ODE by means of the transformation $y(t) = \dfrac{1}{v(t)}$ which yields

$$\dot{v}(t) = p(t)v(t) - 1$$

and whose solution is given by

$$v(t) = v(t_0)e^{\int_{t_0}^{t} p(\tau)d\tau} - e^{\int_{t_0}^{t} p(\tau)d\tau} \int_{t_0}^{t} e^{\int_{t_0}^{\lambda} -p(\tau)d\tau} d\lambda$$

where t_0 is an initial value of t.

In other words, when $q(t) \equiv 0$ the solution of (3) can be explicitly written as

$$y(t) = \frac{y(t_0)e^{-\int_{t_0}^{t} p(\tau)d\tau}}{1 - y(t_0)\int_{t_0}^{t} e^{\int_{t_0}^{\lambda} -p(\tau)d\tau} d\lambda}. \tag{5}$$

In this paper, we treat a special case of the Riccati equation (3) where the function $q(t)$ is not identically zero for all t. More precisely, we consider the case where the function $q(t)$ is positive for all $t \geq t_0$ and is given by

$$q(t) = \frac{q_0 e^{-2\int_{t_0}^{t} p(\tau)d\tau}}{\left(1 + K\sqrt{q_0}\int_{t_0}^{t} e^{-\int_{t_0}^{\lambda} p(\tau)d\tau} d\lambda\right)^2}$$

where $q_0 = q(t_0) > 0$ and K is a constant. We shall give the explicit solution of the Riccati equation (3) when $q(t)$ is as above. Since $q(t)$ depends on $p(t)$, we shall assume, throughout the paper, that the integral $\int_{t_0}^{t} p(\tau)d\tau$ exists. We shall also use the result obtained to provide the solution of the linear second order ODE corresponding to the Riccati equation considered. As far as the authors are aware, the explicit solution of the class of ODEs considered here does not exist in the literature. This lays the foundation for the analysis for that particular class of ODEs in terms of stability, existence of singularities, existence of periodic solutions and so on.

An outline of the paper is as follows: In the next section, the solution of a class of Riccati equations is given along with an example. This result is then used to derive the solution of the class of second order linear homogeneous ODEs corresponding to (3) and a further two examples are provided to illustrate the method. Finally, some conclusions are drawn.

2 Solution of a Class of Riccati Equations

In this section, we give the general solution of a class of Riccati equations. The result is summarised in the following theorem:

Theorem 1 *Consider the Riccati equation*

$$\dot{y}(t) + p(t)y(t) - y^2(t) = q(t) \tag{6}$$

with the initial condition $y(t_0) = y_0$ for some initial value t_0. Assume that $q(t_0) = q_0 > 0$ and that the integral $\int_{t_0}^{t} p(\tau)d\tau$ exists. Assume further that the function $q(t)$ satisfies the relation

$$q(t) = \frac{q_0 e^{-2\int_{t_0}^{t} p(\tau)d\tau}}{\left(1 + K\sqrt{q_0}\int_{t_0}^{t} e^{\int_{t_0}^{\lambda} -p(\tau)d\tau} d\lambda\right)^2}$$

for some constant K. Then, the general solution of (6) is given by

$$y(t) = f(t)\sqrt{q(t)} \tag{7}$$

where the function $f(t)$ is given by

$$f(t) = \begin{cases} -\dfrac{K}{2} + \dfrac{\alpha\left(1 + e^{\alpha\theta_1(t)}\right)}{2\left(1 - e^{\alpha\theta_1(t)}\right)} & \text{if } K^2 > 4 \\ -\dfrac{K}{2} + \dfrac{\beta}{2}\tan\left(\dfrac{\beta\theta_2(t)}{2}\right) & \text{if } K^2 < 4 \\ -\dfrac{K}{2} - \dfrac{1}{\theta_3(t)} & \text{if } K^2 = 4 \end{cases} \tag{8}$$

and the functions $\theta_n(t)$; $n = 1, 2, 3$ are given by

$$\theta_n(t) = c_n + \int_{t_0}^{t} \sqrt{q(\tau)}d\tau \tag{9}$$

where

$$\begin{aligned} c_1 &= \frac{1}{\alpha}\ln\left(\frac{2y_0 + \sqrt{q_0}(K-\alpha)}{2y_0 + \sqrt{q_0}(K+\alpha)}\right) && (10) \\ c_2 &= \frac{2}{\beta}\tan^{-1}\left(\frac{2y_0 + K\sqrt{q_0}}{\beta\sqrt{q_0}}\right) && (11) \\ c_3 &= -\frac{2\sqrt{q_0}}{2y_0 + K\sqrt{q_0}}. && (12) \end{aligned}$$

and

$$\begin{aligned} \alpha &= \sqrt{K^2 - 4} \\ \beta &= \sqrt{4 - K^2}. \end{aligned}$$

Before giving the proof of the above theorem, we shall make a few remarks on the functions $q(t)$ and $f(t)$.

Remarks

1. First of all, note that if $q_0 = 0$ then $q(t) = 0$ for all $t \geq t_0$, and, as mentioned in the introduction, in this case the solution is given by (5).

2. Next, it can be shown that $\sqrt{q(t)}$, which is given by

$$\sqrt{q(t)} = \frac{\sqrt{q_0}e^{-\int_{t_0}^{t} p(\tau)d\tau}}{1 + K\sqrt{q_0}\int_{t_0}^{t} e^{\int_{t_0}^{\lambda} -p(\tau)d\tau} d\lambda},$$

is the solution of the following Bernoulli equation in $\sqrt{q(t)}$:

$$\begin{aligned} \dot{\sqrt{q(t)}} &= -p(t)\sqrt{q(t)} - Kq(t) \\ &= -p(t)\sqrt{q(t)} - K\left(\sqrt{q(t)}\right)^2 && (13) \end{aligned}$$

with initial condition $q(t_0) = q_0$.

3. By differentiating the function (8), it can be shown after some lengthy calculations that

$$\dot{f}(t) = \left(f^2(t) + Kf(t) + 1\right)\sqrt{q(t)}. \tag{14}$$

In other words, (8) is the solution of the above differential equation with some specific initial conditions.

Proof - Theorem 1:

Let us first demonstrate that

$$y_p(t) = f(t)\sqrt{q(t)} \tag{15}$$

is, at least, a particular solution of (6). By differentiating (15) we have

$$\dot{y}_p(t) = \dot{f}(t)\sqrt{q(t)} + f(t)\dot{\sqrt{q(t)}}. \qquad (16)$$

Next, by considering the above remarks and by substituting (14) and (13) into (16) we obtain

$$\begin{aligned}\dot{y}_p(t) &= f^2(t)\,q(t) + q(t) - p(t)f(t)\sqrt{q(t)} \\ &= y_p^2(t) + q(t) - p(t)\,y_p(t)\end{aligned}$$

or alternatively

$$\dot{y}_p(t) + p(t)\,y_p(t) - y_p^2(t) = q(t).$$

Hence, it is clear that (15) is indeed a particular solution of (6).

Now, the general solution of (6) is given by

$$y(t) = y_p(t) + u(t)$$

where $u(t) = \dfrac{1}{v(t)}$ and $v(t)$ satisfies

$$\dot{v}(t) + (2y_p(t) - p(t))\,v(t) = -1.$$

More precisely,

$$u(t) = \frac{u_0 e^{\int_{t_0}^{t}(2y_p(\tau)-p(\tau))d\tau}}{\left(1 - u_0\int_{t_0}^{t} e^{\int_{t_0}^{\lambda}(2y_p(\tau)-p(\tau))d\tau}d\lambda\right)}$$

where $u_0 = u(t_0)$. If c_1, c_2 and c_3 are all chosen as in (10-12), it can be shown that $y_p(t_0) = y_0$. As a result, $u_0 = 0$. It is clear from the above that if $u_0 = 0$, then $u(t) \equiv 0$. Consequently, if c_1, c_2 and c_3 in (8) are all chosen as in (10-12), then $y_p(t)$ becomes the general solution of (6) i.e. $y(t) = y_p(t)$.

Q.E.D.

Remark

Note that for $K < 0$, the function $q(t)$ and hence the solution $x(t)$ might possess some singularities. This will be illustrated by an example in the next section.

Example 1. Consider the initial value problem

$$\dot{y} - \left(3at + \frac{1}{t}\right)y - y^2 = a^2t^2 \qquad (17)$$

where $y(t_0) = y_0$, $t_0 \neq 0$ and where a is some constant. Now, the Riccati equation (17) is of the form (6) with $q(t) = a^2t^2$ and $p(t) = -\left(3at + \dfrac{1}{t}\right)$. It is clear that $q(t_0) > 0$. In addition, it can be shown that $q(t)$ satisfies the relation (13) with $K = 3$. From the result of Theorem 1 we can say that the general solution of (17) is

$$y(t) = \left(-\frac{3}{2} + \frac{\sqrt{5}\left(1 + e^{\sqrt{5}\left(\frac{a}{2}\left(t^2-t_0^2\right)+c_1\right)}\right)}{2\left(1 - e^{\sqrt{5}\left(\frac{a}{2}\left(t^2-t_0^2\right)+c_1\right)}\right)}\right)at$$

where $c_1 = \frac{1}{\sqrt{5}}\ln\left(\frac{2y_0+at_0\left(3-\sqrt{5}\right)}{2y_0+at_0\left(3+\sqrt{5}\right)}\right)$.

3 Solution of a Class of 2nd Order Linear Homogeneous Systems

As mentioned in the introduction, to every Riccati equation there corresponds a linear 2nd order homogeneous equation. In particular, to the Riccati equation (3) there corresponds the homogeneous linear ODE (2). We can therefore use the solution of the previous class of Riccati equations to derive the solution of its corresponding 2nd order ODE. This is given in the following corollary:

Corollary 1. *Consider the initial value problem*

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0 \qquad (18)$$

with given initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$. Assume that $q(t_0) = q_0 > 0$ and that the integral $\int_{t_0}^{t} p(\tau)d\tau$ exists. Assume further that the function $q(t)$ satisfies the relation

$$q(t) = \frac{q_0 e^{-2\int_{t_0}^{t} p(\tau)d\tau}}{\left(1 + K\sqrt{q_0}\int_{t_0}^{t} e^{\int_{t_0}^{\lambda} -p(\tau)d\tau}d\lambda\right)^2}$$

for some constant K. Then the general solution to (18) is given by:

$$x(t) = \begin{cases} x_0\left(\dfrac{1-e^{\alpha\theta_1(t)}}{1-e^{\alpha c_1}}\right)e^{\left(\frac{K}{2}-\frac{\alpha}{2}\right)(\theta_1(t)-c_1)} & \text{if } K^2 > 4 \\ x_0\left(\dfrac{\cos\left(\frac{\beta\theta_2(t)}{2}\right)}{\cos\left(\frac{\beta c_2}{2}\right)}\right)e^{\frac{K}{2}(\theta_2(t)-c_2)} & \text{if } K^2 < 4 \\ x_0\left(\frac{\theta_3(t)}{c_3}\right)e^{\frac{K}{2}(\theta_3(t)-c_3)} & \text{if } K^2 = 4 \end{cases}$$

where c_1, c_2 and c_3 are given by (10-12) in which $y_0 = -\dfrac{\dot{x}_0}{x_0}$.

Proof - Corollary 1:

Employing the transform (4) in (18) yields (6). Now, by Theorem 1 we know the solution of (6) is given by (7) since we are assuming that (13) holds. Solving (4) for x it is clear that

$$x(t) = x_0 e^{-\int_{t_0}^{t} y(\tau)d\tau} = x_0 e^{-\int_{t_0}^{t} f(\tau)\sqrt{q(\tau)}d\tau} \qquad (19)$$

and hence the solution of (18) is (19) where $f(t)$ is given by (8) and where $y_0 = -\frac{\dot{x}_0}{x_0}$.

To determine the explicit form of the solution of (18) it is necessary to determine the following integral

$$I(t) = \int_{t_0}^{t} f(\tau)\sqrt{q(\tau)}d\tau.$$

To do this we set $u = \int_{t_0}^{t}\sqrt{q(\tau)}d\tau$. Then, functions $\theta_n(t); n = 1, 2, 3$ described in (9) can be written

$$\theta_n(t) = c_n + u$$

The function $f(t)$ can therefore be written as a function of u as follows:

$$f(u) = \begin{cases} -\dfrac{K}{2} + \dfrac{\alpha\left(1 + e^{\alpha(c_1+u)}\right)}{2\left(1 - e^{\alpha(c_1+u)}\right)} & \text{if } K^2 > 4 \\ -\dfrac{K}{2} + \dfrac{\beta}{2}\tan\left(\dfrac{\beta(c_2+u)}{2}\right) & \text{if } K^2 < 4 \\ -\dfrac{K}{2} - \dfrac{1}{c_3+u} & \text{if } K^2 = 4 \end{cases}$$

As a result,

$$I(t) = \int_{t_0}^{t} f(u)\,du$$

and from this simplified form it is possible to show, after some lengthy calculations, that

$$I(t) = \begin{cases} \left(\frac{\alpha-K}{2}\right)(\theta_1(t) - c_1) + \ln\left(\frac{1-e^{\alpha c_1}}{1-e^{\alpha\theta_1(t)}}\right) & \text{if } K^2 > 4 \\ -\frac{K}{2}(\theta_2(t) - c_2) + \ln\left(\frac{\cos\left(\frac{\beta c_2}{2}\right)}{\cos\left(\frac{\beta\theta_2(t)}{2}\right)}\right) & \text{if } K^2 < 4 \\ -\frac{K}{2}(\theta_3(t) - c_3) + \ln\frac{c_3}{\theta_3(t)} & \text{if } K^2 = 4 \end{cases}$$

This implies that

$$x(t) = \begin{cases} x_0\left(\dfrac{1-e^{\alpha\theta_1(t)}}{1-e^{\alpha c_1}}\right)e^{\left(\frac{K}{2}-\frac{\alpha}{2}\right)(\theta_1(t)-c_1)} & \text{if } K^2 > 4 \\ x_0\left(\dfrac{\cos\left(\frac{\beta\theta_2(t)}{2}\right)}{\cos\left(\frac{\beta c_2}{2}\right)}\right)e^{\frac{K}{2}(\theta_2(t)-c_2)} & \text{if } K^2 < 4 \\ x_0\left(\frac{\theta_3(t)}{c_3}\right)e^{\frac{K}{2}(\theta_3(t)-c_3)} & \text{if } K^2 = 4 \end{cases}$$

Q.E.D.

Example 2. Consider the initial value problem

$$\ddot{x} + \left(1 + \frac{1}{a}\right)\frac{1}{t}\dot{x} + \frac{1}{a^2t^2}x = 0 \qquad (20)$$

where $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$ with $t_0 > 0$ and a is some constant. By inspection it can be seen that, (20) is of the form (18) with $q(t) = \frac{1}{a^2t^2}$ and $p(t) = \left(1 + \frac{1}{a}\right)\frac{1}{t}$. It is trivial to show that $q(t)$ satisfies the relation (13) with $K = -1$. From the result of Corollary 1 we can say that the general solution of (20) is

$$x(t) = x_0\frac{\cos\left(\frac{\sqrt{3}\left(\frac{1}{a}\ln\left(\frac{t}{t_0}\right)+c_2\right)}{2}\right)}{\cos\left(\frac{\sqrt{3}c_2}{2}\right)}\left(\frac{t_0}{t}\right)^{\frac{1}{2a}}$$

where $c_2 = \frac{2}{\sqrt{3}}\tan^{-1}\left(-\frac{2at_0\dot{x}_0+x_0}{\sqrt{3}x_0}\right)$. Note that the solution possess a singularity at $t = 0$. The above solution holds as long as $t > 0$ because of the $\ln\left(\frac{t}{t_0}\right)$ term appearing in the solution.

Example 3. Consider the initial value problem

$$\ddot{x} - \left(b + 2ae^{bt}\right)\dot{x} + a^2e^{2bt}x = 0 \qquad (21)$$

where $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$ for some initial value t_0 and a and b are constants. The form of (21) is equivalent to that of (18) with $q(t) = a^2e^{2bt}$ and $p(t) = -\left(b + 2ae^{bt}\right)$. Again, it is a simple matter to show that $q(t)$ satisfies the relation (13) with $K = 2$. Corollary 1 shows that the solution of (21) is

$$x(t) = x_0\left(\frac{\frac{a}{b}\left(e^{bt} - e^{bt_0}\right) + c_3}{c_3}\right)e^{\frac{a}{b}\left(e^{bt}-e^{bt_0}\right)}$$

where $c_3 = \frac{2x_0ae^{bt_0}}{2\dot{x}_0 - Kx_0ae^{bt_0}}$.

4 Conclusion

In this paper, we have derived the analytical solution of a class of linear homogeneous ODEs of order 2. We have first given the solution of a particular class of Riccati equations which is then used to characterise the solution of a fairly large class of homogeneous second order linear ODEs. Examples were given in each case to demonstrate the methods at hand. The results obtained will provide a better insight on the class of dynamical systems considered in terms of stability and phase plane analysis.

References

[1] R. Bronson, *Schaum's Outline of Theory and Problems of Differential Equations 2nd Edition*, McGraw-Hill Inc., 1994.

[2] D. Zwillinger, *Handbook of Differential Equations*, Academic Press Inc., 1989.

[3] D.L. Kreider et al, *An Introduction to Linear Analysis*, Addison-Wesley Inc., 1966.

[4] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations 2nd Edition*, Boca Raton, Chapman & Hall/CRC Press, 2003.

[5] A.C. King at al, *Differential Equations: Linear, Nonlinear, Ordinary, Partial*, Cambridge University Press, 2003.

[6] J.C. Butcher, *The Numerical Methods for Ordinary Differential Equations*, John Wiley and Sons Ltd., 2003.

[7] R.M. Mattheij and J. Molenaar, *Ordinary Differential Equations in Theory and Practice*, SIAM, 2002.

[8] J. Hubbard and B. West, *Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations*, New York, Springer-Verlag, 2002.

[9] P. Hartman, *Ordinary Differential Equations*, SIAM, 2002.

[10] E. Hairer et al, *Solving Ordinary Differential Equations I*, New York, Springer-Verlag, 1987.

[11] H.T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, New York, Dover Publications Inc., 1962.