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Classes of Sum-of-Sinusoids Rayleigh Fading Channel Simulators and Their Stationary and Ergodic Properties

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Abstract: – Channel simulators based on Rice's sum-of-sinusoids are playing an important role in mobile fading channel modelling. However, the parameters of the sum-of-sinusoids have to be determined sophisticated to fully exploit the great potential that this powerful procedure has to offer. This paper provides a fundamental work on the stationary and ergodic properties of sum-of-sinusoids-based Rayleigh fading channel simulators. Altogether eight classes of Rayleigh fading channel simulators are introduced, where four of them are new. Each individual class is systematically investigated with respect to its stationary and ergodic properties. This allows us to establish general conditions under which the sum-of-sinusoids procedure results in a stationary and ergodic channel simulator. Moreover, with the help of the proposed classification scheme, several popular parameter computation methods are investigated concerning their usability to design efficient fading channel simulators. The treatment of the problem shows that if and only if the gains and frequencies are constant quantities and the phases are random variables, then the sum-of-sinusoids defines a stationary and ergodic stochastic process. For simplicity reasons, we restrict our investigations to Rayleigh fading channels, but the presented results are of general interest, wherever the principle of Rice's sum-of-sinusoids is employed.

Keywords: – Mobile fading channels, channel modelling, propagation, channel simulators, Rice's sum-of-sinusoids, stochastic processes, deterministic processes.

1 Introduction

The sum-of-sinusoids principle was first introduced in Rice's seminal work [1,2], as a method to model Gaussian noise processes with given temporal correlation properties. With the development of mobile communication systems, this principle became very popular, since it enables the design of efficient and flexible mobile fading channel simulators. However, when Rice's original method is used to compute the model parameters, then the period of the resulting fading process is merely proportional to the number of sinusoids [3]. This is a serious drawback, because it prevents keeping the realization expenditure low. But fortunately, many alternative methods (e.g., [4–9]) have been developed to avoid this drawback.

In present days, the application of the sum-of-sinusoids principle ranges from the development of channel simulators for relatively simple time-variant Rayleigh fading channels [4,5,10] and Nakagami channels [11, 12] over frequency-selective channels [3, 6, 7, 13, 14] up to elaborated space-time narrowband [15, 16] and wideband [17–20] channels. Further applications can be found in research areas dealing with the design of multiple cross-correlated [21] and multiple uncorrelated [9, 22] Rayleigh fading channels. Such channel models are of special interest, e.g., in system performance studies of multiple-input multiple-output systems [23, 24] and diversity schemes [25, Chap. 6], [26]. Moreover, it has been shown that the sum-of-sinusoids principle enables the design of fast fading channel simulators [27] and facilitates the development of perfect channel models [28]. A perfect channel model is a model, whose scattering function can perfectly be fitted to any given measured scattering function obtained from snap-shot measurements carried out in real-world environments. Finally, it should be mentioned that the sum-of-sinusoids principle has successfully been applied recently to the design of burst error models with excellent burst error statistics

[29, 30] and to the development of frequency hopping channel simulators [31, 32].

Apart from the requirement that a sum-of-sinusoids-based channel simulator should have an (approximately) infinite period, several other performance criteria are also important. For example, it is of central importance to know the conditions under which a finite sum of harmonic functions with random parameters results in a stationary and ergodic channel simulator. The solution of this problem is the topic of the present paper. Here, general conditions are stated guaranteeing that the developed fading channel simulator is not only stationary but also ergodic. The primary task of a channel simulator is to generate waveforms or sample functions, the statistics of which is sufficiently close to the desired statistics of a given ideal and generally non-realizable theoretical stochastic channel model, called the reference model. If the channel simulator is ergodic, then each sample function contains the same statistical information. In this case, a single sample function is sufficient to characterize the channel, or, in other words, an averaging over several realizations of sample functions can be avoided. Thus, when ergodic channel simulators are used in system performance studies, the overall simulation time can be reduced drastically. For this reason, it is therefore important to know the conditions under which a sum-of-sinusoids results in an ergodic process.

Since a sum-of-sinusoids depends on three kinds of parameters (gains, frequencies, and phases), where each of which can be a collection of random variables or constants, there exists altogether eight classes of sum-of-sinusoids. Consequently, eight classes of Rayleigh fading channel simulators can be defined. To the best of the authors knowledge, four of them are completely new. In addition to that, one must confess that even for the known classes their stationary and ergodic properties are partly unknown or at least not well understood. The intention of this paper is to close these gaps. Here, the complete set of classes will be defined and systematically investigated with respect to their stationary and ergodic properties. We will see that if and only if the phases are random variables and the gains and frequencies are constant quantities, then the resulting channel simulator is stationary and ergodic. In all cases, where the frequencies are random variables, we obtain a stochastic channel simulator which is stationary but non-autocorrelation-ergodic. The worst case, however, is given when the gains are random variables and the other model parameters are constant. Then, a non-stationary channel simulator is obtained. For simplicity reasons, we focus our attention on Rayleigh fading, but the obtained results can directly be generalized to all kinds of channel simulators employing the principle of Rice's sum-of-sinusoids.

The organization of the paper is as follows. Section 2 describes a non-realizable reference model defined by an infinite number of sinusoids. Limiting the number of sinusoids leads to the simulation models introduced in Section 3. Section 4 reviews briefly the characteristics of stationary and ergodic processes, and presents related performance criteria. Section 5 introduces eight classes of sum-of-sinusoids-based simulation models and investigates their stationary and ergodic properties. Section 6 applies the proposed concept to several parameter computation methods commonly used in practice. Finally, Section 7 draws the conclusion.

2 The Reference Model

For our purpose, it is sufficient to consider Rayleigh fading. A Rayleigh process, $\boldsymbol{\zeta}(t)$, is defined as

$$\boldsymbol{\zeta}(t) = |\boldsymbol{\mu}_1(t) + j\boldsymbol{\mu}_2(t)| \tag{1}$$

where $\boldsymbol{\mu}_1(t)$ and $\boldsymbol{\mu}_2(t)$ are two statistically independent zero-mean real Gaussian processes, each with variance σ_0^2.[1] Hence, the probability density function $p_{\mu_i}(x)$ of $\boldsymbol{\mu}_i(t)$ $(i = 1, 2)$ is given by

$$p_{\mu_i}(x) = \frac{1}{\sqrt{2\pi}\sigma_0}\, e^{-\dfrac{x^2}{2\sigma_0^2}}, \quad x \in \mathbb{R}\,. \tag{2}$$

According to Clarke's [33] popular two-dimensional isotropic scattering model, the autocorrelation function $r_{\mu_i\mu_i}(\tau)$ of $\boldsymbol{\mu}_i(t)$ $(i = 1, 2)$ equals

$$r_{\mu_i\mu_i}(\tau) = \sigma_0^2 J_0(2\pi f_{\max}\tau) \tag{3}$$

where $J_0(\cdot)$ denotes the zeroth order Bessel function of the first kind, and $f_{\max}$ is the maximum

[1] Throughout the paper, we use bold letters to indicate stochastic processes as well as random variables, and normal letters are used for sample functions and realizations (outcomes) of random variables.

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Doppler frequency. The principle of Rice's sum-of-sinusoids [1,2] is based on a superposition of an infinite number of weighted sinusoids with equidistant frequencies and random phases. Using this principle, the Gaussian process $\boldsymbol{\mu}_i(t)$ can be modelled as

$$\boldsymbol{\mu}_i(t) = \lim_{N_i \to \infty} \sum_{n=1}^{N_i} c_{i,n} \cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n}) \quad (4)$$

where N_i denotes the number of sinusoids. According to Rice [1,2], the gains $c_{i,n}$ and frequencies $f_{i,n}$ in the expression above are given by

$$c_{i,n} = 2\sqrt{\Delta f_i \, S_{\mu_i\mu_i}(f_{i,n})} \quad (5)$$

and

$$f_{i,n} = n\Delta f_i \quad (6)$$

respectively, where $i = 1, 2$ and $n = 1, 2, \ldots, N_i$. In (4), the phases $\boldsymbol{\theta}_{i,n}$ are assumed to be random variables having a uniform distribution in the interval $(0, 2\pi]$. The quantity Δf_i appearing in (5) and (6) is chosen in such a way that the relevant one-sided frequency range is completely covered by (6). The symbol $S_{\mu_i\mu_i}(f)$ in (5) denotes the Doppler power spectral density, which is defined as the Fourier transform of the autocorrelation function $r_{\mu_i\mu_i}(\tau)$.

Since the number of sinusoids N_i in (4) is infinite, a software or hardware realization of $\boldsymbol{\mu}_i(t)$ does not exist. Nevertheless, the process $\boldsymbol{\mu}_i(t)$ in (4) is useful because it describes the stochastic reference model. In mobile fading channel modelling, a reference model is important for two reasons. First, the non-realizable stochastic reference model is the starting point for the derivation of a realizable stochastic (or deterministic) simulation model. And second, the reference model enables us to measure the performance of the resulting simulation model described in the next section.

3 Stochastic and Deterministic Simulation Models

Generally, simulation models for fading channels can be classified into two major classes, namely, stochastic and deterministic ones. By employing the sum-of-sinusoids principle, a realizable *stochastic simulation model* is obtained from (4) by using only a finite number of sinusoids N_i. The underlying stochastic process will be denoted as

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} c_{i,n} \cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n}) \quad (7)$$

where the gains $c_{i,n}$ and frequencies $f_{i,n}$ are still constants, and the phases $\boldsymbol{\theta}_{i,n}$ are again uniformly distributed random variables. A comparison with (4) shows directly that the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ tends to the Gaussian process $\boldsymbol{\mu}_i(t)$ as $N_i \to \infty$.

Now, we consider the phases $\theta_{i,n}$ as outcomes (realizations) of a random generator with a uniform distribution in the interval $(0, 2\pi]$. In this case, the phases $\theta_{i,n}$ are real-valued constant quantities, and the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ results in a sample function denoted by

$$\tilde{\mu}_i(t) = \sum_{n=1}^{N_i} c_{i,n} \cos(2\pi f_{i,n} t + \theta_{i,n}) \,. \quad (8)$$

Note that different realizations of the set $\{\theta_{i,n}\}$ result in different sample functions $\tilde{\mu}_i(t)$. It should also be noted that the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ can be interpreted as a family (or an ensemble) of sample functions, i.e.,

$$\tilde{\boldsymbol{\mu}}_i(t) = \{\tilde{\mu}_i(t) \,|\, t \in \mathbb{R}\} \quad (9)$$

where $\mathbb{R}$ denotes the set of real numbers. A sample function is in general deterministic or nondeterministic. To stress that fact that the sample function $\tilde{\mu}_i(t)$ in (8) is completely deterministic, we call $\tilde{\mu}_i(t)$ a deterministic process. Such a process can easily be implemented on a hardware or software platform. The realization of a deterministic process $\tilde{\mu}_i(t)$ in form of a hardware system or a software program is called the *deterministic simulation model*.

Guided by the presentation in [34, pp. 373–374], the relationships between reference models, stochastic simulation models, and deterministic simulation models can be established as shown in Fig. 1.

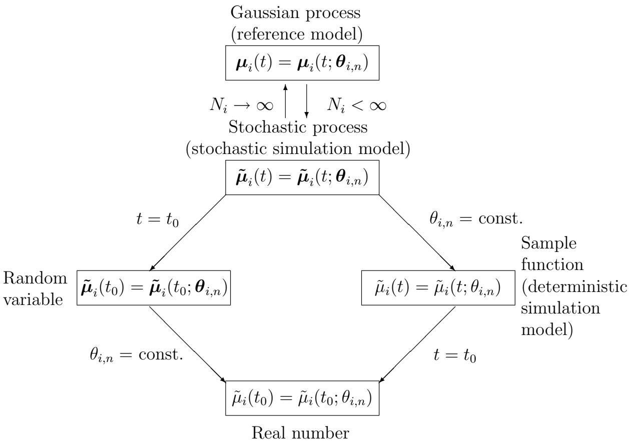

Fig. 1. Relationships between Gaussian processes (reference models), stochastic processes (stochastic simulation models), sample functions (deterministic simulation models), random variables, and real numbers.

This figure illustrates the following interpretations:

1. If $\boldsymbol{\theta}_{i,n}$ is a random variable and $N_i \to \infty$, then $\tilde{\boldsymbol{\mu}}_i(t)$ tends to a Gaussian process $\boldsymbol{\mu}_i(t)$ describing the reference model.

2. If $\boldsymbol{\theta}_{i,n}$ is a random variable and N_i is finite, then $\tilde{\boldsymbol{\mu}}_i(t)$ is a stochastic process, i.e., a family (or an ensemble) of sample functions $\tilde{\mu}_i(t)$. The stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ describes the stochastic simulation model.

3. If $\theta_{i,n}$ is fixed, then $\tilde{\boldsymbol{\mu}}_i(t)$ results in a specific sample function $\tilde{\mu}_i(t) = \tilde{\mu}_i(t; \theta_{i,n})$ called a deterministic process. Its realization is said to be a deterministic simulation model.

4. If $t = t_0$ is fixed and $\boldsymbol{\theta}_{i,n}$ is a random variable, then $\tilde{\boldsymbol{\mu}}_i(t_0) = \{\tilde{\mu}_i(t) \,|\, t = t_0\}$ is a random variable.

5. If both $t = t_0$ and $\theta_{i,n}$ are fixed, then $\tilde{\boldsymbol{\mu}}_i(t_0)$ and $\tilde{\mu}_i(t)$ result in one and the same real number, which is equal to $\tilde{\mu}_i(t_0)$.

For the performance evaluation of the stochastic simulation model, it is important to know the conditions for which the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is stationary and ergodic. A clear understanding of stationary and ergodic processes is important for the intention of the paper. We will therefore review these two terms briefly.

4 Review of Stationary and Ergodic Processes

4.1 Stationarity

A stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is said to be *first-order stationary (FOS)* [34, p. 392] if $\tilde{\boldsymbol{\mu}}_i(t)$ and $\tilde{\boldsymbol{\mu}}_i(t+c)$ have the same statistics for any $c \in \mathbb{R}$. The density of a FOS process is independent of time, i.e.,

$$p_{\tilde{\mu}_i}(x;\, t) = p_{\tilde{\mu}_i}(x;\, t + c) \equiv p_{\tilde{\mu}_i}(x) \tag{10}$$

holds for all values of t and c. This implies that the mean and the variance of $\tilde{\boldsymbol{\mu}}_i(t)$ are independent of time as well.

A stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is said to be *wide-sense stationary (WSS)* [34, p. 388] if $\tilde{\boldsymbol{\mu}}_i(t)$ satisfies the following two conditions:

(i) The mean of $\tilde{\boldsymbol{\mu}}_i(t)$ is constant, i.e.,

$$E\{\tilde{\boldsymbol{\mu}}_i(t)\} = m_{\tilde{\mu}_i} = \text{const.} \tag{11}$$

(ii) The autocorrelation function of $\tilde{\boldsymbol{\mu}}_i(t)$ depends only on the time difference $\tau = t_1 - t_2$, i.e.,

$$\begin{aligned} r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1,\, t_2) &= r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \\ &= E\{\tilde{\boldsymbol{\mu}}_i(t)\tilde{\boldsymbol{\mu}}_i(t+\tau)\} \end{aligned} \tag{12}$$

holds for all values of t_1 and t_2.

4.2 Ergodicity

A stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is said to be *mean-ergodic* if its ensemble average $m_{\tilde{\mu}_i}$ equals the time average $\overline{m}_{\tilde{\mu}_i}$ of $\tilde{\mu}_i(t)$, i.e.,

$$m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \tilde{\mu}_i(t)\, dt\,. \tag{13}$$

The stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is *autocorrelation-ergodic* if its autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ equals the time autocorrelation function $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of $\tilde{\mu}_i(t)$, i.e.,

$$\begin{aligned} r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) &= \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \\ &:= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \tilde{\mu}_i(t)\tilde{\mu}_i(t+\tau)\, dt\,. \end{aligned} \tag{14}$$

The two definitions above imply that mean-ergodic and autocorrelation-ergodic processes are always WSS, since the mean $m_{\tilde{\mu}_i}$ in (13) is constant and the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ in (14) depends only on the time difference $\tau = t_1 - t_2$. On the other hand, however, a stationary process needs not to be ergodic [34].

5 Classification of Channel Simulators

For any given number $N_i > 0$, the sum-of-sinusoids depends on three types of parameters (gains, frequencies, and phases), each of which can be a collection of random variables or constants. However, at least one random variable is required to obtain a stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$—otherwise we get a deterministic process $\tilde{\mu}_i(t)$, as it was pointed out in Fig. 1. Therefore, altogether $2^3 = 8$ classes of sum-of-sinusoids-based simulation models for Rayleigh fading channels can be defined, seven of which are stochastic simulation models and one is completely deterministic. For example, one class of stochastic channel simulators is defined by postulating random values for the gains $\mathbf{c}_{i,n}$, frequencies $\mathbf{f}_{i,n}$, and phases $\boldsymbol{\theta}_{i,n}$. The definition and analysis of the various classes of simulation models with respect to their stationary and ergodic properties will be the topic of this section.

Whenever the gains $\mathbf{c}_{i,1}$, $\mathbf{c}_{i,2}$, $\ldots$, $\mathbf{c}_{i,N_i}$ are random variables, we assume that they are independent identically distributed (i.i.d.). The same shall hold for the sequences of random frequencies $\mathbf{f}_{i,1}$, $\mathbf{f}_{i,2}$, $\ldots$, $\mathbf{f}_{i,N_i}$ and phases $\boldsymbol{\theta}_{i,1}$, $\boldsymbol{\theta}_{i,2}$, $\ldots$, $\boldsymbol{\theta}_{i,N_i}$. From a practical point of view, it is reasonable to assume that the gains $\mathbf{c}_{i,n}$, frequencies $\mathbf{f}_{i,n}$, and phases $\boldsymbol{\theta}_{i,n}$ are mutually independent. We will therefore impose the independence of the random variables $\mathbf{c}_{i,n}$, $\mathbf{f}_{i,n}$, and $\boldsymbol{\theta}_{i,n}$ on our model.

In cases, where the gains $c_{i,n}$ and frequencies $f_{i,n}$ are constant quantities, it is assumed that they are different from zero, so that $c_{i,n} \neq 0$ and $f_{i,n} \neq 0$ hold for all values of $n = 1, 2, \ldots, N_i$ and $i = 1, 2$. We might also impose further constraints on the sum-of-sinusoids model. For example, we require that the absolute values of all frequencies, $|f_{i,n}|$, are different, i.e., (i) $|f_{i,1}| \neq |f_{i,2}| \neq \cdots \neq |f_{i,N_i}|$ for $i = 1, 2$ and (ii) $\{|f_{1,n}|\}_{n=1}^{N_1} \cap \{|f_{2,n}|\}_{n=1}^{N_2} = \emptyset$, where $\emptyset$ denotes the empty set. The former condition (i) is introduced as a measure to avoid intra-correlations, i.e., correlations within $\tilde{\mu}_i(t)$ ($i = 1, 2$), and the latter condition (ii) ensures that the cross-correlation (inter-correlation) of $\tilde{\mu}_1(t)$ and $\tilde{\mu}_2(t)$ is zero.

5.1 Class I Channel Simulators

The channel simulators of Class I are defined by the set of deterministic processes $\tilde{\mu}_i(t)$ [see (8)] with constant gains $c_{i,n}$, constant frequencies $f_{i,n}$, and constant phases $\theta_{i,n}$. Since all model parameters are constants, there is no meaning to examine the stationary and ergodic properties of this class of channel simulators. However, we will benefit from the investigation of the mean and the autocorrelation function of $\tilde{\mu}_i(t)$.

Due to the fact that $\tilde{\mu}_i(t)$ is a deterministic process, its mean $\overline{m}_{\tilde{\mu}_i}$ has to be determined by using time averages instead of statistical averages. Thus, substituting (8) in the right-hand side of (13) and taking into account that $f_{i,n} \neq 0$, we obtain

$$\overline{m}_{\tilde{\mu}_i} = 0\,. \tag{15}$$

Hence, the deterministic process $\tilde{\mu}_i(t)$ of the simulation model has the same mean value as the stochastic process $\boldsymbol{\mu}_i(t)$ of the reference model, i.e., $\overline{m}_{\tilde{\mu}_i} = m_{\mu_i} = 0$.

Similarly, the autocorrelation function $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of $\tilde{\mu}_i(t)$ has to be determined by using time averages. Substituting (8) in the right-hand side of (14) gives the autocorrelation function $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of $\tilde{\mu}_i(t)$ in the following form [3]

$$\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sum_{n=1}^{N_i} \frac{c_{i,n}^2}{2} \cos(2\pi f_{i,n}\tau)\,. \tag{16}$$

One should note that $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ depends on the number of sinusoids N_i, the gains $c_{i,n}$, and the frequencies $f_{i,n}$, but not on the phases $\theta_{i,n}$. It is interesting to note that $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ approaches $r_{\mu_i\mu_i}(\tau)$ as $N_i \to \infty$, if the model parameters $c_{i,n}$ and $f_{i,n}$ are computed according to the method of exact Doppler spread [5].

5.2 Class II Channel Simulators

The channel simulators of Class II are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with constant gains $c_{i,n}$, constant frequencies $f_{i,n}$, and random phases $\boldsymbol{\theta}_{i,n}$, which are uniformly distributed in the interval $(0, 2\pi]$. In this case, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ has exactly the same form as in (7).

The distribution of a sum-of-sinusoids with random phases has first been studied in [35], where it was shown that the first-order density $p_{\tilde{\mu}_i}(x)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is given by

$$p_{\tilde{\mu}_i}(x) = 2\int_0^{\infty}\left[\prod_{n=1}^{N_i} J_0(2\pi c_{i,n}\nu)\right]\cos(2\pi\nu x)\,d\nu\,. \quad (17)$$

Note that (17) is independent of time and depends only on the number of sinusoids N_i and the gains $c_{i,n}$. If all gains are equal to $c_{i,n} = \sigma_0\sqrt{2/N_i}$, then it follows from the central limit theorem of Lindberg-Lévy that the density $p_{\tilde{\mu}_i}(x)$ in (17) approaches to the Gaussian density $p_{\mu_i}(x)$ introduced in (2) if N_i tends to infinity, i.e., $p_{\tilde{\mu}_i}(x) \to p_{\mu_i}(x)$ as $N_i \to \infty$. However, it is widely accepted that the approximation $p_{\tilde{\mu}_i}(x) \approx p_{\mu_i}(x)$ is sufficiently good if $N_i \geq 7$.

From (7), it follows that the mean $m_{\tilde{\mu}_i}$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is constant and equal to 0, because

$$\begin{aligned} m_{\tilde{\mu}_i} &= E\{\tilde{\boldsymbol{\mu}}_i(t)\} \\ &= \sum_{n=1}^{N_i} c_{i,n}\, E\{\cos(2\pi f_{i,n}t + \boldsymbol{\theta}_{i,n})\} \\ &= 0\,. \end{aligned} \quad (18)$$

Substituting (7) in (12) results in the autocorrelation function

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sum_{n=1}^{N_i} \frac{c_{i,n}^2}{2}\cos(2\pi f_{i,n}\tau) \quad (19)$$

which is a function of $\tau = t_1 - t_2$.

From (17)–(19), we may conclude that $\tilde{\boldsymbol{\mu}}_i(t)$ is both FOS and WSS, because the first-order density $p_{\tilde{\mu}_i}(x)$ is independent of time and the conditions (i)–(ii) [see (11)–(12)] are fulfilled. For a specific realization of the random phases $\boldsymbol{\theta}_{i,n}$, it follows from Fig. 1 that the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ results in a deterministic process (sample function) $\tilde{\mu}_i(t)$. In other words, the Class I is a subset of the Class II. We realize that the identity $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ holds, which states with reference to (13) that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic. A comparison of (19) and (16) shows that $\tilde{\boldsymbol{\mu}}_i(t)$ is also autocorrelation-ergodic, since the criterion $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ [see (14)] is fulfilled.

5.3 Class III Channel Simulators

The channel simulators of Class III are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with constant gains $c_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and constant phases $\theta_{i,n}$, i.e.,

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} c_{i,n}\cos(2\pi\mathbf{f}_{i,n}t + \theta_{i,n})\,. \quad (20)$$

Independent of a given specific distribution of $\mathbf{f}_{i,n}$, it is shown in Appendix A that the probability density function $p_{\tilde{\mu}_i}(x;t)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is given by

$$\begin{aligned} p_{\tilde{\mu}_i}(x;t) &= \int_{-\infty}^{\infty}\prod_{n=1}^{N_i}\left[\int_{-\infty}^{\infty} p_{\tilde{\mu}_{i,n}}(y;t)\right. \\ &\qquad \left.\cdot\, e^{j2\pi\nu y}dy\right] e^{-j2\pi\nu x}d\nu \end{aligned} \quad (21)$$

where

$$p_{\tilde{\mu}_{i,n}}(y;t_0) = \begin{cases} \dfrac{\sum\limits_{k=-\infty}^{\infty} p_f(x_k)}{2\pi|t_0 c_{i,n}|\sqrt{1-\left(\dfrac{y}{c_{i,n}}\right)^2}}\,, & |y| < c_{i,n} \\ 0\,, & |y| \geq c_{i,n}\,. \end{cases} \quad (22)$$

In the expression above, $p_f(\cdot)$ denotes the common density of the frequencies $\mathbf{f}_{i,n}$ and the x_k's are the solutions of the equation $y = c_{i,n}\cos(2\pi x t_0 + \theta_{i,n})$. The result in (21) demonstrates that $p_{\tilde{\mu}_i}(x;t)$ is a function of time. Consequently, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is in general not FOS. However, it is also shown in Appendix A that $p_{\tilde{\mu}_{i,n}}(x;t)$ becomes independent of t if t approaches to $\pm\infty$. In this case, the density in (21) can be simplified to

$$p_{\tilde{\mu}_i}(x) \;\to\; 2\int_0^{\infty}\left[\prod_{n=1}^{N_i} J_0(2\pi c_{i,n}\nu)\right]\cos(2\pi\nu x)\,d\nu \quad \text{as} \quad t\to\pm\infty\,. \quad (23)$$

Note that this expression equals the density of the Class II channel simulators, as can immediately be seen by comparing (23) with (17). With regard to practical implications, we may conclude from (23) that a Class III channel simulator behaves approximately like a quasi FOS channel simulator if t is sufficiently large.

To be more specific, we apply the Monte Carlo method [6, 7] in order to compute the model parameters $c_{i,n}$ and $\mathbf{f}_{i,n}$. According to this method, the gains $c_{i,n}$ and frequencies $\mathbf{f}_{i,n}$ are given by

$$\begin{aligned} c_{i,n} &= \sigma_0\sqrt{\frac{2}{N_i}} \quad \text{and} \\ \mathbf{f}_{i,n} &= f_{\max}\sin(\pi\,\mathbf{u_{i,n}}) \end{aligned} \quad (24a,b)$$

respectively, where $\mathbf{u}_{i,n}$ is a random variable which is uniformly distributed in the interval $(0, 1]$. For the phases $\theta_{i,n}$, we still assume that they are constant quantities.

By taking into account that the gains $c_{i,n}$ and phases $\theta_{i,n}$ are constant quantities, the mean $m_{\tilde{\mu}_i}(t)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is obtained by computing the statistical average of (20) with respect to the random characteristics of the frequencies $\mathbf{f}_{i,n}$. Thus, we obtain

$$\begin{aligned} m_{\tilde{\mu}_i}(t) &= E\{\tilde{\boldsymbol{\mu}}_i(t)\} \\ &= \sum_{n=1}^{N_i} c_{i,n}\, E\{\cos(2\pi \mathbf{f}_{i,n} t + \theta_{i,n})\} \\ &= 2\sigma_0 J_0(2\pi f_{\max} t)\sqrt{\frac{2}{N_i}} \sum_{n=1}^{N_i} \cos(\theta_{i,n}) \quad (25) \end{aligned}$$

where we have used the integral representation of the zeroth order Bessel function of the first kind $J_0(\cdot)$ in [36, Eq. (9.1.18)]. Obviously, the mean $m_{\tilde{\mu}_i}(t)$ changes generally with time. To avoid this, we impose the boundary condition $\sum_{n=1}^{N_i} \cos(\theta_{i,n}) = 0$ on the phases $\theta_{i,n}$ of the Class III channel simulators. In this case, the mean of $\tilde{\boldsymbol{\mu}}_i(t)$ is not only constant but also equal to zero, i.e., $m_{\tilde{\mu}_i} = 0$.

In general, the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is not only a function of the time difference $\tau = t_1 - t_2$, because

$$\begin{aligned} r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2) &= E\{\tilde{\boldsymbol{\mu}}_i(t_1)\tilde{\boldsymbol{\mu}}_i(t_2)\} \\ &= \sigma_0^2 J_0(2\pi f_{\max} t_1) J_0(2\pi f_{\max} t_2) \\ &\quad \cdot \frac{2}{N_i} \sum_{n=1}^{N_i} \sum_{\substack{m=1 \\ m\neq n}}^{N_i} \cos(\theta_{i,n})\cos(\theta_{i,m}) \\ &\quad + \sigma_0^2 J_0(2\pi f_{\max}(t_1 + t_2)) \\ &\quad \cdot \frac{1}{N_i} \sum_{n=1}^{N_i} \cos(2\theta_{i,n}) \\ &\quad + \sigma_0^2 J_0(2\pi f_{\max}(t_1 - t_2)) \,. \quad (26) \end{aligned}$$

However, if we impose the boundary conditions $\sum_{n=1}^{N_i} \cos(\theta_{i,n}) = 0$ and $\sum_{n=1}^{N_i} \cos(2\theta_{i,n}) = 0$ on the phases $\theta_{i,n}$ of the channel simulator,[2] then (26) reduces to the expression

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sigma_0^2 J_0(2\pi f_{\max}\tau) \qquad (27)$$

which states that the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ is only a function of the time difference $\tau = t_1 - t_2$. In this case, the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of the stochastic simulation model is identical to the autocorrelation function $r_{\mu_i\mu_i}(\tau)$ of the reference model described by (3). Thus, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is WSS, since the conditions (i)–(ii) [see (11)–(12)] are fulfilled. If, in addition, $t \to \pm\infty$, then $\tilde{\boldsymbol{\mu}}_i(t)$ tends to a FOS process. Also, the condition $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ is fulfilled, implying that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic. But a comparison of (27) and (16) reveals that the inequality $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \neq \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ holds. Consequently, the stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ of the Class III channel simulators are non-autocorrelation-ergodic. The problems of non-autocorrelation-ergodic channel simulators have been discussed in [37].

5.4 Class IV Channel Simulators

The channel simulators of Class IV are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with constant gains $c_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and random phases $\boldsymbol{\theta}_{i,n}$, which are uniformly distributed in the interval $(0,\ 2\pi]$. Thus, the stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ has the following form

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} c_{i,n} \cos(2\pi \mathbf{f}_{i,n} t + \boldsymbol{\theta}_{i,n}) \,. \qquad (28)$$

The assumption that the frequencies $\mathbf{f}_{i,n}$ are random variables has no effect on the density $p_{\tilde{\mu}_i}(x)$ in (17). Hence, the density $p_{\tilde{\mu}_i}(x)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is still given by (17). Also the mean $m_{\tilde{\mu}_i}$ is equal to zero, and, thus, identical to (18). But the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ in (19) has to be averaged with respect to the distribution of the frequencies $\mathbf{f}_{i,n}$, i.e.,

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sum_{n=1}^{N_i} \frac{c_{i,n}^2}{2}\, E\{\cos(2\pi \mathbf{f}_{i,n}\tau)\} \,. \qquad (29)$$

Alternatively, the expression in (29) can be derived by using the result of Example 9-14 in [34, pp. 391–392]. Independent of a specific distribution of $\mathbf{f}_{i,n}$, we can say that $\tilde{\boldsymbol{\mu}}_i(t)$ is FOS and WSS, since the density $p_{\tilde{\mu}_i}(x)$ according to (17) is independent of time and the conditions (i)–(ii) [see (11)–(12)] are fulfilled. Furthermore, the condition $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ is fulfilled, which implies that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic.

When applying the Monte Carlo method [6, 7],

[2] By using trigonometric addition formulas, it can be shown that the system of nonlinear equations, $\sum_{n=1}^{N_i} \cos(\theta_{i,n}) = 0$ and $\sum_{n=1}^{N_i} \cos(2\theta_{i,n}) = 0$, can always be solved if $N_i > 1$. (The proof is omitted here for reasons of brevity.)

the gains $c_{i,n}$ and frequencies $\mathbf{f}_{i,n}$ are given by (24a) and (24b), respectively. Using these equations in (29) results in

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sigma_0^2 J_0(2\pi f_{\max}\tau)\,. \qquad (30)$$

Obviously, the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of the stochastic simulation model is identical to the autocorrelation function $r_{\mu_i\mu_i}(\tau)$ of the reference model described by (3). However, a comparison of (30) and (16) shows that $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \neq \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$. Thus, the stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ of the Class IV channel simulators are non-autocorrelation ergodic. Recall that this is also the case for the channel simulators of Class III.

5.5 Class V Channel Simulators

The channel simulators of Class V are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with random gains $\mathbf{c}_{i,n}$, constant frequencies $f_{i,n}$, and constant phases $\theta_{i,n}$, i.e.,

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} \mathbf{c}_{i,n}\cos(2\pi f_{i,n}t + \theta_{i,n})\,. \qquad (31)$$

The derivation of the density $p_{\tilde{\mu}_i}(x;t)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is similar to the procedure described in Appendix A. For reasons of brevity, we only present here the final result, which can be read as follows

$$p_{\tilde{\mu}_i}(x;\,t) = \int_{-\infty}^{\infty}\left[\prod_{n=1}^{N_i}\int_{-\infty}^{\infty} p_c(y) e^{j2\pi y\nu\cos(2\pi f_{i,n}t+\theta_{i,n})} \cdot dy\right] e^{-j2\pi\nu x}\, d\nu \qquad (32)$$

where $p_c(\cdot)$ denotes the common density function of the gains $\mathbf{c}_{i,n}$. The above expression can be interpreted as follows. The inner integral represents the characteristic function of a single sinusoid $\tilde{\boldsymbol{\mu}}_{i,n}(t) = \mathbf{c}_{i,n}\cos(2\pi f_{i,n}t + \theta_{i,n})$, where the gains $\mathbf{c}_{i,n}$ are i.i.d. random variables described by the density $p_c(\cdot)$. The product form of the characteristic function $\Psi_{\tilde{\mu}_i}(\nu)$ of the sum-of-sinusoids $\tilde{\boldsymbol{\mu}}_i(t)$ is a result of (31), and the outer integral represents the inverse transform of $\Psi_{\tilde{\mu}_i}(\nu)$. From (32), we realize that the density $p_{\tilde{\mu}_i}(x;t)$ is a function of time. Hence, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is not FOS.

In the following, we impose on the stochastic channel simulator the condition that the gains $\mathbf{c}_{i,n}$ are i.i.d. random variables with zero mean and variance σ_c^2, i.e., $E\{\mathbf{c}_{i,n}\} = 0$ and $\mathrm{Var}\{\mathbf{c}_{i,n}\} = E\{\mathbf{c}_{i,n}^2\} = \sigma_c^2$. Then, the mean of $\tilde{\boldsymbol{\mu}}_i(t)$ is constant and equal to zero, because

$$\begin{aligned} m_{\tilde{\mu}_i} &= E\{\tilde{\boldsymbol{\mu}}_i(t)\} \\ &= E\left\{\sum_{n=1}^{N_i}\mathbf{c}_{i,n}\cos(2\pi f_{i,n}t+\theta_{i,n})\right\} \\ &= \sum_{n=1}^{N_i} E\{\mathbf{c}_{i,n}\}\cos(2\pi f_{i,n}t+\theta_{i,n}) \\ &= 0\,. \end{aligned} \qquad (33)$$

The autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1,t_2)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is obtained by substituting (31) in (12). Taking into account that the gains $\mathbf{c}_{i,1}, \mathbf{c}_{i,2}, \ldots, \mathbf{c}_{i,N_i}$ are i.i.d. random variables of zero mean, we find

$$\begin{aligned} r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1,t_2) = \frac{\sigma_c^2}{2}\sum_{n=1}^{N_i} &[\cos(2\pi f_{i,n}(t_1-t_2)) \\ &+ \cos(2\pi f_{i,n}(t_1+t_2)+2\theta_{i,n})]\,. \end{aligned} \qquad (34)$$

Without imposing any specific distribution on $\mathbf{c}_{i,n}$, we can realize that $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1,t_2)$ is not only a function of $\tau = t_1 - t_2$ but also of $t_1 + t_2$. Hence, $\tilde{\boldsymbol{\mu}}_i(t)$ is not even WSS. The condition $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ is fulfilled if $E\{\mathbf{c}_{i,n}\} = 0$, so that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic. A comparison of (34) and (16) shows that $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \neq \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$. Therefore, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is non-autocorrelation-ergodic.

In the following, we study the density $p_{\tilde{\mu}_i}(x;t)$ in (32) for three specific distributions $p_c(y)$ of the gains $\mathbf{c}_{i,n}$.

Case 1: In the first case, we derive the distribution of $\tilde{\boldsymbol{\mu}}_i(t)$ under the condition that the gains are given by (24a), i.e., the gains $c_{i,n}$ are no random variables anymore but constant quantities. This implies that the density function of $c_{i,n}$ is equal to

$$p_c(y) = \delta(y - c_{i,n}) \qquad (35)$$

where $\delta(\cdot)$ being the delta function and $c_{i,n} =$

$\sigma_0\sqrt{2/N_i}$. Inserting (35) in (32) gives us

$$\begin{aligned} p_{\tilde{\mu}_i}(x;\,t) &= \int\limits_{-\infty}^{\infty}\left[\prod_{n=1}^{N_i} e^{j2\pi\nu c_{i,n}\cos(2\pi f_{i,n}t+\theta_{i,n})}\right] \\ &\quad\cdot e^{-j2\pi\nu x}\,d\nu \\ &= \int\limits_{-\infty}^{\infty} e^{j2\pi\nu\sum_{n=1}^{N_i} c_{i,n}\cos(2\pi f_{i,n}t+\theta_{i,n})} \\ &\quad\cdot e^{-j2\pi\nu x}d\nu \\ &= \int\limits_{-\infty}^{\infty} e^{-j2\pi\nu(x-\tilde{\mu}_i(t))}d\nu \\ &= \delta(x-\tilde{\mu}_i(t))\,. \end{aligned} \tag{36}$$

This result describes the instantaneous density of the deterministic process $\tilde{\mu}_i(t)$.

Case 2: In the second case, we assume that the gains $\mathbf{c}_{i,n}$ are given by $\mathbf{c}_{i,n} = \sin(2\pi\mathbf{u}_{i,n})$, where $\mathbf{u}_{i,n}$ denotes again a random variable with a uniform distribution in the interval $(0,1]$. Here, the gains are allowed to take on negative values. Typically, gains are positive, with the negative sign being attributed to a phase in the interval $(0,2\pi]$. From a transformation of random variables [34], we find the density $p_c(y)$ of $\mathbf{c}_{i,n}$ in the form

$$p_c(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}\,, & |y|<1 \\ 0\,, & |y|\geq 1\,. \end{cases} \tag{37}$$

The equation above defines the well-known bathtub shape, like the classical Doppler spectrum for Clarke's isotropic scattering model. This is not surprising, as the definition of the gains is closely related to the Doppler frequencies of the isotropic scattering model. Substituting (37) in (32) and solving the inner integral by using [36, Eq. (9.1.18)] enables us to express the first-order density of $\tilde{\boldsymbol{\mu}}_i(t)$ as

$$\begin{aligned} p_{\tilde{\mu}_i}(x;\,t) &= 2\int\limits_{0}^{\infty}\left[\prod_{n=1}^{N_i} J_0(2\pi\cos(2\pi f_{i,n}t+\theta_{i,n})\nu)\right] \\ &\quad\cdot\cos(2\pi\nu x)\,d\nu\,. \end{aligned} \tag{38}$$

Note that (38) is identical to the density function in (17), when substituting the quantities $c_{i,n}$ by $\cos(2\pi f_{i,n}t+\theta_{i,n})$.

Case 3: Finally, we consider the gains $\mathbf{c}_{i,n}$ as independent Gaussian distributed random variables, each with zero mean and variance σ_c^2, i.e.,

$$p_c(y) = \frac{1}{\sqrt{2\pi}\sigma_c}\,e^{-\dfrac{y^2}{2\sigma_c^2}}\,. \tag{39}$$

Then, after substituting (39) in (32) and using [36, Eq. (7.4.6)], we get the following density function $p_{\tilde{\mu}_i}(x;t)$ of $\tilde{\boldsymbol{\mu}}_i(t)$

$$\begin{aligned} p_{\tilde{\mu}_i}(x;t) =& \int\limits_{-\infty}^{\infty}\left\{\prod_{n=1}^{N_i}\frac{2}{\sqrt{2\pi}\sigma_c}\int\limits_{0}^{\infty}\cdot e^{-y^2/(2\sigma_c^2)}\right. \\ &\left.\cdot\cos[2\pi y\nu\cos(2\pi f_{i,n}t+\theta_{i,n})]\,dy\right\} \\ &\cdot e^{-j2\pi\nu x}\,d\nu \\ =& 2\int\limits_{0}^{\infty}\left[\prod_{n=1}^{N_i} e^{-2[\pi\sigma_c\nu\cos(2\pi f_{i,n}t+\theta_{i,n})]^2}\right] \\ &\cdot\cos(2\pi\nu x)\,d\nu \\ =& 2\int\limits_{0}^{\infty} e^{-2(\pi\sigma_c\nu)^2\sum\limits_{n=1}^{N_i}\cos^2(2\pi f_{i,n}t+\theta_{i,n})} \\ &\cdot\cos(2\pi\nu x)\,d\nu\,. \end{aligned} \tag{40}$$

The remaining integral in the above expression can be solved analytically by using [36, Eq. (7.4.6)] once again. This enables us to express $p_{\tilde{\mu}_i}(x;t)$ in the following closed form

$$p_{\tilde{\mu}_i}(x;t) = \frac{1}{\sqrt{2\pi}\sigma(t)}\,e^{-\dfrac{x^2}{2\sigma^2(t)}} \tag{41}$$

where $\sigma(t) = \sigma_c\sqrt{\sum_{n=1}^{N_i}\cos^2(2\pi f_{i,n}t+\theta_{i,n})}$. The result in (41) reveals that the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ behaves like a zero-mean Gaussian process with a time-variant variance $\sigma^2(t) = \sigma_c^2\sum_{n=1}^{N_i}\cos^2(2\pi f_{i,n}t+\theta_{i,n})$. Note that the frequencies $f_{i,n}$ determine the rate of change of the time-variant variance $\sigma^2(t)$, and that its time average, denoted by $\bar{\sigma}^2$, equals $\bar{\sigma}^2 = N_i\sigma_c^2/2$.

5.6 Class VI Channel Simulators

The channel simulators of Class VI are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with random gains $\mathbf{c}_{i,n}$, constant frequencies $f_{i,n}$, and random phases $\boldsymbol{\theta}_{i,n}$ with a uniform distribution in the interval $(0, 2\pi]$, i.e.,

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} \mathbf{c}_{i,n} \cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n}) \,. \qquad (42)$$

By taking into account that random variables $\mathbf{c}_{i,n}$ and $\boldsymbol{\theta}_{i,n}$ are independent, it is shown in Appendix B that the density $p_{\tilde{\mu}_i}(x)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ can be expressed as

$$\begin{aligned} p_{\tilde{\mu}_i}(x) &= 2\int_0^\infty \left[\int_{-\infty}^{\infty} p_c(y) J_0(2\pi y\nu)\, dy\right]^{N_i} \\ &\quad \cdot \cos(2\pi\nu x)\, d\nu \qquad (43) \end{aligned}$$

where $p_c(\cdot)$ denotes again the density function of the gains $\mathbf{c}_{i,n}$.

The mean $m_{\tilde{\mu}_i}$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is obtained as follows

$$\begin{aligned} m_{\tilde{\mu}_i} &= E\{\tilde{\boldsymbol{\mu}}_i(t)\} \\ &= E\left\{\sum_{n=1}^{N_i} \mathbf{c}_{i,n} \cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n})\right\} \\ &= \sum_{n=1}^{N_i} E\{\mathbf{c}_{i,n}\} E\{\cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n})\} \\ &= 0 \qquad (44) \end{aligned}$$

where we have made use of the fact that $E\{\cos(2\pi f_{i,n} t + \boldsymbol{\theta}_{i,n})\} = 0$.

The autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is obtained by averaging $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ according to (19) with respect to the distribution of the gains $\mathbf{c}_{i,n}$, i.e.,

$$\begin{aligned} r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) &= \sum_{n=1}^{N_i} \frac{E\{\mathbf{c}_{i,n}^2\}}{2} \cos(2\pi f_{i,n}\tau) \\ &= \frac{\sigma_c^2 + m_c^2}{2} \sum_{n=1}^{N_i} \cos(2\pi f_{i,n}\tau) \quad (45) \end{aligned}$$

where σ_c^2 and m_c are the variance and the mean of $\mathbf{c}_{i,n}$, respectively. Since the conditions in (10)–(12) are fulfilled, we have proved that $\tilde{\boldsymbol{\mu}}_i(t)$ is a FOS and WSS process. Also the condition $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ is fulfilled, so that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic. Without imposing any specific distribution on $\mathbf{c}_{i,n}$, we can say that the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is different from the autocorrelation function $\overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of a sample function $\tilde{\mu}_i(t)$, i.e., $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \neq \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$. In this context, the process $\tilde{\boldsymbol{\mu}}_i(t)$ is in general non-autocorrelation-ergodic.

5.7 Class VII Channel Simulators

This class of channel simulators involves all stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with random gains $\mathbf{c}_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and constant phases $\theta_{i,n}$, i.e.,

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} \mathbf{c}_{i,n} \cos(2\pi \mathbf{f}_{i,n} t + \theta_{i,n}) \qquad (46)$$

where, with regards to practical situations, it is assumed that the gains $\mathbf{c}_{i,n}$ and frequencies $\mathbf{f}_{i,n}$ are independent, as stated above.

To find the density $p_{\tilde{\mu}_i}(x)$ of the Class VII channel simulators, we consider—for reasons of simplicity—the case $t \to \pm\infty$ and we exploit the fact that the density of the Class III channel simulators can be considered as the conditional density $p_{\tilde{\mu}_i}(x|\mathbf{c}_{i,n} = c_{i,n})$. Since this density is given by (23) for $t \to \pm\infty$, the desired density $p_{\tilde{\mu}_i}(x)$ of the Class VII channel simulators can be obtained by averaging $p_{\tilde{\mu}_i}(x|\mathbf{c}_{i,n} = c_{i,n})$ over the distribution $p_c(y)$ of the N_i i.i.d. random variables $\mathbf{c}_{i,1}$, $\mathbf{c}_{i,2}$, ..., $\mathbf{c}_{i,N_i}$, i.e.,

$$\begin{aligned} p_{\tilde{\mu}_i}(x) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\tilde{\mu}_i}(x|\mathbf{c}_{i,1} = y_1, \ldots, \mathbf{c}_{i,N_i} = y_{N_i}) \\ &\quad \cdot \left[\prod_{n=1}^{N_i} p_c(y_n)\right] dy_1 \cdots dy_{N_i} \\ &= 2\int_0^{\infty} \left[\int_{-\infty}^{\infty} p_c(y) J_0(2\pi y\nu)\, dy\right]^{N_i} \\ &\quad \cdot \cos(2\pi\nu x)\, d\nu \quad \text{as} \quad t \to \pm\infty \,. \qquad (47) \end{aligned}$$

Note that this result is identical to the density in (43). If t is finite, then we have to repeat the above procedure by using (21) instead of (23). In this case, it turns out that the density $p_{\tilde{\mu}_i}(x;t)$ depends on time t, so that $\tilde{\boldsymbol{\mu}}_i(t)$ is not FOS.

The mean $m_{\tilde{\mu}_i}(t)$ of $\tilde{\boldsymbol{\mu}}_i(t)$ is obtained as follows

$$\begin{aligned} m_{\tilde{\mu}_i}(t) &= E\{\tilde{\boldsymbol{\mu}}_i(t)\} \\ &= E\left\{\sum_{n=1}^{N_i} \mathbf{c}_{i,n} \cos(2\pi \mathbf{f}_{i,n} t + \theta_{i,n})\right\} \\ &= \sum_{n=1}^{N_i} E\{\mathbf{c}_{i,n}\} E\{\cos(2\pi \mathbf{f}_{i,n} t + \theta_{i,n})\} \\ &= m_c \sum_{n=1}^{N_i} E\{\cos(2\pi \mathbf{f}_{i,n} t + \theta_{i,n})\} \qquad (48) \end{aligned}$$

where m_c denotes the mean value of the i.i.d. random variables $\mathbf{c}_{i,n}$. Hence, $m_{\tilde{\mu}_i}(t)$ is only zero, if $m_c = 0$ and/or the distribution of the random frequencies $\mathbf{f}_{i,n}$ is such that $\sum_{n=1}^{N_i} E\{\cos(2\pi\mathbf{f}_{i,n}t + \theta_{i,n})\}$ can be forced to zero. The latter condition is fulfilled, if the distribution of $\mathbf{f}_{i,n}$ is an even function and if $\sum_{n=1}^{N_i} \cos(\theta_{i,n}) = 0$. Note that this statement has also been made below (25).

The autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2)$ of a Class VII channel simulator is obtained by averaging the right-hand side of (34) with respect to $\mathbf{f}_{i,n}$, i.e.,

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2) = \frac{\sigma_c^2}{2} \sum_{n=1}^{N_i} [E\{\cos(2\pi\mathbf{f}_{i,n}(t_1 - t_2))\} + E\{\cos(2\pi\mathbf{f}_{i,n}(t_1 + t_2) + 2\theta_{i,n})\}] \tag{49}$$

where σ_c^2 denotes the variance of $\mathbf{c}_{i,n}$. Obviously, the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2)$ depends on $t_1 - t_2$ and $t_1 + t_2$. However, if the conditions: (i) $E\{\mathbf{c}_{i,n}\} = m_c = 0$, (ii) the density of $\mathbf{f}_{i,n}$ is even, and (iii) $\sum_{n=1}^{N_i} \cos(2\theta_{i,n}) = 0$ are fulfilled, then $r_{\tilde{\mu}_i\tilde{\mu}_i}(t_1, t_2)$ is only a function of $\tau = t_1 - t_2$. In this case, we obtain

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \frac{\sigma_c^2}{2} \sum_{n=1}^{N_i} E\{\cos(2\pi\mathbf{f}_{i,n}\tau)\}\,. \tag{50}$$

Now, let $\mathbf{f}_{i,n}$ be given by (24b) and $\sigma_c^2 = 2\sigma_0^2/N_i$. Then the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ in (50) equals the autocorrelation function $r_{\mu_i\mu_i}(\tau)$ of the reference model in (3), i.e.,

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \sigma_0^2 J_0(2\pi f_{\max}\tau)\,. \tag{51}$$

From the investigations in this subsection, it turns out that $\tilde{\boldsymbol{\mu}}_i(t)$ is WSS and mean-ergodic, if the above mentioned boundary conditions are fulfilled. If the boundary conditions are fulfilled and $t \to \pm\infty$, then $\tilde{\boldsymbol{\mu}}_i(t)$ tends to a FOS process. In no case, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is autocorrelation-ergodic, since the condition $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ is violated.

5.8 Class VIII Channel Simulators

The Class VIII channel simulators are defined by the set of stochastic processes $\tilde{\boldsymbol{\mu}}_i(t)$ with random gains $\mathbf{c}_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and random phases $\boldsymbol{\theta}_{i,n}$, which are uniformly distributed in the interval $(0, 2\pi]$. In this case, the stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$ is of type

$$\tilde{\boldsymbol{\mu}}_i(t) = \sum_{n=1}^{N_i} \mathbf{c}_{i,n} \cos(2\pi\mathbf{f}_{i,n}t + \boldsymbol{\theta}_{i,n}) \tag{52}$$

where it is assumed that all random variables are statistically independent.

Concerning the density function $p_{\tilde{\mu}_i}(x)$, we can resort to the result in (43), which was obtained for a sum-of-sinusoids with random gains $\mathbf{c}_{i,n}$ and random phases $\boldsymbol{\theta}_{i,n}$. This result is still valid, because the random behavior of the frequencies $\mathbf{f}_{i,n}$ has no influence on $p_{\tilde{\mu}_i}(x)$ in (43). Also the mean $m_{\tilde{\mu}_i}$ is identical to (18), i.e., $m_{\tilde{\mu}_i} = 0$. To ease the derivation of the autocorrelation function $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ of $\tilde{\boldsymbol{\mu}}_i(t)$, we average $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$ in (19) with respect to the distributions of $\mathbf{c}_{i,n}$ and $\mathbf{f}_{i,n}$. This results in

$$r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) = \frac{\sigma_c^2 + m_c^2}{2} \sum_{n=1}^{N_i} E\{\cos(2\pi\mathbf{f}_{i,n}\tau)\}\,. \tag{53}$$

Without assuming any specific distribution of $\mathbf{c}_{i,n}$ and $\mathbf{f}_{i,n}$, we can say that $\tilde{\boldsymbol{\mu}}_i(t)$ is both FOS and WSS, since the conditions in (10)–(12) are fulfilled. Also, we can easily see that the condition $m_{\tilde{\mu}_i} = \overline{m}_{\tilde{\mu}_i}$ is fulfilled, proving the fact that $\tilde{\boldsymbol{\mu}}_i(t)$ is mean-ergodic. But $\tilde{\boldsymbol{\mu}}_i(t)$ is non-autocorrelation-ergodic, since $r_{\tilde{\mu}_i\tilde{\mu}_i}(\tau) \neq \overline{r}_{\tilde{\mu}_i\tilde{\mu}_i}(\tau)$.

6 Application to Parameter Computation Methods

For any given number $N_i > 0$, we have seen that the sum-of-sinusoids depends on three types of parameters (gains, frequencies, and phases), each of which can be a random variable or a constant. However, at least one random variable is required to obtain a stochastic process $\tilde{\boldsymbol{\mu}}_i(t)$—otherwise the sum-of-sinusoids defines a deterministic process $\tilde{\mu}_i(t)$. From the discussions in Section 5, it is obvious that altogether $2^3 - 1 = 7$ classes of stochastic Rayleigh fading channel simulators and one class of deterministic Rayleigh fading channel simulators can be defined. The analysis of the various classes with respect to their stationary and ergodic properties has been given in the previous section. The results are summarized in Table I. This table illustrates also the relationships between the eight classes of simulators. For

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example, the Class VIII simulator is a superset of all the other classes and a Class I simulator can belong to any of the other classes. To the best of the authors knowledge, the Classes III, V, VI, and VII have never been introduced before. The new Class VI—and under certain conditions also the new Classes III and VII—have the same stationary and ergodic properties as the well-known Class IV, which is often used in practice.

Table 1
Classes of Deterministic and Stochastic Channel Simulators and Their Statistical Properties

Class	Gains $c_{i,n}$	Frequ. $f_{i,n}$	Phases $\theta_{i,n}$	First-order stat.	Wide-sense stat.	Mean-erg.	Auto-corr. erg.
I	const.	const.	const.	–	–	–	–
II	const.	const.	RV	yes	yes	yes	yes
III	const.	RV	const.	no/ yes$^{a,\,b,\,c,\,d}$	no/ yes^{a,b,c}	no/ yes^{a,b}	no
IV	const.	RV	RV	yes	yes	yes	no
V	RV	const.	const. .	no	no	no/ /yes^{e}	no
VI	RV	const.	RV	yes	yes	yes	no
VII	RV	RV	const.	no/ yes^{a,b,c,d}	no/ yes^{a,b,c}	no/ yes$^{e \text{ or } a,b}$	no
VIII	RV	RV	RV	yes	yes	yes	no

[a]If the density of $\mathbf{f}_{i,n}$ is an even function.
[b]If the boundary condition $\sum_{n=1}^{N_i} \cos(\theta_{i,n}) = 0$ is fulfilled.
[c]If the boundary condition $\sum_{n=1}^{N_i} \cos(2\theta_{i,n}) = 0$ is fulfilled.
[d]Only in the limit $t \to \pm\infty$.
[e]If the gains $\mathbf{c}_{i,n}$ have zero mean, i.e., $E\{\mathbf{c}_{i,n}\} = 0$.

In the following, we apply the above concept to some selected parameter computation methods. Starting with the original Rice method [1, 2], we realize by considering (5) and (6) that the gains $c_{i,n}$ and frequencies $f_{i,n}$ are constant quantities. Due to the fact that the phases $\boldsymbol{\theta}_{i,n}$ are random variables, it follows from Table I that the resulting channel simulator belongs to Class II. Such a channel simulator enables the generation of stochastic processes which are not only FOS but also mean- and autocorrelation-ergodic. On the other hand, if the Monte Carlo method [6, 7] or the method due to Zheng and Xiao [9] is applied, then the gains $c_{i,n}$ are constant quantities and the frequencies $\mathbf{f}_{i,n}$ and phases $\boldsymbol{\theta}_{i,n}$ are random variables. Consequently, the resulting channel simulator can be identified as a Class IV channel simulator, which is FOS and mean-ergodic, but unfortunately non-autocorrelation-ergodic. For a given parameter computation method, the stationary and ergodic properties of the resulting channel simulator can directly be examined with the help of Table I. It goes without saying that the above concept is easy to handle and can be applied to any given parameter computation method. For some selected methods [1,2,4–10,14], the obtained results are presented in Table II.

Table 2
Overview of Parameter Computation Methods and the Statistical Properties of the Resulting Stochastic Channel Simulators

Parameter computation method	Class	First-order stat.	Wide-sense stat.	Mean-erg.	Auto-correl.-erg.
Rice method [1, 2]	II	yes	yes	yes	yes
Monte Carlo method [6, 7]	IV	yes	yes	yes	no
Jakes method [4] (with random phases)	II	yes	yes	yes	yes
Harmonic decomposition technique [14]	II	yes	yes	yes	yes
Method of equal distances [8]	II	yes	yes	yes	yes
Method of equal areas [8]	II	yes	yes	yes	yes
Mean-square-error method [8]	II	yes	yes	yes	yes
Method of exact Doppler spread [5]	II	yes	yes	yes	yes
Lp-norm method [5]	II	yes	yes	yes	yes
Method proposed by Zheng and Xiao [9]	IV	yes	yes	yes	no
Improved method by Zheng and Xiao [10]	VIII	yes	yes	yes	no

Of great popularity is the Jakes method [4]. When this method is used, then the gains $c_{i,n}$, frequencies $f_{i,n}$, and phases $\theta_{i,n}$ are constant quantities. Consequently, the Jakes channel simulator is per definition completely deterministic. To obtain the underlying stochastic channel simulator, it is advisable to replace the constant phases[3] $\theta_{i,n}$ by random phases $\boldsymbol{\theta}_{i,n}$. According to Table I, the mapping $\theta_{i,n} \mapsto \boldsymbol{\theta}_{i,n}$ transforms a Class I type channel simulator into a Class II type one, which is FOS, mean-ergodic, and autocorrelation-ergodic. It should be pointed out here that our result is in contrast with the analysis in [39], where it has been claimed that Jakes' simulator is nonstationary. However, this is not surprising when we take into account that the proof in [39] is based

[3]It should be noted that the phases $\theta_{i,n}$ in Jakes' channel simulator are equal to 0 for all $i = 1, 2$ and $n = 1, 2, \ldots, N_i$ [38].

on the assumption that the gains $\mathbf{c}_{i,n}$ are random variables. But this assumption cannot be justified in the sense of the original Jakes method, where all model parameters are constant quantities.

Nevertheless, the Jakes simulator has some disadvantages [38], which can be avoided, e.g., by using the method of exact Doppler spread [5] or the even more powerful L_p-norm method [5]. Both methods enable the design of deterministic channel simulators, where all parameters are fixed including the phases $\theta_{i,n}$. The investigation of the stationary and ergodic properties of deterministic processes makes no sense, since the concept of stationarity and ergodicity is only applicable to stochastic processes. A deterministic channel simulator can be interpreted as an emulator for sample functions of the underlying stochastic channel simulator. According to the concept of deterministic channel modelling, the corresponding stochastic simulation model is obtained by replacing the constant phases $\theta_{i,n}$ by random phases $\boldsymbol{\theta}_{i,n}$ (see Fig. 1). Important is now to realize that all stochastic channel simulators derived in this way from deterministic channels simulators are Class II channel simulators with the known statistical properties as they are listed in Table I. This must be taken into account when considering the results shown in Table II.

7 Conclusion

In this paper, the stationary and ergodic properties of Rayleigh fading channel simulators using the sum-of-sinusoids principle have been analyzed. Depending on whether the model parameters are random variables or constant quantities, altogether one class of deterministic channel simulators and seven classes of stochastic channel simulators have been defined, where four of them have never been studied before. It turned out that if and only if the phases are random variables and the gains and frequencies are constant quantities, then the resulting channel simulator is stationary and ergodic. If the frequencies are random variables, then the stochastic channel simulator is always non-autocorrelation-ergodic but stationary if certain boundary conditions are fulfilled. The worst case, however, is given when the gains are random variables and the other parameters are constants. Then, a non-stationary channel simulator is obtained. To arrive at these conclusions, it was necessary to investigate all eight classes of channel simulators. The results presented here are also of fundamental importance for the performance assessment of existing design methods. Moreover, the results give strategic guidance to engineers for the development of new parameter computation methods enabling the design of stationary and ergodic channel simulators, because the proposed scheme might prevent them from introducing random variables for the gains and frequencies. Although we have restricted our investigations to Rayleigh fading channel simulators, the obtained results are of general importance to all types of channel simulators, wherever the sum-of-sinusoids principle is employed.

Appendix A

In this appendix, we derive the density of a sum-of-sinusoids with constant gains $c_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and constant phases $\theta_{i,n}$. In the first step, we compute the density $p_{\tilde{\mu}_{i,n}}(x)$ of a single sinusoid at the time instant $t = t_0$. Hence,

$$\tilde{\boldsymbol{\mu}}_{i,n}(t_0) = c_{i,n} \cos(2\pi \mathbf{f}_{i,n} t_0 + \theta_{i,n}) \tag{54}$$

describes a random variable. Applying the concept of transformation of random variables [34] allows us to express the density of $\tilde{\boldsymbol{\mu}}_{i,n}(t)$ as

$$p_{\tilde{\mu}_{i,n}}(y; t_0) = \begin{cases} \dfrac{\sum\limits_{k=-\infty}^{\infty} p_f(x_k)}{2\pi |t_0 c_{i,n}| \sqrt{1 - \left(\dfrac{y}{c_{i,n}}\right)^2}}, & |y| < c_{i,n} \\ 0, & |y| \geq c_{i,n} \end{cases} \tag{55}$$

where $p_f(\cdot)$ denotes the common density of the frequencies $\mathbf{f}_{i,n}$ and the x_k's are the solutions of the equation $y = c_{i,n} \cos(2\pi x t_0 + \theta_{i,n})$. Note that in each interval of length $1/|t_0|$ there are two solutions. Furthermore, we notice that

$$x_{k+2} - x_k = \frac{1}{|t_0|} \to 0 \quad \text{as } |t_0| \to \infty \tag{56}$$

and consequently

$$\lim_{|t_0| \to \infty} \frac{1}{2|t_0|} \sum_{k=-\infty}^{\infty} p_f(x_k) = \int_{-\infty}^{\infty} p_f(x)\, dx = 1. \tag{57}$$

Hence, in the limit $|t_0| \to \infty$, the density

$p_{\tilde{\mu}_{i,n}}(y;t_0)$ can be expressed as

$$\begin{aligned}\lim_{|t_0|\to\infty} & p_{\tilde{\mu}_{i,n}}(y;t_0)\\ &= p_{\tilde{\mu}_{i,n}}(y)\\ &= \begin{cases} \dfrac{1}{\pi|c_{i,n}|\sqrt{1-\left(\dfrac{y}{c_{i,n}}\right)^2}}\,, & |y|<c_{i,n}\\ 0\,, & |y|\geq c_{i,n}\,. \end{cases} \end{aligned} \quad (58)$$

The characteristic function $\Psi_{\tilde{\mu}_{i,n}}(\nu;t_0)$ of the random variable $\tilde{\boldsymbol{\mu}}_{i,n}(t_0)$ is defined by the transformation

$$\Psi_{\tilde{\mu}_{i,n}}(\nu;t_0) = \int_{-\infty}^{\infty} p_{\tilde{\mu}_{i,n}}(y;t_0)e^{j2\pi\nu y}dy\,. \quad (59)$$

Since the random variables $\mathbf{f}_{i,n}$ are assumed to be statistically independent, it follows from (54) that the random variables $\tilde{\boldsymbol{\mu}}_{i,n}(t_0)$ are also statistically independent (as this holds for any transformation of independent random variables [34]). Thus, the characteristic function $\Psi_{\tilde{\mu}_i}(\nu;t_0)$ of the sum of random variables

$$\begin{aligned}\tilde{\boldsymbol{\mu}}_i(t_0) &= \sum_{n=1}^{N_i}\tilde{\boldsymbol{\mu}}_{i,n}(t_0)\\ &= \sum_{n=1}^{N_i} c_{i,n}\cos(2\pi\mathbf{f}_{i,n}t_0+\theta_{i,n})\end{aligned} \quad (60)$$

can be expressed as the product

$$\begin{aligned}\Psi_{\tilde{\mu}_i}(\nu;t_0) &= \prod_{n=1}^{N_i}\Psi_{\tilde{\mu}_{i,n}}(\nu;t_0)\\ &= \prod_{n=1}^{N_i}\int_{-\infty}^{\infty} p_{\tilde{\mu}_{i,n}}(y;t_0)e^{j2\pi\nu y}dy\,.\end{aligned} \quad (61)$$

Finally, the density $p_{\tilde{\mu}_i}(x;t_0)$ can be obtained from the above result by taking the inverse transform

$$\begin{aligned}p_{\tilde{\mu}_i}(x;t_0) &= \int_{-\infty}^{\infty}\Psi_{\tilde{\mu}_i}(\nu;t_0)e^{-j2\pi\nu x}d\nu\\ &= \int_{-\infty}^{\infty}\left[\prod_{n=1}^{N_i}\int_{-\infty}^{\infty} p_{\tilde{\mu}_{i,n}}(y;t_0)e^{j2\pi\nu y}dy\right]\\ &\quad\cdot e^{-j2\pi\nu x}d\nu\,.\end{aligned} \quad (62)$$

Note that this expression holds for all values of $t_0\in\mathbb{R}$. Therefore, we can replace t_0 by t, so that we obtain the following equation for the density of a sum-of-sinusoids with random frequencies

$$\begin{aligned}p_{\tilde{\mu}_i}(x;t) &= \int_{-\infty}^{\infty}\left[\prod_{n=1}^{N_i}\int_{-\infty}^{\infty} p_{\tilde{\mu}_{i,n}}(y;t)e^{j2\pi\nu y}dy\right]\\ &\quad\cdot e^{-j2\pi\nu x}d\nu\,.\end{aligned} \quad (63)$$

Thus, the density $p_{\tilde{\mu}_i}(x;t)$ of a sum-of-sinusoids $\tilde{\boldsymbol{\mu}}_i(t)$ with constant gains $c_{i,n}$, random frequencies $\mathbf{f}_{i,n}$, and constant phases $\theta_{i,n}$ is generally a function of time t. Consequently, $\tilde{\boldsymbol{\mu}}_i(t)$ is not FOS. In the limit $t\to\pm\infty$, however, $p_{\tilde{\mu}_i}(x;t)$ becomes independent of t, as can be seen by substituting (58) in (63) and solving the inner integral by using [40, Eq. (3.715.19)]. In this case, we obtain

$$\begin{aligned}\lim_{t\to\pm\infty} p_{\tilde{\mu}_i}(x;t) &= p_{\tilde{\mu}_i}(x)\\ &= 2\int_0^{\infty}\left[\prod_{n=1}^{N_i}J_0(2\pi c_{i,n}\nu)\right]\\ &\quad\cdot\cos(2\pi\nu x)\,d\nu\,.\end{aligned} \quad (64)$$

Thus, for $t\to\pm\infty$, the density of the Class III channel simulators equals the density of the Class I/II channel simulators.

Appendix B

In this appendix, we derive the density function $p_{\tilde{\mu}_i}(x)$ of a sum-of-sinusoids with random gains $\mathbf{c}_{i,n}$, constant frequencies $f_{i,n}$, and random phases $\boldsymbol{\theta}_{i,n}$, which are uniformly distributed in the interval $(0,2\pi]$. It is assumed that the gains $\mathbf{c}_{i,n}$ and the phases $\boldsymbol{\theta}_{i,n}$ are statistically independent.

Our starting point is the solution of the Problem 6-38 in [34, p. 239]. In this reference, we can find the probability density function of a single sinusoid $\tilde{\boldsymbol{\mu}}_{i,n}(t) = \mathbf{c}_{i,n}\cos(2\pi f_{i,n}t+\boldsymbol{\theta}_{i,n})$ in the following form

$$\begin{aligned}p_{\tilde{\mu}_{i,n}}(x) &= \frac{1}{\pi}\int_{-\infty}^{-|x|}\frac{p_c(y)}{\sqrt{y^2-x^2}}\,dy\\ &\quad+\frac{1}{\pi}\int_{|x|}^{\infty}\frac{p_c(y)}{\sqrt{y^2-x^2}}\,dy\end{aligned} \quad (65)$$

where $p_c(\cdot)$ is the common density function of the gains $\mathbf{c}_{i,n}$. Let us denote the characteristic function of $p_{\tilde{\mu}_{i,n}}(x)$ by $\Psi_{\tilde{\mu}_{i,n}}(\nu) = \Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu)+\Psi^{(2)}_{\tilde{\mu}_{i,n}}(\nu)$, where $\Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu)$ and $\Psi^{(2)}_{\tilde{\mu}_{i,n}}(\nu)$ are the characteristic functions of the density defined by the first and second integral in (65), respectively. The characteristic function $\Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu)$ can be expressed as

$$\Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{-|x|}\frac{p_c(y)}{\sqrt{y^2-x^2}}\,e^{j2\pi x\nu}\,dydx$$
$$= \frac{1}{\pi}\int_{-\infty}^{0}\int_{-y}^{y}\frac{p_c(y)}{\sqrt{y^2-x^2}}\,e^{j2\pi x\nu}\,dxdy$$
$$= \frac{1}{\pi}\int_{-\infty}^{0}p_c(y)\int_{-y}^{0}\frac{\cos(2\pi x\nu)}{\sqrt{y^2-x^2}}\,dxdy$$
$$+\frac{1}{\pi}\int_{-\infty}^{0}p_c(y)\int_{0}^{y}\frac{\cos(2\pi x\nu)}{\sqrt{y^2-x^2}}\,dxdy\,. \quad (66)$$

In the first term of (66), we substitute x by $-y\cos(\theta)$, and in the second term, we replace x by $y\cos(\theta)$. This leads to the expression

$$\Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu) = \frac{2}{\pi}\int_{-\infty}^{0}p_c(y)\int_{0}^{\pi/2}\cos(2\pi y\nu\cos\theta)\,d\theta dy \quad (67)$$

from which, by using the integral representation of the zeroth-order Bessel function of the first kind [40, Eq. (3.715.19)]

$$J_0(z) = \frac{2}{\pi}\int_{0}^{\pi/2}\cos(z\cos\theta)\,d\theta \quad (68)$$

the result

$$\Psi^{(1)}_{\tilde{\mu}_{i,n}}(\nu) = \int_{-\infty}^{0}p_c(y)J_0(2\pi y\nu)\,dy \quad (69)$$

immediately follows. Similarly, it can be shown that $\Psi^{(2)}_{\tilde{\mu}_{i,n}}(\nu)$ is given by

$$\Psi^{(2)}_{\tilde{\mu}_{i,n}}(\nu) = \int_{0}^{\infty}p_c(y)J_0(2\pi y\nu)\,dy\,. \quad (70)$$

Hence, it turns out that

$$\Psi_{\tilde{\mu}_{i,n}}(\nu) = \int_{-\infty}^{\infty}p_c(y)J_0(2\pi y\nu)\,dy\,. \quad (71)$$

Since the gains $\mathbf{c}_{i,1}$, $\mathbf{c}_{i,2}$, $\ldots$, $\mathbf{c}_{i,N_i}$ are assumed to be statistically independent, the characteristic function $\Psi_{\tilde{\mu}_i}(\nu)$ of $p_{\tilde{\mu}_i}(x)$ is given by

$$\Psi_{\tilde{\mu}_i}(\nu) = \prod_{n=1}^{N_i}\Psi_{\tilde{\mu}_{i,n}}(\nu)$$
$$= \left[\int_{-\infty}^{\infty}p_c(y)J_0(2\pi y\nu)\,dy\right]^{N_i}. \quad (72)$$

Finally, the density function $p_{\tilde{\mu}_i}(x)$ is obtained by computing the inverse transformation of the characteristic function $\Psi_{\tilde{\mu}_i}(\nu)$ given above, i.e.,

$$p_{\tilde{\mu}_i}(x) = \int_{-\infty}^{\infty}\Psi_{\tilde{\mu}_i}(\nu)\,e^{-j2\pi x\nu}\,d\nu$$
$$= 2\int_{0}^{\infty}\left[\int_{-\infty}^{\infty}p_c(y)J_0(2\pi y\nu)\,dy\right]^{N_i}\cdot\cos(2\pi\nu x)\,d\nu\,. \quad (73)$$

References:

[1] S. O. Rice, "Mathematical analysis of random noise," *Bell Syst. Tech. J.*, vol. 23, pp. 282–332, Jul. 1944.

[2] S. O. Rice, "Mathematical analysis of random noise," *Bell Syst. Tech. J.*, vol. 24, pp. 46–156, Jan. 1945.

[3] M. Pätzold, *Mobile Fading Channels.* Chichester: John Wiley & Sons, 2002.

[4] W. C. Jakes, Ed., *Microwave Mobile Communications.* Piscataway, NJ: IEEE Press, 1994.

[5] M. Pätzold, U. Killat, F. Laue, and Y. Li, "On the statistical properties of deterministic simulation models for mobile fading channels," *IEEE Trans. Veh. Technol.*, vol. 47, no. 1, pp. 254–269, Feb. 1998.

[6] P. Höher, "A statistical discrete-time model for the WSSUS multipath channel," *IEEE Trans. Veh. Technol.*, vol. 41, no. 4, pp. 461–468, Nov. 1992.

[7] H. Schulze, "Stochastische Modelle und digitale Simulation von Mobilfunkkanälen," in *U.R.S.I/ITG Conf. in Kleinheubach 1988,* Germany (FR), Proc. Kleinheubacher Reports of the German PTT, Darmstadt, Germany, 1989, vol. 32, pp. 473–483.

[8] M. Pätzold, U. Killat, and F. Laue, "A deterministic digital simulation model for Suzuki processes with application to a shadowed Rayleigh land mobile radio channel," *IEEE Trans. Veh. Technol.*, vol. 45, no. 2, pp. 318–331, May 1996.

[9] Y. R. Zheng and C. Xiao, "Improved models for the generation of multiple uncorrelated Rayleigh fading waveforms," *IEEE Communications Letters*, vol. 6, no. 6, pp. 256–258, Jun. 2002.

[10] Y. R. Zheng and C. Xiao, "Simulation models with correct statistical properties for Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 51, no. 6, pp. 920–928, Jun. 2003.

[11] T.-M. Wu and S.-Y. Tzeng, "Sum-of-sinusoids-based simulator for Nakagami-m fading channels," in *Proc. 58th IEEE Semiannual Veh. Techn. Conf., VTC'03-Fall*, Orlando, Florida, USA, Oct. 2003, pp. 158–162.

[12] N. Youssef, C.-X. Wang, and M. Pätzold, "A study on the second order statistics of Nakagami-Hoyt mobile fading channels," *IEEE Trans. Veh. Technol.*, vol. 54, no. 4, pp. 1259–1265, Jul. 2005.

[13] K.-W. Yip and T.-S. Ng, "Efficient simulation of digital transmission over WSSUS channels," *IEEE Trans. Commun.*, vol. 43, no. 12, pp. 2907–2913, Dec. 1995.

[14] P. M. Crespo and J. Jiménez, "Computer simulation of radio channels using a harmonic decomposition technique," *IEEE Trans. Veh. Technol.*, vol. 44, no. 3, pp. 414–419, Aug. 1995.

[15] J.-H. Yoo, C. Mun, J.-K. Han, and H.-K. Park, "Spatiotemporally correlated deterministic Rayleigh fading model for smart antenna systems," in *Proc. IEEE 50th Veh. Technol. Conf., VTC'99-Fall*, Amsterdam, The Netherlands, Sept. 1999, pp. 1397–1401.

[16] J.-K. Han, J.-G. Yook, and H.-K. Park, "A deterministic channel simulation model for spatially correlated Rayleigh fading," *IEEE Communications Letters*, vol. 6, no. 2, pp. 58–60, Feb. 2002.

[17] M. Pätzold and N. Youssef, "Modelling and simulation of direction-selective and frequency-selective mobile radio channels," *International Journal of Electronics and Communications*, vol. AEÜ-55, no. 6, pp. 433–442, Nov. 2001.

[18] M. Pätzold, "System functions and characteristic quantities of spatial deterministic Gaussian uncorrelated scattering processes," in *Proc. 57th IEEE Semiannual Veh. Technol. Conf., VTC 2003-Spring*, Jeju, Korea, Apr. 2003, pp. 256–261.

[19] G. Wu, Y. Tang, and S. Li, "Fading characteristics and capacity of a deterministic downlink MIMO fading channel simulation model with non-isotropic scattering," in *Proc. 14th IEEE Int. Symp. on Personal, Indoor and Mobile Radio Communications, PIMRC'03*, Beijing, China, Sept. 2003, pp. 1531–1535.

[20] C. Xiao, J. Wu, S.-Y. Leong, Y. R. Zheng, and K. B. Letaief, "A discrete-time model for spatio-temporally corralated MIMO WSSUS multipath channels," in *Proc. 2003 IEEE Wireless Communications and Networking Conference, WCNC'03*, New Orleans, Lousiana, USA, Mar. 2003, vol. 1, pp. 354–358.

[21] C.-X. Wang and M. Pätzold, "Efficient simulation of multiple correlated Rayleigh fading channels," in *Proc. 14th IEEE Int. Symp. on Personal, Indoor and Mobile Radio Communications, PIMRC 2003*, Beijing, China, Sept. 2003, pp. 1526–1530.

[22] C.-X. Wang and M. Pätzold, "Methods of generating multiple uncorrelated Rayleigh fading processes," in *Proc. 57th IEEE Semiannual Veh. Technol. Conf., VTC 2003-Spring*, Jeju, Korea, Apr. 2003, pp. 510–514.

[23] G. H. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Pers. Commun.*, vol. 6, pp. 311–335, Mar. 1998.

[24] A. J. Paulraj and C. B. Papadias, "Space-time processing for wireless communications," *IEEE Signal Processing Magazine*, vol. 14, no. 5, pp. 49–83, Nov. 1997.

[25] G. L. Stüber, *Principles of Mobile Communications.* Boston, MA: Kluwer Academic Publishers, 2nd edition, 2001.

[26] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, no. 8, pp. 1451–1458, Oct. 1998.

[27] M. Pätzold, R. García, and F. Laue, "Design of high-speed simulation models for mobile fading channels by using table look-up techniques," *IEEE Trans. Veh. Technol.*, vol. 49, no. 4, pp. 1178–1190, Jul. 2000.

[28] M. Pätzold and Q. Yao, "Perfect modeling and simulation of measured spatio-temporal wireless channels," in *Proc. 5th Int. Symp. on Wireless Personal Multimedia Communications, WPMC'02*, Honolulu, Hawaii, Oct. 2002, pp. 563–567.

[29] C.-X. Wang and M. Pätzold, "A novel generative model for burst error characterization in Rayleigh fading channels," in *Proc. 14th IEEE Int. Symp. on Personal, Indoor and Mobile Radio Communications, PIMRC 2003*, Beijing, China, Sept. 2003, pp. 960–964.

[30] C.-X. Wang and M. Pätzold, "A generative deterministic model for digital mobile fading channels," *IEEE Comm. Letters*, vol. 8, no. 4, pp. 223–225, Apr. 2004.

[31] M. Pätzold, F. Laue, and U. Killat, "A frequency hopping Rayleigh fading channel simulator with given correlation properties," in *Proc. IEEE Int. Workshop on Intelligent Signal Processing and Communication Systems, ISPACS'97*, Kuala Lumpur, Malaysia, Nov. 1997, pp. S8.1.1–S8.1.6.

[32] C.-X. Wang, M. Pätzold, and B. Itsarachai, "A deterministic frequency hopping Rayleigh fading channel simulator designed by using optimization techniques," in *Proc. 13th IEEE Int. Symp. on Personal, Indoor and Mobile Radio Communications, PIMRC 2002*, Lisbon, Portugal, Sept. 2002, pp. 478–483.

[33] R. H. Clarke, "A statistical theory of mobile-radio reception," *Bell Syst. Tech. Journal*, vol. 47, pp. 957–1000, Jul. /Aug. 1968.

[34] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes.* New York: McGraw-Hill, 4th edition, 2002.

[35] W. R. Bennett, "Distribution of the sum of randomly phased components," *Quart. Appl. Math.*, vol. 5, pp. 385–393, May 1948.

[36] M. Abramowitz and I. A. Stegun, Eds., *Pocketbook of Mathematical Functions.* Frankfurt/Main: Harri Deutsch, 1984.

[37] M. Pätzold, U. Killat, F. Laue, and Y. Li, "On the problems of Monte Carlo method based simulation models for mobile radio channels," in *Proc. IEEE 4th Int. Symp. on Spread Spectrum Techniques & Applications, ISSSTA'96*, Mayence, Germany, Sept. 1996, pp. 1214–1220.

[38] M. Pätzold and F. Laue, "Statistical properties of Jakes' fading channel simulator," in *Proc. IEEE 48th Veh. Technol. Conf., VTC'98*, Ottawa, Ontario, Canada, May 1998, pp. 712–718.

[39] M. F. Pop and N. C. Beaulieu, "Limitations of sum-of-sinusoids fading channel simulators," *IEEE Trans. on Communications*, vol. 49, no. 4, pp. 699–708, Apr. 2001.

[40] I. S. Gradstein and I. M. Ryshik, *Tables of Series, Products, and Integrals*, vol. I and II. Frankfurt: Harri Deutsch, 5th edition, 1981.