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A Developed Algorithm for Solving Constrained Linear Quadratic Problems with Time Delay

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Abstract: Discrete time linear quadratic optimization problem with time delay and system constraints is investigated. After reformulating the problem, a new procedure is proposed for its solution. It is shown that with the developed algorithm the dimensionality of the problem will not increase due to time delays. Moreover, no extra multipliers are required to handle system constrains. Simulations results are given to illustrate the capability of the proposed procedure in solving this problem.

Keywords: discrete systems, optimization theory, inequality constraints, time delay systems, computational algorithms.

1 Introduction

Most practical control systems are subject to constraints on states and/or controls. These constraints are imposed by physical conditions, and to provide efficient plant operations. Moreover, some physical systems may acquire time delays in states, and/or control for different reasons such as, communication, transportation, computational time, …etc. The solution of such control systems is challenging due to numerical difficulties encountered in finding global optimal solution.

Recently, this problem has attracted many researchers [1-4]. It has been shown that system stability and good performance can only be achieved with non-linear control law. The most prominent approaches for designing such a controller fall into either anti-windup class of techniques [5,6], or model predictive control (MPC) [7-9]. Irrespective of the drawbacks of anti-windup schemes, they are used in most SISO systems, whilst model predictive control has become the acceptable standard for complex constrained multi-variable control problems.

Hassan et al [10, 11] have developed new techniques for solving continuous time Linear Quadratic control Problems (LQP) with constraints, which has been extended to discrete time systems in [12]. The developed procedures do not require additional multipliers are needed to handle the inequality constraint, and their convergence is fast enough.

Although the above problem is complex enough, it is expected that its complexity will increase if the system includes time delays in the states and/or controls in addition to the existing constraints. This is simply due to the fact that time delays will increase the computational burden and hence, may slow down the convergence to the optimal solution.

In this paper, discrete time LQP with time delays in the states and/or controls, as well as system constraints, is considered. The developed technique in [12] is extended to handle the problem at hand. Time delays are treated without increasing the dimensionality of the system. To do that, the problem is reformulated to another optimization one, which when solved, leads to the optimal solution of the original one. Then, based on the derived necessary conditions of optimality, an algorithm is developed to solve the resulted set of equations which leads to the optimal solution. It is worth mentioning that, the obtained control strategy is open loop. However, the convergence is fast enough which allows its application on-line in many real life problems.

The rest of the paper is divided into the following sections. Discrete LQP with time delays and system constraints is described and reformulated in Section 2. The new developed procedure and its associated algorithm are presented in Section 3. To illustrate the applicability of the proposed technique, an example is given and simulated in Section 4. Finally, the paper is concluded in Section 5.

2 Problem Formulation

Let us consider a linear quadratic discrete optimization problem with time delays in the states and controls. Moreover, it is assumed that a subset of the state, and/or control variables are subject to boundary constraints. Thus, we have:

$$\min J = \frac{1}{2}\left\|x(k_f) - x^d(k_f)\right\|_S + \sum_{k=k_O}^{k_f-1} \frac{1}{2}\left[\left\|x(k) - x^d(k)\right\|_Q^2 + \left\|u(k)\right\|_R^2\right] \quad (1)$$

subject to:

$$x(k+1) = A_o\, x(k) + \sum_{i=0}^{\theta} A_i\, x(k-i) + \sum_{j=0}^{\gamma} B_j\, u(k-j) + d(k) \quad (2)$$

with:

$$x(k_o) = x_o,\ x(k) = 0 \quad for\ \ k < 0$$
$$and\ \ u(k) = 0 \quad for\ k < 0$$
$$\underline{x}_j \le x_j(k) \le \overline{x}_j\ ;\ j \in \{1,2,......,l\},$$
$$l \le n;\ l \overset{\Delta}{=} \{the\ set\ of\ constrained\ states\} \quad (3)$$
$$\underline{u}_r \le u_r(k) \le \overline{u}_r\ ;\ r \in \{1,2,......,p\},$$
$$p \le m;\ p \overset{\Delta}{=} \begin{Bmatrix} the\ set\ of\ constrained \\ control\ \ variables \end{Bmatrix} \quad (4)$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, $x^d \in R^n$ is the desired state trajectory, $d \in R^n$ is a constant or time varying known input, k_f is the final time, $A_i \in R^{nxn}$; $i \in \{0,1,....,\theta\}$ and $B_j \in R^{nxm}$; $j \in \{0,1,....,\gamma\}$ are the system matrices, $S, Q \in R^{nxn}$ are positive semi- definite weighting matrices and finally, $R \in R^{mxm}$ is a positive definite weighting matrix for the control. Also, it is assumed that $(\underline{x}, \overline{x})$, $(\underline{u}, \overline{u})$ are the lower and upper bounds of the states and control variables respectively.

The above problem can be rewritten in the following equivalent form:

$$\min J = \frac{1}{2}\left\|x^o(k_f) - x^d(k_f)\right\|_S^2 + \frac{1}{2}\left\|x(k_f) - x^o(k_f)\right\|_{S_2}^2 + \sum_{k=k_O}^{k_f-1} \frac{1}{2}[\left\|x^o(k) - x^d(k)\right\|_Q^2 + \left\|x(k) - x^o(k)\right\|_{Q_2}^2 + \left\|u(k)\right\|_R^2] \quad (5)$$

subject to:

$$x(k+1) = A_o\, x(k) + \sum_{i=1}^{\theta} A_i\, x^o(k-i) + \sum_{j=0}^{\gamma} B_j\, u(k-j) + d(k) \quad (6)$$

with $\quad x(k_o) = x_o \quad ,$

$$x(k) = x^o(k) \quad (7)$$
$$\underline{x}_j \le x_j^o(k) \le \overline{x}_j \quad (8)$$
$$\underline{u}_r \le u_r(k) \le \overline{u}_r \quad (9)$$

Note that, in the above formulation, the two terms added in the cost function, namely, $\frac{1}{2}\left\|x(k_f) - x^o(k_f)\right\|_{S_2}^2$, and $\frac{1}{2}\left\|x(k) - x^o(k)\right\|_{Q_2}^2$ are merely convexing terms to speed up convergence. They will not cause any additional cost to the objective function (5) at the end of convergence process, if the problem has a solution, since $x^{o\nu}(k) \to x^\nu(k)\ as\ \nu \to \infty$ where ν is the iteration number.

3 The Developed Procedure

Relaxing for the moment the constraints given by (8), (9), hence the Lagrangian of the reformulated problem is given by:

$$L = \frac{1}{2}\left\|x^o(k_f) - x^d(k_f)\right\|_S^2 + \frac{1}{2}\left\|x(k_f) - x^o(k_f)\right\|_{S_2}^2 + \pi^T(k_f)(x(k_f) - x^o(k_f)) + \sum_{k=k_O}^{k_f-1}\{\ \frac{1}{2}\left\|x^o(k) - x^d(k)\right\|_Q^2 + \frac{1}{2}\left\|x(k) - x^o(k)\right\|_{Q_2}^2 + \frac{1}{2}\left\|u(k)\right\|_R^2 + \lambda^T(k+1)[A_o x(k) + \sum_{i=1}^{\theta} A_i\, x^o(k-i) + \sum_{j=0}^{\gamma} B_j\, u(k-j) + d(k) - x(k+1)] + \pi^T(k)[(x(k) - x^o(k)\ \ ]\ \} \quad (10)$$

where $\lambda(k) \in R^n$ is the co-state vector and $\pi(k) \in R^n$ is the Lagrange multiplies corresponding to the equality constraint (7).

Defining the Hamiltonian $H(x, u, x^o, \lambda, \pi)$ as follows:

$$H(x, u, x^o, \lambda, \pi) \overset{\Delta}{=} \frac{1}{2}\left\|x^o(k) - x^d(k\ \ )\right\|_Q^2 + \frac{1}{2}\left\|x(k) - x^o(k)\right\|_{Q_2}^2 + \frac{1}{2}\left\|u(k)\right\|_R^2 + \lambda^T(k+1)A_o x(k) + \sum_{i=1}^{\theta} \lambda^T(k+1+i)A_i\, x^o(k) + \sum_{j=0}^{\gamma} \lambda^T(k+1+j)B_j\, u(k) + \lambda^T(k+1)d(k) + \pi^T(k)(x(k) - x^o(k)) \quad (11)$$

Therefore, (10) can be rewritten in the form:

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$$L=\frac{1}{2}\left\|x^o(k_f)-x^d(k_f)\right\|_S^2+\frac{1}{2}\left\|x(k_f)-x^o(k_f)\right\|_{S_2}$$
$$-\lambda^T(k_f)x(k_f)+\lambda^T(k_o)x(k_o)$$
$$+\pi^T(k_f)(x(k_f)-x^o(k_f))$$
$$+\sum_{k=k_O}^{k_f-1}[H(x,u,x^o,\lambda,\pi)-\lambda^T(k)x(k)] \qquad (12)$$

The necessary conditions of optimality give:

$$\frac{\partial H}{\partial u(k)}=0 \quad \Rightarrow$$
$$u(k)=-R^{-1}\sum_{j=0}^{\gamma}B_j^T\lambda(k+1+j)=\Gamma(\lambda) \qquad (13)$$

with $\lambda(k)=0 \quad for \quad k>k_f$

However, to satisfy the constraints given by (9), the control vector which minimizes the Hamiltonian is given by [13]:

$$u_i(k)=\begin{cases}\underline{u}_i & if \quad \Gamma_i(\lambda)<\underline{u}_i\\ \Gamma_i(\lambda) & if \quad \underline{u}_i\le\Gamma_i(\lambda)\le\overline{u}_i\\ \overline{u}_i & if \quad \Gamma_i(\lambda)>\overline{u}_i\end{cases} \qquad (14)$$

where i indicates the i^{th} element in the array.

$$\frac{\partial H}{\partial x(k)}=\lambda(k) \Rightarrow$$
$$\lambda(k)=A_o^T\lambda(k+1)+Q_2[x(k)-x^o(k)]$$
$$+\pi(k) \qquad (15)$$

with

$$\lambda(k_f)=S_2[x(k_f)-x^o(k_f)]+\pi(k_f) \qquad (16)$$

$$\frac{\partial H}{\partial\lambda(k+1)}=x(k+1) \Rightarrow$$
$$x(k+1)=A_ox(k)+\sum_{i=1}^{\theta}A_i\,x^o(k-i)$$
$$+\sum_{j=0}^{\gamma}B_j\,u(k-j)+d(k) \qquad (17)$$

with $x(k_o)=x_o$

and $u(k)=0$, $x(k)=0 \quad for \ k<0$

$$\frac{\partial H}{\partial x^o(k)}=0 \Rightarrow$$
$$x^o(k)=(Q+Q_2)^{-1}[Qx^d(k)+Q_2x(k)$$
$$-\sum_{i=1}^{\theta}A_i\lambda(k+1+i)+\pi(k)]$$
$$=f(x,\lambda,\pi,k) \qquad (18)$$

$$\frac{\partial H}{\partial x^o(k_f)}=0 \Rightarrow$$
$$x^o(k_f)=(S+S_2)^{-1}[Sx^d(k_f)+S_2x(k_f)$$
$$+\pi(k_f)]=G(x,\pi,k_f) \qquad (19)$$

Again to satisfy system constraints given by (8), one gets:

$$x_i^o(k)=\begin{cases}\underline{x}_i & if \quad f_i(x,\lambda,\pi,k)<\underline{x}_i\\ f_i(x,\lambda,\pi,k) & if \quad \underline{x}_i\le f_i(x,\lambda,\pi,k)\le\overline{x}_i\\ \overline{x}_i & if \quad f_i(x,\lambda,\pi,k)>\overline{x}_i\end{cases} \qquad (20)$$

and

$$x_i^o(k_f)=\begin{cases}\underline{x}_i & if \quad G_i(x,\pi,k_f)<\underline{x}_i\\ G_i(x,\pi,k_f) & if \quad \underline{x}_i\le G_i(x,\pi,k_f)\le\overline{x}_i\\ \overline{x}_i & if \quad G_i(x,\pi,k_f)>\overline{x}_i\end{cases} \qquad (21)$$

Finally we have:

$$\frac{\partial H}{\partial\pi}=x(k)-x^o(k) \qquad (22)$$

from which, the updating algorithm for $\pi(k)$ is given by:

$$\pi^{\nu+1}(k)=\pi^{\nu}(k)+\alpha\ l^{\nu}(k) \qquad (23)$$

Since the variable $\pi(k)$ has to be maximized, the iteration constant α has to be positive. The value of the vector $l^{\nu}(k)$ can be determined either by using steepest descent approach, or conjugate gradient technique.

Based on the necessary conditions of optimality given above, the following algorithm is proposed to solve the problem at hand:

Step (1): Initialize the vectors $\pi^{\nu}(k),x^{o\nu}(k),\lambda^{\nu}(k)$,and put the iteration number $\nu=1$.

Step (2): Calculate the control vector $u^{\nu}(k)$ using (14).

Step (3): Calculate the state vector $x^{\nu}(k)$ using (17).

Step (4): Calculate the error criteria:

$$error = \sqrt{\sum_{k=k_o}^{k_f} \left\| x^{\nu}(k) - x^{\nu o}(k) \right\|^2}$$

If error < ε, where ε is a pre-specified small constant, record the trajectories and stop. Otherwise, go to the next step.

Step (5): Update the trajectories of the Lagrange multiplier vector $\pi^{\nu+1}(k)$ using (23).

Step (6): Calculate the co-state vector $\lambda^{\nu+1}(k)$, and $x^{o\nu+1}(k)$ using (15), (16), (20), and (21).

Step (7): Put $\nu = \nu + 1$ and go to step (2) above.

Based on the above algorithm, it is worth to conduct the following remarks:

1. It is not required to solve the difficult TPBVP.
2. No additional multiplier (Kuhn-Tacker) parameters are needed to satisfy inequality constraints since the satisfaction of (23) insures the satisfaction of the inequality constraints.
3. The algorithm is simple, since we have to solve linear vector difference equations to get $x(k)$ and $\lambda(k)$ and then make a direct substitution to get $u(k), \pi(k)$ and $x^o(k)$.
4. The matrices Q_2, S_2 play a very important role in the convergence process of the algorithm.

Having developed an algorithm to solve linear quadratic discrete optimization problems with time delays and system constraints, in the following Section, an illustrative example is given to show the applicability of the proposed procedure.

4 Illustrative Example

To illustrate the preceding results, let us consider the following linear time delay system:

$$\min J = \frac{1}{2}\int_0^t (\|x\|_Q^2 + \|u\|_R^2)\ dt \qquad (24)$$

subject to:

$$\dot{x} = A_o x(t) + A_1 x(t-\tau) + Bu(t) \qquad (25)$$

with $x^T(o) = [5.5 \quad 3.5]$

$$\underline{x} \le x(t) \le \overline{x}$$

$$\underline{u} \le u(t) \le \overline{u}$$

where

$$A_o = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 0 & 0.1 \\ -0.2 & -0.3 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

$Q = I_2$, $R = 0.5$.

The above problem is discretized using an interval $\Delta T = 0.1$, τ is taken as a fixed number of discrete intervals.

For illustration purposes, the above problem is solved using different values of the delay τ, Q_2, R, final time, k_f, and upper and lower bounds of the states and control. Table (1) summarizes the results obtained for the simulated cases. Fig.1, and 2 show the state and control trajectories for case (1). Whilst Fig.3 and 4 show the results of case (4).

5 Conclusion

In this paper, linear quadratic discrete optimization problem with time delays and system constraints is investigated. A new algorithm is developed for handling such a complicated problem. It is shown that within the proposed procedure, it is neither required to increase the dimensionality of the state vector due to time delays nor to use Kuhn-Tacker parameters for the inequality constraints. Extensive simulation results showed that, if the problem has a feasible optimal solution, the algorithm can hit this solution while satisfying system constraints. It has been noticed that, different factors affect the convergence rate of this algorithm. More investigation is required in the future to detect how these factors can be specified to get the best convergence rate of the algorithm.

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References

[1] Bemporad, F. Borrelli and M. Morari, “Min-max control of constrained uncertain discrete-time linear systems”, IEEE Transaction on AC, Vol.48, pp.1600-1606, 2003.

[2] Bemporad, M. Morari, V. Dua and E.N. Pistikopoulos, “The explicit linear quadratic regulator for constrained Systems”, Automatica, Vol.38, pp.3-20, 2002.

[3] A.Bemporad, F. Borrelli and M. Morari, “The explicit solution of constrained LP-based receding horizon control”, presented at Decision and Control, 2000, Proceedings of the 39” IEEE Conference, 2000.

[4] A.G. Tsirukis and M. Morari, “Controller design with actuator constraints”, Presented at Decision and Control, 1992, proceedings of the 31st IEEE Conference, 1992.

[5] G. F. Mulder, M.V. Kothare and M. Morari, "Multivariable anti-windup controller synthesis using iterative linear Matrix inequality" Proceedings of European Control Conference, 1999.

[6] A.Teel and N. Kapoor, "The L2 anti-windup problems, its definition and solution", European Control Conference, Brussels, Belgium, 1997.

[7] S. J. Qin and T.A. Badgwell, "An overview of industrial model predictive control technology", Chemical Process Control, 1997, 93:232-256.

[8] A.Zheng and M. Morari, "Stability of model predictive control with mixed constraints", IEEE Transaction AC, Vol.40, pp.1818-1823, 1995.

[9] Zheng and M.Morari, "Global stability of linear discrete-time systems with bounded controls, A model predictive control approach". Presented at American Control Conference, 1994.

[10] M. F. Hassan and N.B. Almutair, "New algorithm for complex optimization problem with inequality constraints" Journal of Autosoft Intelligent Automation and Soft Computing, Vol. 11, pp 1-31, 2005..

[11] M.F. Hassan "Solution of linear quadratic constraint problem via coordinating approach", International Journal of Innovative Computing, Information and Control (To appear).

[12] M. F. Hassan and A. K. Ali, "A new algorithm for discrete LQP with inequality constraints in states and control". Presented for publication.

[13] D.E.Kirk, " Optimal control theory: An introduction", Englewood Cliffs, N.J. Prentice Hall1970.

Table 1: Summery of Simulation Results

Case No.	**T_f**	**Q_2**	**R**	**τ**	**x_{min}**	**x_{max}**	**u_{min}**	**u_{max}**	**No of itrations**
1	6	1	0.5	1	open	open	open	open	100
2	6	1.5	0.5	1	$x(2) \geq -2.5$	open	open	open	5305
3	6	11	0.5	1	$x(2) \geq -2.5$	open	open	open	5432
4	6	1	0.5	1	$x(2) \geq -2.5$	open	open	≤ 33.0	225226
5	6	1.5	0.5	1	$x(2) \geq -2.0$	open	open	open	8122
6	12	1.5	0.5	5	$x(2) \geq -1.0$	open	open	open	11363
7	6	1	0.5	10	$x(2) \geq -2.5$	open	open	open	4708
8	6	1	1.0	1	$x(2) \geq -2.5$	open	open	open	10526
9	6	1.5	0.5	1	open	$x(1) < 5.85$	open	open	685012
10	6	1.5	0.1	1	open	$x(1) < 5.85$	open	open	136474

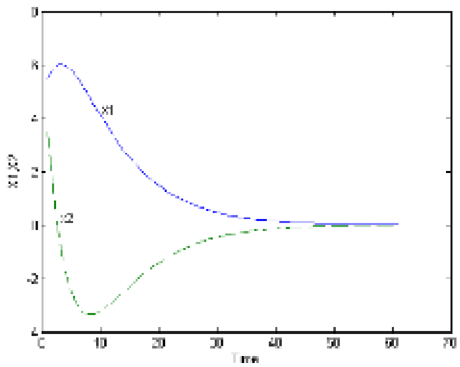

Fig.1: State Trajectories for Case (1)

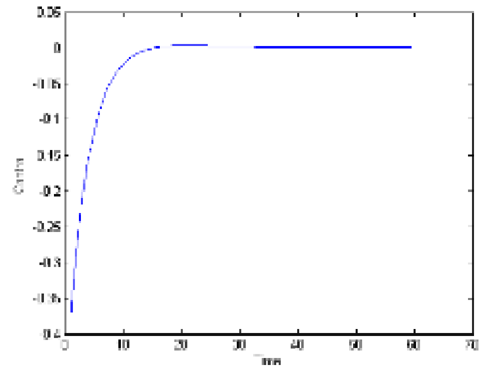

Fig.2: Control Trajectories for Case (1)

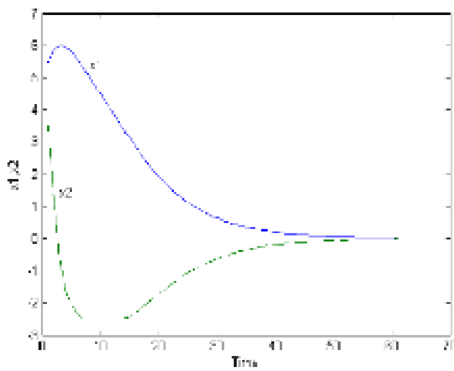

Fig.3: State Trajectories for Case (4)

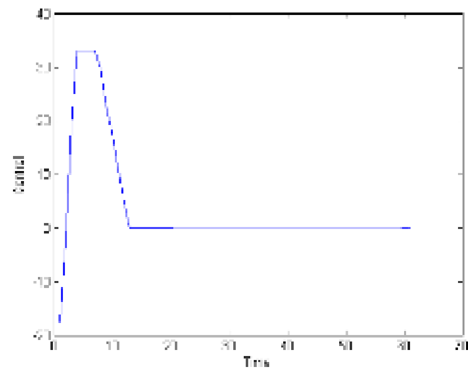

Fig.4: Control Trajectories for Case (4)