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GLOBAL ASYMPTOTIC STABILITY OF POINT TIME-DELAY SYSTEMS WITH UNCERTAINTIES WITHIN POLYTOPES

by M. De la Sen

Instituto de Investigacion y Desarrollo de Procesos

Facultad de Ciencias. Universidad del Pais Vasco

Campus de Leioa (Bizkaia). Aptdo. 644 de Bilbao. 48080-Bilbao. SPAIN

Abstract.- This brief paper discusses a linear fractional representation (LFR) of parameter-dependent nonlinear systems with real rational nonlinearities and point-delayed dynamics. Sufficient conditions for robust global asymptotic stability independent of the delays are investigated in terms of testing a finite number of linear matrix inequalities when the (perhaps time-varying) uncertain parameter vector lies within a known polytope containing the origin. Such inequalities are obtained from the stability analysis via Lyapunov Stability Theory by taking advantage of the characterization of the uncertainties within polytopes.

Index Terms.- Time-delayed dynamics, parameter-dependent systems, point delays, robust stability.

I. INTRODUCTION

Time-delay systems are very common in nature like, for instance, related to transportation problems, population growing, signal transmission or neural network-based models (see, for instance [1-12]). The stability and stabilization of such systems has been widely studied in the literature in connection, for instance, with Lyapunov theory or frequency domain methods (see, for instance, [1-8] and references therein). A part of the related results are referred to as stability independent of the delays since they are independent of the sizes of the delays. In this paper, the global asymptotic stability independent of a single point delay is investigated provided that the dynamic system is subject to internal (i.e. in the state) delays and subject to uncertain rational real-valued (and perhaps, time-varying) nonlinearities parametrized within a known polytope containing the origin. The problem statement and the main robust stability result are developed in Section II via Lyapunov' s second method. Such a main result basically consists of testing the positive negativeness of a set of matrices which are directly obtained from calculations related to the vertices of the polytope that parametrizes the uncertainties. Finally, two simple illustrative examples are given in Section III.

II. STABILITY ANALYSIS USING LYAPUNOV FUNCTIONS

Consider the parameter-dependent system of point delay $h \geq 0$

$$\dot{x}(t) = A(\theta(t))x(t) + A_1 x(t-h) + B(\theta(t))u(t) \quad (1.a)$$

$$y(t) = C(\theta(t))x(t) + D(\theta(t))u(t) \quad (1.b)$$

where $x(t) \in R^n$, $u(t) \in R^{n_u}$ and $y(t) \in R^{n_y}$ are the state, input and measurable signals respectively and A, A_1, C and D are real-valued rational functions of time-varying parameter vector

$\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_m(t)]^T \in \Theta$

for all $t \geq 0$. The parameter set Θ is assumed to be a polytope containing the origin such that (1.a) has a mild solution for all time for all $\theta(t)$ for any given absolute continuous function $\varphi : [-h, 0] \to R^n$ ($x(0) = \varphi(0)$) of initial conditions. This is not restrictive since the results obtained in this paper are also applicable if formulated over any polytope containing the parameter vector. Since A and A_1 are real-valued rational functions of $\theta(t)$, there exist associate Linear Fractional representations (LFR):

$$A(\theta) = A_0(\theta) + B_{q0}\Delta_0(\theta)\left(I_d - D_{pq0}\Delta_0(\theta)\right)^{-1}C_{p0} \quad (2)$$

$$A_1(\theta) = A_{01}(\theta) + B_{q1}\Delta_1(\theta)\left(I_{d_1} - D_{pq1}\Delta_1(\theta)\right)^{-1}C_{p1} \quad (3)$$

for appropriate matrix functions A_0, A_{01}, B_{qi}, C_{qi}, D_{pqi} and Δ_i (i=0,1) of appropriate sizes, where I_j denotes the j-identity matrix. The subscript is deleted when the size of the identity matrix follows directly from context. The related free-delay case has been investigated in [13] and references there in. For well-posedness, it is assumed that both above inverses exist over Θ. Thus, a state -space realization of the unforced (1) is:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_{01} x(t-h) + B_{q0} q_0(t) + B_{q1} q_1(t-h) \\ p_i(t) &= C_{pi} x(t) + D_{pqi} q_i(t) = \left(I - D_{pqi}\Delta_i(\theta)\right)^{-1} C_{pi}\, x(t) \\ q_i(t) &= \Delta_i(\theta) p_i(t) = \Delta_i(\theta)\left(I - D_{pqi}\Delta_i(\theta)\right)^{-1} C_{pi}\, x(t) \\ \Delta_i(\theta) &= \mathrm{Diag}\left(\theta_1 I_{s_{1i}}, \dots, \theta_m I_{s_{mi}}\right) \end{aligned} \quad (4)$$

where $q_i, p_i \in R^{d_i}$ and the degrees of the LFR are $s_i = \underset{1 \le k \le r}{Max}\left(s_{ki}\right)$ for i=0,1. Note that the variables $q_{(.)}$ are normalized variables for the variables $p_{(.)}$ according to the size of the current uncertainty parameter vector $\theta(t)$ through the normalized matrix $\Delta_{(.)}$.

If the unforced system (i.e. for $u \equiv 0$) is globally asymptotically stable independent of the delay for all parametrization in Θ then both A_0 and $\left(A_0 + A_{01}\right)$ are stability matrices (i.e. with all their eigenvalues in Re s < 0) since Θ includes zero and the system is asymptotically stable for the limit delays h=0 and $h \to \infty$. The robust stability margin of (1) is defined in a natural way as $\sigma = \mathrm{Sup}\left\{\rho > 0\right\}$ such that (1) is robustly stable over $\rho\,\Theta$ for all $\rho \in [0, \sigma]$. Since the parameter set Θ is a polytope of v_Θ vertices $Ver(\Theta) = \left\{\Theta^{(i)}; i = \overline{1, v_\Theta}\right\}$ then $\Delta_i = \left\{\Delta_i(\theta) : \theta \in \Theta\right\}$ (i=0,1 are polytopes of v_i vertices $\Delta_i^{(k)}$; k = $\overline{1, v_i}$ (i=0,1). The number of those vertices depends, in general, on the number of vertices of Θ and on i=0,1. The total number of vertices $\Delta^{(k_0, k_1)}$, for $k_i = \overline{1, v_i}$, i=0,1 of the polytope $\Delta = \Delta_0 \times \Delta_1$ which parametrizes the matrices of the dynamic system (1) is upper-bounded by v= $v_0 \times v_1$. The following result is proved in Appendix A.

Theorem 1. The (unforced) system (2) is globally asymptotically stable independent of the delay h if there exist real matrix functions $P = P^T > 0$, $S = S^T > 0$ and matrices $G_{\Delta i}(\Delta(\theta)) \in C^{n \times d_i}$, $H_{\Delta i}(\Delta(\theta)) \in C^{d_i \times d_i}$ for each $\Delta_i(\theta) \in \Delta_i$; i=0,1 as $\theta \in \Theta$ such that the square (2n + $d_0 + d_1$) real matrix function $Q(\Delta(\theta)) = Q^T(\Delta(\theta))$ = Block Matrix $\left[Q_{ij}(\Delta(\theta)); i, j = \overline{1,3}\right]$ of block matrices defined as follows is negative definite for all $\theta \in \Theta$:

$$Q_{11} = A_{00}^T P + PA_{00} + S + G_{\Delta 0} C_{p0} + C_{p0}^T G_{\Delta 0}^* \;; \quad Q_{12} = Q_{21}^T = PA_{01}$$

$$Q_{13} = Q_{31}^T = \left[\, P\left(B_{q0}\Delta_0\right) + G_{\Delta 0}\left(D_{pq0}\Delta_0\right) - G_{\Delta 0} + C_{p0}^T H_{\Delta 0}^* \,,\; P\left(B_{q1}\Delta_1\right) \right]$$

$$Q_{22} = \left[G_{\Delta 1} C_{p1} + C_{p1}^{T} G_{\Delta 1}^{*} - S \right]$$
$$Q_{23} = Q_{32}^{T} = \text{Block Matrix} \left[0,\ G_{\Delta 1}\left(D_{pq1}\Delta_1\right) - G_{\Delta 1} + C_{p1}^{T} H_{\Delta 1}^{*} \right]$$
$$Q_{33} = \text{Block Diag} \left[H_{\Delta 0}\left(D_{pq0}\Delta_0\right) + \left(D_{pq0}\Delta_0\right)^{T} H_{\Delta 0}^{*} - H_{\Delta 0} - H_{\Delta 0}^{*},\ H_{\Delta 1}\left(D_{pq1}\Delta_1\right) + \left(D_{pq1}\Delta_1\right)^{T} H_{\Delta 1}^{*} - H_{\Delta 1} - H_{\Delta 1}^{*} \right] \quad (5)$$
❑

If $G_{\Delta(.)}$ and $H_{\Delta(.)}$ are restricted to have special forms such that $Q(\Delta(\theta))$ is convex for Θ being convex then it suffices that $Q(\Delta(\theta)) = Q^{T}(\Delta(\theta)) < 0$ for the values of $\Delta_i(\theta)$ at all their vertices ; i.e. to replace matrices $\Delta_i(\theta)$ in (5) by all its vertices $\Delta_i^{(k_i)}$; $k_i = \overline{1, v_i}$; $i = 0,1$, to guarantee the global asymptotic stability. In that way, stability is guaranteed if the requested positive negativeness is fulfilled by (at most) v real symmetric matrices. The following corollaries are also useful to test the global asymptotic stability independent of the delays in practical situations. Their proofs are very similar to that of Theorem 1.

Corollary 1. The (unforced) system (2) is globally asymptotically stable independent of the delay h if there exist real matrices $P = P^{T} > 0$, $S = S^{T} > 0$, $M_i = M_i^{T} > 0$ and matrices $G_{\Delta i} \in C^{n \times d_i}$, $H_{\Delta i} \in C^{d_i \times d_i}$ for each $\Delta_i^{(k)} \in \Delta_i$; $k = \overline{1, v_i}$, i=0,1 such that $v = v_0 \times v_1$ square $(2n + d_0 + d_1)$ symmetric real matrices $Q'(k_0, k_1)$ = Block Matrix $\left[Q'_{ij}(k_0, k_1); i, j = \overline{1,3} \right]$ of block matrices defined as follows is negative definite:

$$Q'_{11} = A_{00}^{T} P + PA_{00} + S + C_{p0}^{T} M_0 C_{p0} \ ; \quad Q'_{12} = Q'^{T}_{21} = PA_{01}$$
$$Q'_{13}(k_0, k_1) = Q'^{T}_{31}(k_0, k_1) = \left[P\left(B_{q0}\Delta_0^{(k_0)}\right) + C_{p0}^{T} M_0 \left(D_{pq0}\Delta_0^{(k_0)}\right),\ P\left(B_{q1}\Delta_1^{(k_1)}\right) \right]$$
$$Q'_{22} = \left[C_{p1}^{T} M_1 C_{p1} - S \right]$$
$$Q'_{23}(k_0, k_1) = Q'^{T}_{32}(k_0, k_1) = \text{Block Matrix} \left[0,\ C_{p1}^{T} M_1 C_{p1} - S \right]$$
$$Q'_{33}(k_0, k_1) = \text{Block Diag} \left[-M_0 + \left(D_{pq0}\Delta_0^{(k_0)}\right)^{T} M_0 \left(D_{pq0}\Delta_0^{(k_0)}\right),\ \ldots,\ -M_1 + \left(D_{pq1}\Delta_1^{(k_1)}\right)^{T} M_1 \left(D_{pq1}\Delta_1^{(k_1)}\right)^{T} \right]$$

for $k_i = \overline{0, v_i}\,;\, i = \overline{0,1}$ (6) ❑

Corollary 2. Assume unity LFR degrees (i.e. $s_0 = s_1 = 1$) of the LFR´s (2)-(3). Thus Corollary 1 also holds if M_i is replaced by, in general distinct, symmetric positive definite real matrices $M_i^{(k_i)}$ at any of the v test matrices (6) for each $k_i = \overline{1, v_i}\,;\, i = 0,1$. ❑

Remark 1. Note that Corollary 2 is stronger than Corollary 1 since $v = v_0 \times v_1$ different $M_{(.)}$-matrices, rather than two, are allowed in the v set of tests of negative definiteness to guarantee stability. Note also that both Corollaries 1- 2 automatically hold if all the tests do not fail for a unique M- matrix. ❑

Note that the extension of all the above results to the case of presence of multiple point delays is direct by completing the sizes and composition of the matrices for the stability tests with the necessary block matrices associated to the various extra delays.

III. EXAMPLES

Example 1: Consider the first-order system with parameter-dependent uncertainty

$$\dot{x}(t) = A(\theta(t))\, x(t) + A_1(\theta(t))\, x(t-h) \tag{7}$$

where $\theta(t) \in [\underline{\theta}, \overline{\theta}]$ a single (perhaps time-varying) uncertain real parameter and A (.) and B(.) are rational functions of θ (t) given by

$$A = a_0 + \frac{b_0\theta}{1-d_0\theta} = \frac{0.6017\,\theta - 1}{1-0.4517\,\theta}\ ;$$

$$A_1 = a_1 + \frac{b_1\theta}{1-d_1\theta} = \frac{0.1125\ -0.337\,\theta}{1-0.1\,\theta}$$

with uncertainty independent values $a_0 = -1$ and $a_1 = 0.1125$, respectively. The uncertainty-free problem is asymptotically stable independent of the delay h since $a_0 < 0$ and $|a_0| > |a_1|$. Assume that $\overline{\theta} = -\underline{\theta} = 0.7$. Thus, Corollary 1 is tested with the 4×4 real symmetric matrices obtained from the two distinct (matrix) vertices at $\theta = \pm\overline{\theta}$ of the (convex) symmetric matrix function:

$$\begin{bmatrix} 2a_0 p + s + m_0 & p a_1 & (p b_0 + m_0 d_0)\theta & p b_1\theta \\ p a_1 & m_1 - s & 0 & m_1 - d_1\theta \\ (p b_0 + m_0 d_0)\theta & 0 & m_0(d_0^2\theta^2 - 1) & 0 \\ p b_1\theta & m_1 - d_1\theta & 0 & m_1(d_1^2\theta^2 - 1) \end{bmatrix}$$

Those vertices are obtained from evaluating all the distinct possible combinations at the positions (1,3), (1,4) and (2,4) at $\pm\overline{\theta}$ using the structure and symmetry properties of the matrix function since the remaining position take identical numerical values at all the potential vertices for θ $=\pm\overline{\theta}$. For the values p=s=2.11, $m_0 = m_1 = 1.11$. Corollary 1 ensures that the eighth matrices are negative definite and, thus, the system is globally asymptotically stable independent of the delay. The stability test might be performed also via Corollary 2 by using distinct positive real numbers $\overline{m}_0, \overline{m}_1, \underline{m}_0, \underline{m}_1$ at the related matrix vertices generated from using boundary values of θ at the positions (1, 3) and (2,4) since the LFR degrees are $s_{0,1} = 1$.

Example 2: An unforced second-order neural network with point delays belonging to a similar class to that analyzed in [12] is given by

$$\dot{x}_i(t) = -\sum_{j=1}^{2} a_{ij}\, x_i(t) + \sum_{j=1}^{2} w_{ij}(\theta(t))\, x_j(t-h)\ ;\ i = 1, 2 \tag{8}$$

where, contrarily to the class discussed in [12], the structure of the delay-free part is not necessarily diagonal. The network is of intervalized parameters if those parameters very within prescribed intervals. The stability analysis proposed in Section II may be used to discuss the situation arising when the adjusted weights are rational functions of time-varying parameters subject to saturations. Consider the particular neural network of the class (8) described in matrix form via (7) with $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$; and

$$A_1 = \begin{bmatrix} 0 & 0 \\ \dfrac{0.022 + 0.044\theta_1}{1 - 1.3\theta_1} & \dfrac{0.04 + 0.019\theta_2}{1 - 0.56\theta_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.215 & 0.043 \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} \dfrac{1}{1 - d_1\theta_1} & 0 \\ 0 & \dfrac{1}{1 - d_2\theta_1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix}$$

where only the delayed dynamics depends on a two-dimensional parametrical vector function $\theta(t) = \left[\theta_1(t), \theta_2(t)\right]^T$ which takes values in some subset $S_\theta = \left[\underline{\theta}_1, \bar{\theta}_1\right] \times \left[\underline{\theta}_2, \bar{\theta}_2\right]$ of R^2. Note that $s_0 = 0$, since the delay-free dynamics is constant, and $s_1 = 1$ .There are four distinct matrices for stability checking if Corollary 1 is used. The system is found to be asymptotically stable independent of the delay for $\bar{\theta}_1 = -\underline{\theta}_1 = 0.3$; $\bar{\theta}_2 = -\underline{\theta}_2 = 0.7$.

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APPENDIX A

Proof of Theorem 1. Consider the Lyapunov´s - Krasovsky functional,[2]:

$$V(t) = x^T(t)\,Px(t) + \int_{-h}^{0} x^T(t+\tau)\,Sx(t+\tau)\,d\tau \qquad \text{(A.1)}$$

for the unforced system (2) for some real positive definite symmetric matrices P and S. Taking time-derivatives in (A.1) along any state trajectory yields:

$$\dot{V}(t) = \bar{x}^T(t)\,Q_0\,\bar{x}(t) = \bar{x}^T(t)\,Q\,\bar{x}(t) \qquad \text{(A.2)}$$

with

$$\bar{x}^T(t) = \left(x^T(t),\, x^T(t-h),\, p_0^T(t),\, p_1^T(t-h)\right) \qquad \text{(A.3)}$$

and $Q(\Delta(\theta))$ is defined by block matrices (5), and

$$Q_0(\Delta(\theta)) = Q_0^T(\Delta(\theta)) = \text{Block Matrix}\left[Q_{0ij}(\Delta(\theta));\quad i,j=\overline{1,3}\right] \qquad \text{(A.4a)}$$

and

$$Q_{011} = A_{00}^T P + PA_{00} + S \;;\quad Q_{012} = Q_{021}^T = PA_{01}$$
$$Q_{013} = Q_{031}^T = \text{Block Matrix}\left[P(B_{q0}\Delta_0),\; P(B_{q1}\Delta_1)\right];\quad Q_{022} = -S$$
$$Q_{023} = Q_{032}^T = 0 \;;\quad Q_{33} = 0 \qquad \text{(A.4b)}$$

Thus, the proof follows if $Q_0(\Delta(\theta)) < 0$, or if $Q(\Delta(\theta)) < 0$ provided that furthermore $\bar{x}^T(t)(Q_0 - Q)\bar{x}(t) = 0$, for some matrix functions $G_{\Delta i}; H_{\Delta i}$ (i=0,1) of θ, through Δ, and all $\Delta \in \Delta$ (i.e. for all $\theta \in \Theta$). The constraint $\bar{x}^T(t)(Q_0 - Q)\bar{x}(t) = 0$ holds since from (4)

$$x^T(t-h_i)G_{\Delta i}\,p_i(t-h) = x^T(t-h_i)G_{\Delta i}C_{pi}\,x(t-h_i) + x^T(t-h_i)G_{\Delta i}(D_{pqi}\Delta_i)\,p_i(t-h_i)$$

$$p_i^T(t-h_i)H_{\Delta i}\,p_i(t-h_i) = p_i^T(t-h_i)H_{\Delta i}C_{pi}\,x(t-h_i) + p_i^T(t-h_i)H_{\Delta i}(D_{pqi}\Delta_i)\,p_i(t-h_i) \qquad \text{(A.5)}$$

for i =0, 1 with $h_0 = 0$ what implies that $z_i^T(t)\,\overline{M}_i(\Delta(\theta))\,z_i(t) = 0$ for any complex matrices $G_{\Delta i}(\Delta(\theta))$ and $H_{\Delta i}(\Delta(\theta))$, for i=0,1: of appropriate sizes where $z_i(t) = \left(x^T(t-h_i),\, p_i^T(t-h_i)\right)^T$ for i=0,1; and

$$\overline{M}_i(\Delta(\theta))$$

$$= \begin{bmatrix} G_{\Delta i} C_{pi} + C_{pi}^T G_{\Delta i}^* & G_{\Delta i}\left(D_{pqi}\Delta_i\right) - G_{\Delta i} + C_{pi}^T H_{\Delta i}^* \\ \left(D_{pqi}\Delta_i\right)^T G_{\Delta i}^* - G_{\Delta i}^* + H_{\Delta i}^* C_{pi} & H_{\Delta i}\left(D_{pqi}\Delta_i\right) + \left(D_{pqi}\Delta_i\right)^T H_{\Delta i}^* - H_{\Delta i} - H_{\Delta i}^* \end{bmatrix}$$

for i=0,1 (A.6)

Thus, the result follows if $Q(\Delta(\theta)) < 0$ for some design matrices $G_{\Delta i}(\Delta(\theta)) \in C^{n \times d_i}$, $H_{\Delta i}(\Delta(\theta)) \in C^{d_i \times d_i}$ for each $\Delta_i(\theta) \in \Delta_i$; i=0,1 as $\theta \in \Theta$. ❑

Proof of Corollary 1. Note from (6) that $Q(\Delta(\theta)) = Q'(k_0, k_1) < 0$ at the vertices $\Delta_0^{(k_0)}$ and $\Delta_1^{(k_1)}$ of the polytopes Δ_0 and Δ_1; for $k_0 = \overline{1, v_0}$; $k_1 = \overline{1, v_1}$ provided that $Q(\Delta(\theta))$ is defined for the choices $G_{\Delta i} = C_{pi}^T M_i / 2$ and $H_{\Delta i} = \left[\left(D_{pqi}\Delta_i\right)^T + I\right] C_{pi}^T M_i / 2$ for i = 0, 1. Furthermore, the matrix function $Q(\Delta(\theta))$ is convex in $\Delta(\theta))$ if $M_i = M_i^T > 0$ (i = 0, 1) so that Theorem 1 only needs to be tested at the vertices of the polytope $\Delta = \Delta_0 \times \Delta_1$. ❑

Proof of Corollary 2. Since the LFR' s degrees $s_i (i = 0,1)$ are unity, then the following identity holds for convex hulls (denoted by **Co**) of sets of matrices:

$$\mathbf{Co}\left\{ A_i\left(\Delta_i^{(k_i)}\right); k_i = \overline{1, v_i} \right\} = \left\{ A_i(\Delta_i(\theta)) : \Delta_i \in \mathbf{Co}\left\{ \Delta_i^{(k_i)}; k_i = \overline{1, v_i} \right\} \right\}$$

for each i = 0, 1. Now, the proof is similar as that of the Corollary 1 under the matrix replacements $M_i \to M_i^{(k_i)}$ for $k_i = \overline{1, v_i}$; $i = 0, 1$. ❑