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Analysis of Periodicity Solutions in Discrete Cobweb Model

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ABSTRACT

Periodicity of the solution of the *2-d* discrete map of the anticipative cobweb has been studied in detail. The periodic conditions of the model has been analytically determined by the application of *z-transform*. Specification of the periodicity regions according to the standard characteristic equations has been stated. The *Farey* tree sequence has been applied at the change of discrete map parameter. Initial linear *2-d* discrete map has been changed with the nonlinear rule. Gained periodicity conditions for initial map have been considered in the nonlinear case. The periodicity conditions of the initial map have been preserved in the nonlinear case.

Keywords: - cobweb, anticipation, anticipative system, nonlinear system, period, Farey tree

1. INTRODUCTION

Cobweb model has been extensively studied by the researchers on account of its importance in the field of the dynamics of complex systems. The fact that the cobweb model originates from the field of discrete dynamics is rather an advantage since the systems of difference equations are often easier to grasp. For example in his enduring scholarly value work on the studies of Dynamic Systems Luenberger [10] on the first place addresses difference and later on differential equations. The classical knowledge about the cobweb model states, that the Price and Quantity are related however, the structure of the classical cobweb model can be represented in the different way. By transforming the cobweb model to SD form the model could become non-autonomous depending on the variable Δt. The following two equations represent the different formulation of the cobweb model:

$$Q_s(k+1) = c + d\frac{Q_s(k)-a}{b} \quad (1)$$

$$P(k+1) = \frac{c + dP(k) - a}{b} \quad (2)$$

This reformulation represents Q_s and P as the non-related quantities. The only bound that exists are the coefficients.

The anticipative formulation of the cobweb equations are based on the fact, that P and Q_s depend only on the parameter values a, b, c, d and p i.e. the initial conditions [1, 2, 3, 4]. System of cobweb equations enables the determination of entire anticipative (future event) chain while equation:

$$P(k-1) = \frac{bP(k) - c + a}{d} \quad (3)$$

enables the determination of feedback (past event) chain. The representation of the Feedback ∼ Anticipative chain is shown in Fig. 1. The dynamics of interest is therefore the chains dynamics which is dependant on the parameters $a(t), b(t), c(t), d(t)$ and $p(t)$. Both chains are actually dependant on strategy dynamics which could be formulated as the $f(a, b, c, d, p, t)$.

Application of hyperincursive algorithm and inspection of gained equations with Dubois' [1] formulation of logistic growth yields the following set of equations for the hyperincursive cobweb model:

$$P(k+2) = \frac{d}{b}\Big(A - \Big(\frac{bB - c + a}{d}\Big)\Big) \quad (4)$$

$$Q_s(k+2) = \frac{d}{b}\Big(C - a - \frac{b}{d}\Big(D - c\Big)\Big) \quad (5)$$

with initial conditions:

$$P_0(k+1) = \frac{p-a}{b} \quad (6)$$

$$P_0(k) = \frac{bP_0(k+1) + a - c}{d} \quad (7)$$

$$Q_{s0}(k+1) = p \quad (8)$$

$$Q_{s0}(k) = a + \frac{b}{d}(Q_{s0}(k+1) - c) \quad (9)$$

The coefficients A and B in Eq.4 could be replaced by the terms $P(k+1)$ or $P(k)$ while coefficients C and D in Eq.5 by $Q_s(k+1)$ or $Q_s(k)$. This yields 16 different combinations of system defined by Eq.4 and Eq.5 that should be studied. The system combination further examined will have the following terms: $A = P(k+1), B = P(k), C = Q_s(k+1)$ and $D = Q_s(k)$. This yields the following set of equations:

$$P(k+2) = \frac{d}{b}\Big(P(k+1) - \Big(\frac{bP(k) - c + a}{d}\Big)\Big)$$

$$Q_s(k+2) = \frac{d}{b}\Big(Q_s(k+1) - a - \frac{b}{d}\Big(Q_s(k) - c\Big)\Big)$$

This could be reformulated in order to show the dependency of the future-present-past events:

$$P(k) = \frac{bP(k-1) + a - c}{d} + \frac{b}{d}P(k+1)$$

$$Q_s(k) = \frac{b}{d}Q_s(k+1) + \frac{b}{d}Q_s(k-1) + a - \frac{bc}{d}$$

which states that the value of the present is dependent on the past as well as on the future. This paradoxical statement is realizable since the formulation of feedback anticipative chain could be stated. Fig. 1 has two delay chains, one for P and one for Q_s. One might notice, that the level and rate elements are dependant only on the coefficients and initialization values.

The developed model should enable us to examine the properties of the cobweb model and also to consider it's structural and incursive perspective. There are several approaches in modification and analysis of cobweb dynamics [5, 6, 7, 8].

2. PERIODICITY OF THE SYSTEM

The z-transform is the basis of an effective method for solution of linear constant-coefficient difference equations. It essentially automates the process of determining the coefficients of the various geometric sequences that comprise a solution [10]. The application of z-transform on the Eq.10 and Eq.10 with initial conditions stated by Eqs. 6 ∼ 9 gives:

$$Y(z) = \frac{-y_1 z + y_0 dz - y_0 z^2}{-1 + dz - z^2} \quad (10)$$

Inverse z-transform yields the following solution:

$$\begin{aligned} Y^{-1}(z) = {} & 2^{-1-n}\, y_0 \left(d - \sqrt{-4+d^2}\right)^n - \\ & - \frac{y_1 \,(d - \sqrt{-4+d^2})^n}{2^n \sqrt{-4+d^2}} + \\ & + \frac{2^{-1-n}\, y_0\, d\,(d - \sqrt{-4+d^2})^n}{\sqrt{-4+d^2}} + \\ & + 2^{-1-n}\, y_0\,(d + \sqrt{-4+d^2})^n + \\ & + \frac{y_1 \,(d + \sqrt{-4+d^2})^n}{2^n \sqrt{-4+d^2}} - \\ & - \frac{2^{-1-n}\, y_0\, d\,(d + \sqrt{-4+d^2})^n}{\sqrt{-4+d^2}} \end{aligned} \quad (11)$$

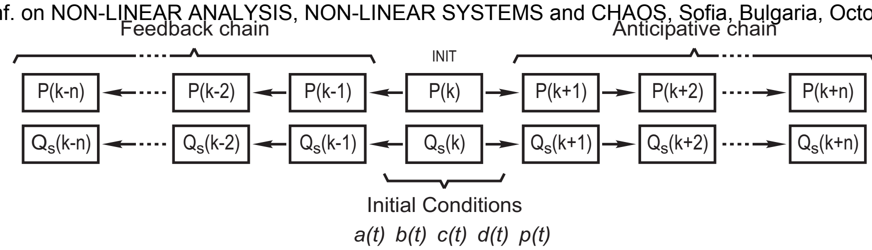

Figure 1: Feedback ∼ Anticipative chain

In order to gain conditions for the periodic response of the system the following equation should be solved:

$$Y^{-1}(z) \quad = \quad y_0 \tag{12}$$

Let us compute a numerical example of periodic solution applying the *z-transform*. The period examined will be the period of 9 i.e. $n = 9$. In Eq.12 one should put the condition $n = 9$. One of the possible solutions for the initial condition worth of examination is the following:

$$d = \frac{1}{\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{\frac{1}{3}}} + \left(\frac{1}{2}(-1+i\sqrt{3})\right)^{\frac{1}{3}} \tag{13}$$

The term $(-1+i\sqrt{3})^{\frac{1}{3}}$ (let us denote the term as z^*) could be expressed in the following way by three different imaginary values in polar form:

$$z_1^* \quad = \quad \sqrt[3]{2}\Big(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\Big) \tag{14}$$

$$z_2^* \quad = \quad \sqrt[3]{2}\Big(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\Big) \tag{15}$$

$$z_3^* \quad = \quad \sqrt[3]{2}\Big(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\Big) \tag{16}$$

By putting Eq.14, Eq.15 and Eq.16 into Eq.13 and performing trigonometric reduction one gets the following solutions:

$$d_1 = 2\cos\frac{2\pi}{9} \quad d_2 = 2\cos\frac{4\pi}{9} \quad d_3 = 2\cos\frac{8\pi}{9} \tag{17}$$

By inspecting the Eq.13 and considering the equation for the roots of complex numbers [9]:

$$\sqrt[n]{z} = \sqrt[n]{r}\Big(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\Big) \tag{18}$$

the general form of the solution for the parameter d could be assumed:

$$d = 2\cos\frac{2\pi m}{n} \tag{19}$$

where n is the period and $m = 1, 2, 3, ..., n-1$. Similar procedure could be performed for the arbitrary period n. More general solutions which regards also the parameter b which was fixed for the purpose of determination of solutions is:

$$d = 2b\cos\frac{2\pi m}{n} \tag{20}$$

Tab. 1 shows the solutions for the parameter d up to the period $n = 10$. The periodic solutions determine the shape of the polygon in *2-d* mapping of solutions. Solutions geometry is important at the examination of complex nonlinear dynamical systems [12]. Numerical values of the solutions for parameter d are important since this values also confirm the findings of Sonis [11] about the domain of attraction for 2D dynamics by n-dimensional linear bifurcation analysis. One of the important conditions gained by the proposed inspection is the value of the period $n = 10$ which is in close relation to the period $n = 5$. The value of parameter d is $d = \frac{1}{2}(1+\sqrt{5})$ with numerical value 1.61803... This solution represents the "*Golden Ratio*" (ϕ). Some of the different representations of solution for parameter d value at period $n = 10$ are:

$$d_{10} = \phi = 2\cos\frac{\pi}{5} = \frac{1}{2}(1+\sqrt{5}) = 1.618033... \tag{21}$$

The first solution of parameter d at period $n = 10$ connects the considered discrete system with the

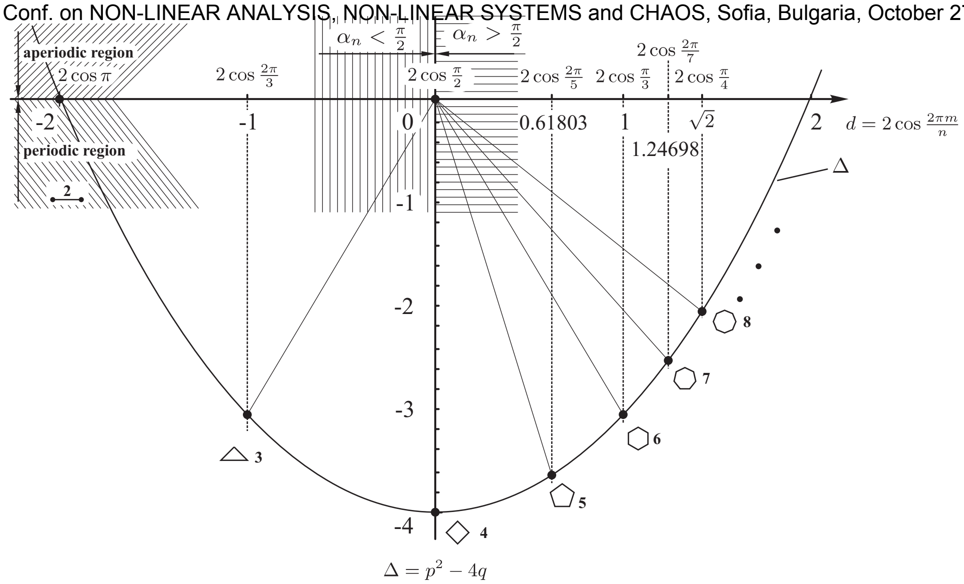

Figure 2: Periodicity of the discrete 2d map — Classification of solutions according to the determinant

Fibonacci numbers given by the infinite series:

$$d_{10} = \phi = 1 + \sum_{u=1}^{\infty} \frac{(-1)^{u+1}}{F_u F_{u+1}} \tag{22}$$

The fact, that the periodicity conditions of the examined discrete system incorporates the golden ratio number ϕ could be observed in other studies [5] of complex nonlinear expansions of the basically cob-web systems e.g. Brock and Hommes "Almost Homoclinic Tangency Lemma". One should expect that the symmetric response in $n - mapping$ should follow the pattern with the match in certain point of solution with the ϕ condition. The source of the mentioned condition is presented by the preceding procedure. (The value of parameter d for mentioned period $n = 5$ is $d = \frac{\sqrt{5}-1}{2} = 0.61803...$ often called the "*Golden Mean*".)

3. SYNCHRONIZATION PATTERNS

In order to derive the stability results on our two-dimensional discrete map,

$$\begin{cases} P_{1_{k+1}} = f(P_{1_k}, P_{0_k}) \\ P_{0_{k+1}} = g(P_{1_k}, P_{0_k}) \end{cases} \tag{23}$$

where P_{1_k} and P_{0_k} represent the components of the iteration process at time k the Jacobian matrix J should be evaluated at assumed equilibrium points:

$$J = \begin{pmatrix} \frac{\partial f(P_1^*, P_0^*)}{\partial P_1} & \frac{\partial f(P_1^*, P_0^*)}{\partial P_0} \\ \frac{\partial g(P_1^*, P_0^*)}{\partial P_1} & \frac{\partial g(P_1^*, P_0^*)}{\partial P_0} \end{pmatrix} \tag{24}$$

The range of the cyclical behavior is determined by the classical imaginary solution of the dynamical system which is in our case defined by

Table 1: Synchronization parameter d values of periodicity conditions up to period 10

Period n	Argument $\Omega = 2\pi\frac{m}{n}$	Num. value $d = 2\cos\frac{2\pi m}{n}$
2*	π	-2.00000
3	$\frac{2\pi}{3}$	-1.00000
4**	$\frac{\pi}{2}$	0.00000
5	$\frac{2\pi}{5}$	0.61803
	$\frac{4\pi}{5}$	-1.61803
6	$\frac{\pi}{3}$	1.00000
7	$\frac{2\pi}{7}$	1.24698
	$\frac{4\pi}{7}$	-0.44504
	$\frac{6\pi}{7}$	-1.80194
8	$\frac{\pi}{4}$	1.41421
	$\frac{3\pi}{4}$	-1.41421
9	$\frac{2\pi}{9}$	1.53209
	$\frac{4\pi}{9}$	0.34730
	$\frac{8\pi}{9}$	-1.87939
10	$\frac{\pi}{5}$	1.61803
	$\frac{3\pi}{5}$	-0.61803

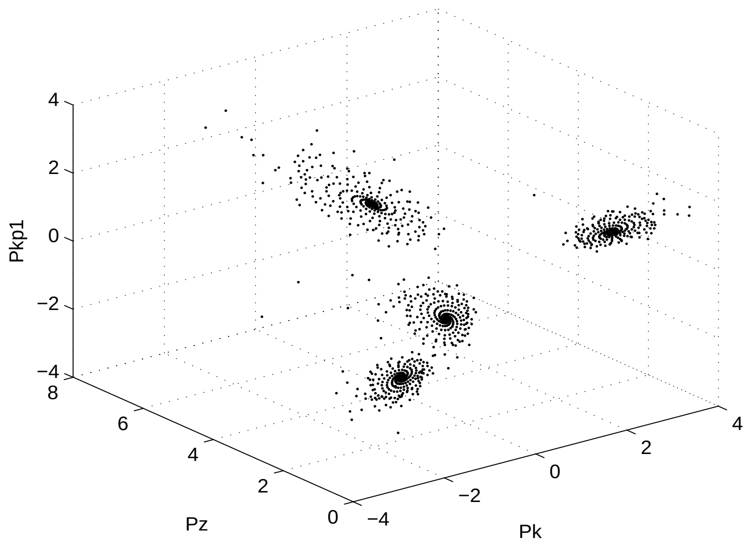

Figure 3: Emergence of Four Synchronous Attractors in the nonlinear case where $d = 0.26131278$ and $b_1 = 0.33$

the characteristic equation

$$\lambda = \frac{-2b + d \pm \sqrt{-4b^2 + d^2}}{2b} \tag{25}$$

Stability result corresponds to the polynomial $\lambda^2 = trJ\lambda - detJ$ where periodic solutions will be considered. One should consider e.g. [11] for details. Discrete map stated by Eq.23 should be analyzed according to the variation of parameter d and the determinant $\Delta = p^2 - 4q$. According to the Tab. 1 gained by the *z-transform* the following classification of the periodic solution could be drawn, shown in Fig. 2. One of the questions that arose at the analysis of similar *2d* systems is the question about the rule that determines the periodicity. In our case the change of parameter d causes the system to switch between different equilibriums. The ordering of the equilibriums is determined by the general Eq. 20. The rational fraction $\frac{m}{n}$, which is in our case transformed by the Eq. 20 to the value of the parameter d, corresponds to the Farey sequence which could be represented by the Farey tree. Fig. 2 represents the classification of the periodicity values. Aperiodic region is determined by the condition $\Delta > 0$ and the periodicity by the $\Delta < 0$. The vertical classification at $d < 0$ determines the angles which are determined by the three points in the 2-d map in our case, $\alpha_n < \frac{\pi}{2}$; $d > 0$, the angles of the map are $\alpha_n > \frac{\pi}{2}$. The strongest periodicity points are determined by the polygon structures in 2-d mapping. In the Fig. 2 the polygons are marked near the hyperbola starting with *digon*, *triangle* etc. Other periodicity is the subset of the main sections which is determined by the $\sum \alpha_n$ and the Farey tree. The emergence of the system periodic stability in the shape of n-sided polygon could be observed not only in economical systems [15]; the n-sided polygon and the Farey tree organization of the equilibria could be observed in the technical systems as for example in laser control as the paradigm of the chaotic system [17].

3.1. Bifurcation Analysis - Extension to Nonlinear systems

Periodicity conditions in previous section are general and could be transferred from linear systems to nonlinear, see for example [11, 13]. In order to analyze the pseudo-bifurcation response of the initial 2-d discrete map the bifurcation was performed for the change in parameter d in the range

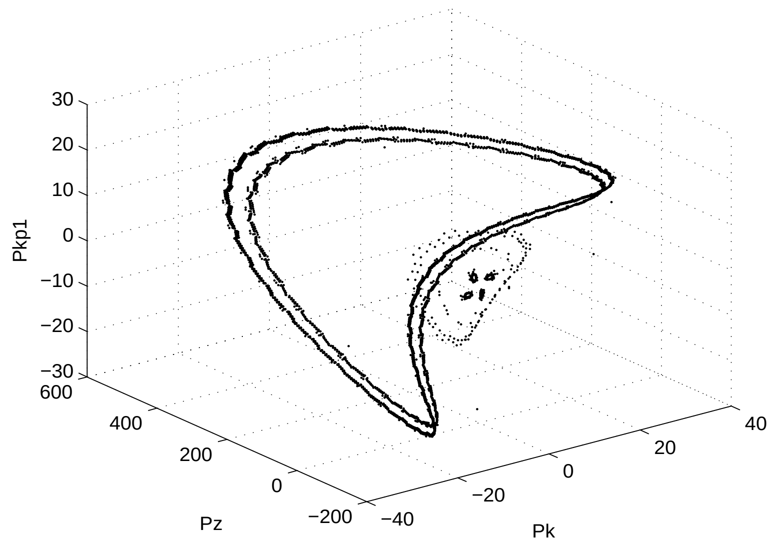

Figure 4: Preservation of Four Synchronous Attractors in the nonlinear case where $d = 0.26151152$ and $b_1 = 0.33$

$d \in [-2, 2]$ which covers the whole periodicity area of the studied model. In such case the strong periodicity points are indicated and correspond to the Farey tree sequence of polygons. The periodicity response is manifested as the sequence of mayor gaps in the bifurcation with the pole at the origin of *X-axis*. The analysis performed so far leads to the following proposition:

Proposition 3.1 *Periodicity conditions in the linear 2-d map of the cobweb anticipative system exists in the nonlinear expansion of the system.*

Let us consider the following two expansions of the model (several other expansions could also be applied, see for example [6, 8]; let us define the adaptive nonlinear multiplicator rule R as:

$$R = \begin{cases} \frac{P_{k+1}-P_k}{P_k} & \text{if} \quad -1 < \frac{P_{k+1}-P_k}{P_k} < 1 \\ 1 & \text{if} \quad \frac{P_{k+1}-P_k}{P_k} \geq 1 \\ -1 & \text{if} \quad \frac{P_{k+1}-P_k}{P_k} \leq -1 \end{cases} \tag{26}$$

Since the nonlinear rule has been applied the characteristic nonlinear bifurcation diagram evolves. The response of the system at the period $p = 6$ as one of the polygon rules which should provide the periodicity of the considered system has been preserved in such altered system. The beginning of bifurcation in corresponds to the value of parameter $d = 1$ which has been indicated in the analysis of the initial 2-d discrete map. Period six is followed by the $p = 7$ and $p = 8$. The underlying Farey sequence define the adapted nonlinear *2-d* discrete map. Such evidences are also find in other works in nonlinear system analysis for example [5] or in the recent works of dr. T. Puu [15, 14].

Consider another generic alteration of the initial anticipative cobweb model:

$$\begin{aligned} P_K(k+1) &= P_K + P_{KP1}(k) - \\ &- \left(P_K(k) + \frac{1}{P_Z(k)P_K(k)}\right) \\ P_{KP1}(k+1) &= P_{KP2}(k) \\ P_{KP2}(k+1) &= \frac{d}{b}\left(P_{KP1} - \frac{bP_K(k) - c + a}{d}\right) \\ P_Z(k+1) &= P_Z(k) + P_K(k)P_{KP1}(k) - \\ &- vP_Z(k) \end{aligned} \tag{27}$$

Slight modification of initial Hicks' model [16] gives the interesting response. The system can be represented in three dimensions which reveals the periodicity of the system for which the previously determined conditions of Farey tree generally still holds. Fig. 3 shows the *3d* bifurcation diagram for the altered model. One can see the four attractors which are simultaneous and represent the four possible equilibrium states for the trade dynamics. The 4-cycle characteristic is preserved at the alteration of the parameter v which could be observed in the Fig. 4. The four dots on the center-right side of the figure represents the four-cycle characteristic of the response. The larger orbits indicate the steep change in the modus of the system.

4. CONCLUSION

The solution of the periodicity conditions for the *2-d* discrete linear cobweb map provided the means to determine the periodicity conditions. Analytical approach with *z-transformation* is an adequate proposition for determination of the periodic solutions. The emergence of Farey tree as the rational fraction representation yields the organisation of the periodicity solutions. The developed model shows that by the statement of general rule of the system the synchronization of entire feedback-anticipative chain could be gained by setting the appropriate strategy in the form of parameters value set which should be time dependant. The bifurcation experiment with the nonlinear mapping provided the example of periodicity transposition to the systems of higher complexities.

5. *ACKNOWLEDGEMENT*

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