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Dynamical Systems on Cantorian spacetime and applications

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Abstract: - In this paper we analyze some dynamical systems, whose motion is not on continuous path, but on a fractal one. Starting from El Naschie's space time and E-Infinity theory we show a mathematical approach based on a potential theory to describe the interaction system-support. We study some relevant force fields on Cantorian space and analyze the differences with respect to the analogous case on continuum. Furthermore, we consider the idea that a Cantorian space could explain some relevant stochastic and quantum processes, if the space acts as a harmonic oscillating support. This means that a quantum process could sometimes be explained as a classical one, but on a non differential and discontinuous support, that is without invoking quantum mechanics. We consider the validity of this point of view, that in principle could be more realistic, because it describes the real nature of matter and space. Indeed, the presently observed large scale structure reflects the phenomenology of the microscopic world. The consequence of this point of view could be extended in many fields such as biomathematics, structural engineering, physics, astronomy, biology and so on.

Key-Words: - chaos, dynamical systems

1 Introduction

Nature shows us structures with scaling rules, where clustering properties from cosmological to nuclear objects reveals a form of hierarchy. Moreover, many systems show an oscillatory behaviour. In the previous papers, the authors consider the compatibility of a Stochastic Self-Similar, Fractal Universe with the observation and the consequences of this model. In particular, it has been demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N)=(h/Mc)N^{\{\phi\}}$, where R is the radius of the structures, h is the Planck constant, M is the total Mass of the self-gravitating system, c the speed of light, N the number of nucleons within the structures, and $\phi=((\sqrt{5}-1)/2)$ is the Golden Mean value [1]. It appears that the Universe has a memory of its quantum origin as suggested by R.Penrose with respect to quasi-crystal [2]. Particularly, the model is related to Penrose tiling and thus to $\varepsilon^{(\infty)}$ theory (Cantorian space-time theory) as proposed by El Naschie [3], [4] as well as with Connes Noncommutative Geometry [5].

Reading El Naschie's papers and the previous contribution, it clearly appears that the E-Infinity theory is a new framework for understanding and describing Nature. Probably, the main point of the theory is the fact that everything we see or measure is resolution dependent. As reported by El Naschie, in E-infinity view, spacetime is an infinite dimensional fractal that happens to have D=4 as the expectation value for the topological dimension [7]. In detail, the topological dimension 3+1=4 means that in our low energy resolution, the world appears to us as if it were four-dimensional. The vision presented by El Naschie for the micro-world and by Iovane for the macro-physics, suggests a radical change based on Cantorian spacetime. Here $\varepsilon^{(\infty)}$ Cantorian space-time is the physical spacetime, where nature manifests its transfinitness; while as we have seen in [6] Hilbert's space $H^{(\infty)}$ is a mathematical framework to describe the interaction between the observer and the dynamical system under measurement.

The present formulation, based on the non classical Cantorian geometry and topology of spacetime, automatically solves the paradoxical outcome of the two-slit experiment. As we will see in detail, the measurement, from a mathematical point of view, is equivalent to a projection of $\varepsilon^{(\infty)}$ on $H^{(\infty)}$ based on a 3+1 Euclidean space. As predicted in some papers by El Naschie, the mathematical solution of the two-slit experiment is the physical realization of Gödel's indecidability.

In the present paper we study the behaviour of a harmonic force field on Cantorian space and analyze the differences with respect to the analogous case of a continuum. The idea that we want to stress in this paper is that a Cantorian space could explain some relevant stochastic and quantum processes, if the space acts as a harmonic oscillating support, such as it happens in Nature. In other word, the vision is that an apparent uncertainty, linked with a fractal support rather than a continuous one, can produce an uncertainty on the motion of a physical object, which is explained via a stochastic or quantum process. This means that a quantum process, in some cases, could be explained as a classical one, but on a non continuous and fractal support. Consequently, an external observer looking at the

motion of a particle under a fixed solicitation can measure an unusual behaviour with respect to a continuous support, that is obvious with respect to the knowledge of the fractal support behaviour. In this case, he can make the hypothesis of an uncertainty or a stochasticity in the process (motion), while there is just really ignorance with respect to the support on which the motion takes place.
We show how the Heisenmberg's uncertainty principle can be translated from the processes and systems to the support, where we have classical dynamics. We consider the validity of this point of view, that in principle could be more realistic, since it describes the real nature of the matter, force and spacetime, which does not only exist in Euclidean space or curved only, but in a Cantorian spacetime as well. To do this we will introduce the creation and annihilation operators to create and destroy holes whose behaviour is comparable with quantum harmonic oscillators. In a sense, instead of considering the motion of a system in quantum mechanics, we have also another chance; that is, we can consider the classical motion of a system but on a Cantorian support, that shows the behaviour of a harmonic quantum chain.In conclusion, we arrive at a possible genesis of the multifractal space $\varepsilon^{(\infty)}$.

2 Cantor space and E-infinity Cantorian space

In [6] the author presented some results to show the link between E-Infinity and Hilbert (and Sobolev) spaces. Here we introduce some well known results in the functional analysis and summarize some results in [6] to focus our attention to dynamical systems on the Cantorian $\varepsilon^{(\infty)}$.
Let Ω be a nonempty open set in R^m.

Definition 1: *Let us consider $p \in R$ with $1 \le p < \infty$, we pose*

$$L^p(\Omega,\mu) = \{ f : \Omega \to R, \text{ with } f \text{ measurable } |f|^p \in L^1(\Omega,\mu) \}, \quad (1)$$

with

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \, d\mu \right)^{1/p} \quad (2)$$

Moreover, we pose

$$L^\infty(\Omega,\mu) = \{ f : \Omega \to R \mid f \text{ is measurable and } \exists c \in R : |f(x)| \le c \text{ almost everywhere in } \Omega \} \quad (3)$$

with

$$\|f\|_{L^\infty} = \inf \{ c ; |f(x)| \le c : \Omega \to R \text{ almost everywhere in } \Omega \}. \quad (4)$$

Definition 2: *Let us consider a finite or infinite complete vector space H on the field of complex numbers. In this space, a scalar product is defined so that for $\psi(x), \phi(x) \in H$:*

$$h_1: \quad \langle \psi, \varphi \rangle = \int_\Omega \psi^* \varphi \, d\mu = \left(\int_\Omega \varphi^* \psi \, d\mu \right)^* = \left(\langle \varphi^*, \psi \rangle \right)^*; \quad (5)$$

$$h_2: \quad \langle \psi, a_1\varphi_1 + a_2\varphi_2 \rangle = a_1 \langle \psi, \varphi_1 \rangle + a_2 \langle \psi, \varphi_2 \rangle;$$

$$h_3: \quad \langle \psi, \psi \rangle = \int_\Omega \psi^* \psi \, d\mu = \int_\Omega |\psi|^2 \, d\mu;$$

$$h_4: \quad \langle \psi, \psi \rangle = 0 \Leftrightarrow \psi = 0.$$

Remark 1: *$L^2(\Omega)$ with a scalar product is a Hilbert's space such as Sobolev's Space H^1 on L^2; we will introduce in the following.*
From the Riesz-Frechet representation theorem it follows that a state function can be written as $\psi = \sum_{i=1}^{\infty} \langle \psi, e_i \rangle e_i$ with $|\psi|^2 = \sum_{i=1}^{\infty} |\langle \psi, e_i \rangle|^2$.

Definition 3: *The functional space*

$$W^{q,p}(\Omega) = \left\{ f \in L^p(\Omega) : D^\alpha f = \frac{\partial^\alpha f}{\partial^{\alpha_1} x_1 .. \partial^{\alpha_k} x_k} \in L^p \right| \quad (6)$$

$$\left| \alpha \right| \le q, \alpha = \sum_{i=1}^{k} \alpha_i \Big\},$$

with the norm

$$\|f\|_{W^{q,p}} = \|f\|_{L^p} + \sum_{1 \le \alpha \le q} \|D^\alpha f\|_{L^p} \quad (7)$$

is named Sobolev's space.

Definition 4: *We denote $H^1(\Omega) = W^{1,2}(\Omega)$.*

Theorem 1: *The space $H^1(\Omega)$ is a separable Hilbert's space.*

We can call $H^2(\Omega) = W^{2,2}(\Omega)$. Consequently, we understand the functions of state $\psi \in H^2$.

Definition 5: *We can also define $W^{q,\infty}(\Omega)$ as*

$$W^{q,\infty}(\Omega) = \left\{ f \in L^\infty(\Omega) : D^\alpha f = \frac{\partial^\alpha f}{\partial^{\alpha_1} x_1 .. \partial^{\alpha_k} x_k} \in L^\infty(\Omega) \right| \quad (8)$$

$$\left| \alpha \right| \le q, \alpha = \sum_{i=1}^{k} \alpha_i \Big\},$$

with the norm

$$\|f\|_{W^{q,\infty}} = \max_{0\le\alpha\le q}\left\|D^{\alpha} f\right\|_{L^{\infty}} \qquad (9)$$

The mathematical tools introduced above are general: they can be used in quantum mechanics but not only in it. In [6] it has been showed their application to E-Infinity. From a mathematical point of view, in quantum mechanics we consider Hilbert's space on complex field C, while on Cantorian spacetime the basic field is R. Furthermore, while in quantum mechanics we have an uncertainty on the state or on the process (it depends if you use Schrödinger or Heisenberg representation), here the uncertainty is linked to the geometry and topology of the spacetime and more in deep to the resolution through which we make the observations. Consequently, also by using the same mathematics, the physical interpretation of nature and motion is completely different. Indeed, while in quantum mechanics we have Heisenberg's uncertainty, in E-Infinity we use a probabilistic approach to describe the motion on "complex" path on Cantor set. The main difference is that if we know the exactly geometry and topology of the support on which the motion happens we have no uncertainty. In addition, our approach as we will show, allows us to link the probability of a possible path directly with a potential energy function linked to the geometry and topology of the support (both material and energetic support).
Now if we consider E-Infinity, the following results can be reached.

Definition 6: *By using the same notation and meaning of the previous definitions and theorems, the following definitions hold*
$\left(\varepsilon^{(\infty)}, M, \mu_H\right)$ *is a Measure space;*
$L^1\left(\varepsilon^{(\infty)}, \mu_H\right)$ *is a Functional space of integrable function;*
$L^2\left(\varepsilon^{(\infty)}, \mu_H\right)$ *is a Functional space of square integrable function;*
$H^q\left(\varepsilon^{(\infty)}, \mu_H\right) = W^{q,2}\left(\varepsilon^{(\infty)}, \mu_H\right)$ *is a Sobolev space corresponding to integrable function;*
Naturally, for physical applications among H^q Sobolev's space $H^2(\varepsilon^{(\infty)},\mu_H)$ is the most relevant, we named it $H^{(\infty)}$.
In the descriptive set theory and the theory of polish spaces it is shown that [8]:

Definition 7: *When a space $A^{\mathbb{N}}$ is viewed as the product of infinitely many copies of A with discrete topology and is completely metrizable and if A is countable, then the space is said to be polish.*

In particular, when A={0,1},|A|=2, then we call $C=2^{IN}$ Cantor space. For A^{-1} defined in an interval $A^{-1} \subset]0,1[$ then $C_F=A^{IN}$ is called a fuzzy Cantor space. If $|A^{-1}|=(\sqrt{5}-1)/2$ and N=n-1, where $-\infty\le n\le\infty$, then $C_F = \varepsilon^n$ is the E-infinity Cantorian space. Mohamed El Naschie in [9] showed the relationship between the Cantor space C and ε^{∞} As He reports: " the relationship comes from the cardinality problem of a Borel set in polish spaces Thus we call a subset of a topological space a Cantor set if it is homeomorphic to the Cantor space".

2.1 Preliminaries

We denote by $D(\Omega)$ the set of $C^{\infty}(\Omega)$ functions with compact support in Ω, $D(\Omega) := C^{\infty}_C(\Omega)$.

Definition 8: *A distribution is a linear mapping $T \to \langle T,\varphi\rangle$ from $D(\Omega)$ to R, which is (sequentially) continuous, i.e. if $\varphi_n \to \varphi$ in $D(\Omega)$, then $\langle T,\varphi_n\rangle \to \langle T,\varphi\rangle$. The set of all distributions is called $D'(\Omega)$.*

Each $L^1(\Omega)$ function, say $f\in L^1(\Omega)$ can be regarded as a distribution setting

$$\langle f,\varphi\rangle = \int_{\Omega}\varphi(x)f(x)dx.$$

But $D'(\Omega)$ is much larger, for instance one may consider the Dirac mass centred at 0, with $0\in\Omega$, δ_0 defining

$$\langle \delta_0,\varphi\rangle := \int_{\Omega}\varphi(x)\delta_0 = \varphi(0).$$

Definition 9: *A sequence $\{T_n\}$ in $D'(\Omega)$ converges to $T\in D'(\Omega)$ if*

$$\langle T_n,\varphi\rangle \to \langle T,\varphi\rangle, \text{ for every } \varphi\in D(\Omega).$$

Definition 10: *Let Ω be an open set in R^m. Let $T\in D'(\Omega)$. Then the derivative of T with respect to x_j is defined as*

$$\left\langle \frac{\partial T}{\partial x_j},\varphi\right\rangle = -\left\langle T,\frac{\partial\varphi}{\partial x_j}\right\rangle$$

for every $\varphi\in D(\Omega)$.

If $T\in D'(\Omega)$, the support of T is the smallest set K outside which T vanishes, in the sense that $\varphi=0$ outside K, i.e. $<T,\varphi>=0$.
We also recall that the derivative operator, defined above is (sequentially) continuous, in the sense that if a sequence of distributions $\{T_n\}$ converges to T in $D'(\Omega)$, then the sequence $\{DT_n\}$ still converges to DT.
Assuming that $S,T\in D'(R^m)$, either S or T has compact support, then the convolution of S and T is defined by

$$\langle S*T,\varphi\rangle = \langle S(x)T(y),\varphi(x+y)\rangle \qquad (10)$$

and convolution is easily seen to be a commutative operation.

Theorem 2: *Let S be in $D'(R^m)$, and assume that $T_n\to T$ in $D'(R^m)$ and one of the following holds:*
i) The supports of all the T_n are contained in a common compact set;

ii) S has compact support;
iii) m=1 and the supports of the T_n and of S are bounded on the same side, independently of n.

Then $T_n * S \to T * S$ $D'(R^m)$.

For further details about the Theory of Distributions we refer to [10].

In the remainder of this section we recall well known facts of measure theory for reader's convenience. This section is very much inspired by [11].

Let X be a non empty set and M a σ-algebra in X (closed to $\varnothing$,X complementation and countable union).

Definition 11: *Let (X,M) be a measure space and $\mu: M \to [0,\infty]$. We say that μ is a positive measure if $\mu(\varnothing)=0$ and μ is σ-additive, i.e., for any sequence $\{E_h\}$ of pairwise disjoint elements of M,*

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h).$$

A positive measure μ such that μ(X)=1 is called a probability measure.

Definition 12: *Let X be a locally compact and separable metric space, B(X) its Borel σ-algebra (σ-algebra generated by open sets), and consider the measure space (X,B(X)).*
A positive measure on (X,B(X)) is called a Borel measure. If a Borel measure is finite on compact sets, it is called a positive Radon measure.

By $[M_{loc}(X)]^m$ it is usually denoted the space of the R^mvalued Radon measures on X.

Definition 13: *Let $\mu \in [M_{loc}(X)]^m$ and let $\{\mu_h\}_h \subset [M_{loc}(X)]^m$; the sequence $\{\mu_h\}_h$ locally weakly $*$ converges to μ if*

$$\lim_{h\to\infty} \int_X u d\mu_h = \int_X u d\mu$$

for every $u \in C_c(X)$; if μ and μ_h are finite, we say that $\{\mu_h\}_h$ weakly $*$ converges to μ if

$$\lim_{h\to\infty} \int_X u d\mu_h = \int_X u d\mu$$

Theorem 3: *If $\{\mu_h\}_h$ is a sequence of finite Radon measures on the locally compact and separable metric space X with $\sup\{|\mu_h|(X): h \in N\} < +\infty$, then it has a weakly $*$ converging subsequence. Moreover the map $\mu \to |\mu|(X)$ is lower semicontinuous with respect to the weak $*$ convergence.*

Remark 2: *It is useful for our aims to recall that if X coincides with a non empty open set Ω in R^m, then any Radon measure in M(Ω) is a distribution, $\langle \mu, \varphi \rangle = \int_X \varphi d\mu$ for every $\varphi \in D(\Omega)$.*

Definition 14: *Let (X, ε) and (Y,F) be measure spaces, and let $\varphi: X \to Y$ be such that $\varphi^{-1}(F) \in \varepsilon$ whenever $F \in F$. For any positive measure μ on (X, ε) we define a measure $\varphi_{\#}\mu(F)$ in (Y,F) by*

$$\varphi_{\#}\mu(F) := \mu(\varphi^{-1}(F))$$

for every $F \in F$.

Given any Radon measure ν on the measure space (X, ε), and any subset *G* in ε, with the symbol ν⌊F, we mean the measure ν acting on $G \cap E$, for any $E \in \varepsilon$.

The notions of Hausdorff measure and dimension will be needed in the sequel.

Consider the metric space (R^m,d), where d is the metric induced from the Euclidean norm. Let $A \subset R^m$ be bounded. By *A* we denote the set of sequences of subsets $\{A_i \subset A\}$, such that $A = \bigcup_i^{\infty} A_i$.

Let $0 < \varepsilon < +\infty$, and $0 \le s < +\infty$. We define

$$H_{\varepsilon}^{s}(A) = \inf\left\{ \sum_{i=1}^{\infty} (diam A_i)^s : \{A_i\},\ diam A_i < \varepsilon \text{ for every } i \in IN \right\}$$

Clearly $H_{\varepsilon}^{s}(A)$ increases as $\varepsilon \to 0$, hence

$$H^s(A) = \lim_{\varepsilon \to 0} H^s{}_{\varepsilon}(A) \tag{11}$$

is well posed.

The next Theorem is proven in [12].

Theorem 4: *Let m be a positive integer. Let A be a bounded subset of (R^m,d). Then there exists a unique real number $dim_H \in [0,m]$ such that*

$$H^s(A) = \begin{cases} \infty & \text{if } s < \dim_H \text{ and } s \in [0,+\infty| \\ 0 & \text{if } s > \dim_H \text{ and } s \in [0,+\infty| \end{cases}$$

We can construct the Cantor set by using a uniterative procedure (see [13] for details).

Let us consider a set of intervals $A = \{A^{(1)}, A^{(2)}, ..., A^{(n)}\}$ with

$$\begin{aligned} A^{(0)} &= [a,b] \\ A^{(1)} &= [a, a+(b-a)/3] \cup [a+2(b-a)/3, b] \\ &.... \quad .... \\ A^{(n)} &= [a, a+(b-a)/3^n] \cup ... \cup [a+2(b-a)/3^n, b] \end{aligned} \tag{12}$$

where *a,b* are real numbers.

If w is the level corresponding to $A^{(w)}$ the number of extreme points is 2^w.

For a=0 and b=1 we obtain the Cantor space $C=\bigcap_{n\in N} C_n$ with the following well know properties:

- C is compact, with null Reinmannian measure;
- There are no intervals in C;
- C has the cardinality of continuum.

For our purpose we consider the set A. In particular, at the level w the length of a segment is $k_{w+1}=(b-a)/3^w$ with $w=0,1,2,..,n-1$.

To evaluate the extremes for each a level w without using an iterative procedure we show the following algebraic method.

Let us introduce the vector $g_m=(g_{m,1},g_{m,2},...,g_{m,2^n})$ with $m=1,...,n-1$ and with 2^n components, whose values are:

$$g_{m,z}=\begin{cases}2 & if \quad z=2^m(1+2s)+h\\ 1 & if \quad z=2^m(1+2s)\\ 0 & elsewhere\end{cases} \qquad (13)$$

with $s=0,...,2^{n-m-1}-1$ and $h=1,...,2^m-1$. For $m=n$ we have a vector with all components equal to zero except the component at the place 2^n, that is equal to 1, i.e., $g_n=(0,0,...,0,1)$. Consequently, the coordinates of the extremes at a fixed level n are given by the following vector

$$A^{(n)}=a+\sum_{p=0}^{n-1}k_{n-p}g_{p+1}, \qquad (14)$$

where $\mathbf{a}=(a,....,a)$ is a constant vector; in detail, for the individual coordinate we have the following result.

Proposition 1: The coordinate of an extreme at the level n is

$$A_z^{(n)}=a+\sum_{p=0}^{n-1}k_{n-p}g_{p+1,z}. \qquad (15)$$

The two expressions (14) and (15) give us a uniterative method to evaluate the vector of extremes and a fixed extremal at each a level. As example we see

$$A_1^{(1)}=a,\quad A_2^{(1)}=b;$$

$$A_1^{(2)}=a,\quad A_2^{(2)}=a+k_2,\quad A_3^{(2)}=a+2k_2,\quad A_4^{(2)}=b;$$

$$A_1^{(3)}=a,\quad A_2^{(3)}=a+k_3,\quad A_3^{(3)}=a+2k_3,\quad A_4^{(3)}=a+k_2,\quad A_5^{(3)}=a+2k_2,\quad A_6^{(3)}=a+2k_2+k_3,\quad A_7^{(3)}=a+2k_2+2k_3,\quad A_8^{(3)}=b;$$

$$\begin{aligned}
A_1^{(4)}&=a\\ A_2^{(4)}&=a+k_4\\ A_3^{(4)}&=a+2k_4\\ A_4^{(4)}&=a+k_3\\ A_5^{(4)}&=a+2k_3\\ A_6^{(4)}&=a+2k_3+k_4\\ A_7^{(4)}&=a+2k_3+2k_4\\ A_8^{(4)}&=a+k_2\\ A_9^{(4)}&=a+2k_2\\ A_{10}^{(4)}&=a+2k_2+k_4\\ A_{11}^{(4)}&=a+2k_2+2k_4\\ A_{12}^{(4)}&=a+2k_2+k_3\\ A_{13}^{(4)}&=a+2k_2+2k_3\\ A_{14}^{(4)}&=a+2k_2+2k_3+k_4\\ A_{15}^{(4)}&=a+2k_2+2k_3+2k_4\\ A_{16}^{(4)}&=b
\end{aligned}$$

It clearly appears how the previous definition and two propositions can be also used on the set A, which generalizes the Cantor space on the interval $[a,b]$ instead of [0,1].

We also recall that $dim_H(A)=((log2)/(log3))$ and $H^{\dim_H}(A)=1$, (see [14]).

Starting from the construction of A presented above we can define a sequence of probability measures on the (locally compact and separable) metric space (R,d), where d is the Euclidean metric, namely

$$\begin{aligned}
&\mu_1=\frac{1}{2}(\delta_0+\delta_1)\\
&\mu_2=\frac{1}{2^2}\left(\delta_0+\delta_{\frac{1}{3}}+\delta_{\frac{2}{3}}+\delta_1\right)\\
&\mu_2=\frac{1}{2^3}\left(\delta_0+\delta_{\frac{1}{9}}+\delta_{\frac{2}{9}}+\delta_{\frac{1}{3}}+\delta_{\frac{2}{3}}+\delta_{\frac{7}{9}}+\delta_{\frac{8}{9}}+\delta_1\right)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\mu_n=\frac{1}{2^n}\sum_{i=1}^{2^n}\delta_{c_i^n}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \qquad (16)$$

where c_i^n is the i-th point, which leads to the construction of A at level n.

By virtue of Theorem 3 the sequence $\{\mu_n\}$ admits a weakly $*$ converging subsequence $\{\mu_{k_n}\}$. From Definitions 9 and 13 it follows that the sequence $\{\mu_{n_k}\}$ converges also in the sense of distributions.

In order to identify the limit probability measure μ, we consider the primitives $\int du_n = f_n$, (we recall that if a sequence $\{T_n\} \subset D'(R)$ converges to T in $D'(R)$ then the sequence $\int T_n$ still converges to $\int T$, cf. [10].) where $f : R \to [0,1]$ is the step function below

$$f_n(x) = \begin{cases} 0 & if \quad x < 0 \\ \frac{1}{2^n} & if \quad 0 \le x < \frac{1}{3^n} \\ \frac{1}{2^{n-1}} & if \quad \frac{1}{3^n} \le x < \frac{2}{3^n} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ 1 - \frac{1}{2^n} & if \; 1 - \frac{1}{3^n} \le x < 1 \\ 1 & if \quad x \ge 1 \end{cases} \tag{17}$$

The sequence $\{f_n\}$ converges uniformly to the Cantor-Vitali function *f*.

It can be easily shown that *f* is increasing and continuous with 'classical' derivative coinciding with 0 a.e. On the other hand, one can prove that the distributional derivative of *f*, namely *Df* is a probability measure μ supported on *C*, and it results

$$\mu = H^{\frac{\log 2}{\log 3}} \lfloor A. \tag{18}$$

Hence

$$f(t) = H^{\frac{\log 2}{\log 3}}([0,t] \cap A) \text{ for any } t \ge 0 \tag{19}$$

(see [11] for details).

Consequently we can say that the sequence of distributional derivatives $\{Df_n\}$, namely $\{\mu_n\}$ converges in the sense of distributions to the derivative of f, the probability measure *Df* in (18), i.e.

$$Df_n = \mu_n \to Df = H^{\frac{\log 2}{\log 3}} \lfloor C \text{ in } D'(R) \tag{20}$$

We also emphasize that this measure is the only probability measure on *C* which satisfies a scaling property as *C* itself does, namely

$$\mu = \frac{1}{2}[s_{1\#}(\mu) + s_{2\#}(\mu)] \tag{21}$$

As a consequence we can also say that $H^{\frac{\log 2}{\log 3}} \lfloor A$ is the limit in the sense of Definition 13 of the whole sequence $\{\mu_n\}$.

Next taking any distribution (potential) $V \in D'(R)$ satisfying the assumptions of Theorem 2, we may define, again keeping in mind the scheme above, a sequence of 'potentials' $\{V_n\}$, defined as

$$V_n(x) := \frac{1}{2^n} \sum_{i=1}^{2^n} V(x - c_i) \tag{22}$$

where c_i is a generical point as in *A*.

Clearly we may rewrite (22) as

$$V_n = V * \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{c_i} = \pi * \mu_n \tag{23}$$

Again Theorem 2 and convergence (20) give us that

$$V_n = V * H^{\frac{\log 2}{\log 3}} \lfloor A \text{ in } D'(R) \tag{24}$$

This argument proves the following theorem

Theorem 3 *Let $\{\mu_n\}$ be the sequence of probability measures in (16) and let V be any distribution in D'(R) satisfying the assumption of Theorem 2. Then (24) holds, with V_n defined in (22).*

Remark 3 *We stress that if the potential V is more regular than what is required by Theorem 2, the convergence in (24) can be shown to be much stronger.*

Remark 4 *It is worthwhile mentioning that the argument above can be easily adapted to other kinds of Fractals, more general than A. The potential V can be, as already mentioned, very general, thus leaving the opportunity to describe many physical problems. For instance, a Gaussian potential will work for describing a barrier or an obstacle on the support where the motion happens. Furthermore the sequence $\{\mu_n\}$ presented in (16) can be replaced by any other probability measures' sequence converging to $H^{\frac{\log 2}{\log 3}} \lfloor A$. Our choice risen from the reason of working out a basic case. Clearly other choices are possible, even not probability measures' sequences, but just uniformly bounded ones, thus leading to describe other limit measures μ still supported on the same fractals but with different weigh.*

3 The physical scenario and the unification of the fundamental interactions in $\varepsilon^{(\infty)}$ Cantorian

Are there other direct or indirect consequences of the fact that the real spacetime is infinite dimensional hierarchical?

In [1], [15], [16] the author demonstrated that the Compton wavelength rule is a special case of a self-similar law. In detail, the following theorem occurs.

Theorem 6: *The structures of the Universe appear as if they were a classically self-similar random process at all astrophysical scales. The characteristic length scale has a self-similar expression*

$$R(N) = \frac{h}{Mc} N^{1+\phi} = \frac{h}{m_n c} N^{\phi} \tag{25}$$

where the mass M is the mass of the structure, m_n is the mass of a nucleon, N is the number of nucleons into the structure and ϕ is the Golden Mean value. In terms of Plankian quantities the length scale can be recast in $R_P(N) = \frac{l_P}{m_P}\sqrt{\frac{\hbar c}{G}} N^{(1+\phi)}$. *The previous expression reflects the quantum (stochastic) and relativistic memory of the Universe at all scales, which appears as hierarchy in the clustering properties.*

From the previous theorem it follows that it exists a fundamental length

$$R_P = \frac{l_P}{m_P}\sqrt{\frac{\hbar c}{G}}; \tag{26}$$

it can be seen as the minimum resolution, under which we have stochastic fluctuations of the geometry and topology of spacetime. Moreover, for $N \to 1 \Rightarrow R_P(N) \to R_P$ with fixed hierarchical jumps, corresponding to fixed fundamental scale linked with the global geometry and topology of spacetime.

As showed in [16], it is easy to find the following general expression, which links the energy at macroscopic scale with the microscopic one.

Theorem 7: *The mass and the extension of a body are connected with its quantum properties, through to the relation*

$$E_{E,N}(N) = E_P N^{1+\phi}, \tag{27}$$

that links Plank's energy and Einstein's one.

Moreover

$$E_{E,N}(N) = E_P N^{1+\phi}, \Leftrightarrow E_E = \widetilde{E}_{P,N}(N) \tag{28}$$

where

$$\widetilde{E}_{P,N}(N) = h\widetilde{\upsilon}, \tag{29}$$

with $\widetilde{\nu}(N) = \nu N^{(1+\phi)}$.

The quantum (stochastic) and relativistic memory is reflected at all scales and manifests itself through a clusterization principle of the mass and extension of the body.

From the previous relations it follows that it exists a characteristic frequency of each scale. It is linked up its mass and extension.

Could some deviations exist in the Newtonian law at low or very large distances?

Let us consider a gravitational potential $V=V_N(r)$, which is a continuous function of the distance, but also depends (discreetly) on the number of the components or better on the resolution $R(N)$, i.e.

$$V_N(r) \propto \frac{1}{r} e^{-r/R(N)} \tag{30}$$

For $N=1$, $V_N(r)$ is the Yukawa potential, while for $N \to \infty$ or $R(N)$ very large $V_N(r) \to 1/r$.

By comparing our potential with the Yukawa potential, we see that in our case the role of the action radius λ is played by the length scale (i.e. by the resolution). As well known from some general properties of the relativistic field theory, there are some constrains about the alternatives to $V_N(r) \propto \frac{1}{r}$ [17]. Indeed, the unique alternative to (30) is given by a combination of potential of the (30)-type. This means considering different gravitational fields. Moreover, the behaviour at large scale of the total potential will be dominated by the term with the greatest length scale factor $R(N)$. This point of view is similar to the case of higher order theories of gravity, where typically the following interaction Lagrangian functional is assigned

$$L \propto \sqrt{-g}\,(R + \sum_{j=0}^{k} a_j R\,\Box^j R) \tag{31}$$

where R is Ricci's curvature scalar and $\Box$ is d'Alembert's operator. It can be shown that in the Post-Newtonian limit, we get

$$V_N(r) \propto \frac{1}{r}\left[1 + \sum_{i=0}^{n} \beta_i e^{-r/\lambda_i}\right] \tag{32}$$

In other words, any higher order correction to Einstein's gravity gives a Yukawa's contribution to the Newtonian potential. From these considerations, it clearly appears that another hypothesis to the potential (30) is the following

$$V_N(r) \propto \frac{1}{r} + \frac{\alpha}{r} e^{-r/R(N)_i} \tag{33}$$

For $R(N) \to \infty, V_N(r) \to 1/r$, while for $r << R(N) \to V_N(r) \propto ((1+\alpha))/r)$.

How can we measure α and which is its order of magnitude? A full analysis is given in [18], here we can only say that as

we will show in the next sections it exists a deep link between the geometry and the topology of the spacetime (support) and the dynamical system moving on it.

4 A toy model: an elastic force field on Cantorian space

The idea that we want to stress here is that a Cantorian space could explain some relevant stochastic and quantum processes, if the space acts as a harmonic oscillating support, such as it happens in nature. In other word, the vision is that an apparent uncertainty, linked with a fractal support rather than a continuous one, can produce an uncertainty on the motion of a physical object, which is explained thanks to stochastic or quantum process. This means that a quantum process in some cases could be explained as a classical one, but on a non continuous and fractal support.

To show this result we have to consider a classical harmonic oscillator on a fractal support and then we have to consider it not as the subject of our study, but as the scenario where a process could happen.

Let us suppose that an elastic force field $f(q)=-\alpha q$ is on the interval $A^{(1)}=[a,b]=[A_1^{(1)},A_2^{(1)}]$, moreover α is a nonnegative elastic constant; it is well known the motion equation for a point of mass m coming from the Newtonian equation $f=ma$ is,

$$\ddot{q}^{(1)}+\left(\omega^{(1)}\right)^2 q=0, \tag{34}$$

where $\omega^{(1)}=\alpha/m$ and with the solution

$$q^{(1)}=A_2^{(1)}\cos(\omega^{(1)}t)$$

where we have assumed the initial condition $q^{(1)}(t=0)=A_2^{(1)}=b$ and $\dot{q}(t=0)=0$. This can be seen as the level 1 of a recursive procedure, where we consider a fractal support. In fact, at the level 2 we have

$$\begin{aligned}A^{(2)}&=[a,a+(b-a)/3]\cup[a+2(b-a)/3,b]\\&=[A_1^{(2)},A_1^{(2)}+k_2]\cup[A_1^{(2)}+2k_2,A_4^{(2)}]\\&=[A_1^{(2)},A_2^{(2)}]\cup[A_3^{(2)}+,A_4^{(2)}]\end{aligned}$$

where $k_2=(A_2^{(1)}-A_1^{(1)})/3=(b-a)/3$; consequently we consider the force field

$$f^{(2)}(q)=\begin{cases}-\varepsilon^{(2)}\alpha q & \forall q\in[A_1^{(2)},A_2^{(2)}]\\0 & \forall q\in[A_2^{(2)},A_3^{(2)}]\\-\varepsilon^{(2)}\alpha q & \forall q\in[A_3^{(2)},A_4^{(2)}]\end{cases} \tag{35}$$

where $\varepsilon^{(2)}$ is a parameter which takes into account the reduction of the support and to preserve the total energy. We will determine it in the following.

The total path of the level 2 can be see as composed by three sub-path: the first and the third with an oscillation motion, while the second where we find a uniform motion. In fact, it is easy to find for the third sub-path

$$q_3^{(2)}(t)=A_4^{(2)}\cos(\omega^{(2)}t),$$

with $(\omega^{(2)})^2=\varepsilon^{(2)}(\omega^{(1)})^2$, where we have considered the initial conditions $q_3^{(2)}(t=0)=A_4^{(2)}=b$ and $\dot{q}_3^{(2)}(t=0)=0$. Analogously, we get for the second sub-path

$$q_2^{(2)}(t)=\beta_2^{(2)}t+A_3^{(2)},$$

where we have used the following initial condition $q_3^{(2)}(t_1)=q_2^{(2)}(t_1)=A_3^{(2)}$, and so

$$A_3^{(2)}=A_4^{(2)}\cos(\omega^{(2)}t_1)\rightarrow t_1=\frac{1}{\omega^{(2)}}\cos^{-1}(A_3^{(2)}/A_4^{(2)})$$

and

$$\begin{aligned}&\dot{q}_2^{(2)}(t=t_1)=\dot{q}_3^{(2)}(t=t_1)\equiv\beta_2^{(2)}=\\&=-A_4^{(2)}\omega^{(2)}\sin\cos^{-1}(A_3^{(2)}/A_4^{(2)}).\end{aligned}$$

On the first path, we find again an oscillation motion

$$q_1^{(2)}(t)=A_2^{(2)}\cos(\omega^{(2)}t+B_2^{(2)}),$$

with the following initial condition $q_2^{(2)}(t_2)=q_1^{(2)}(t_2)=A_2^{(2)}\rightarrow t_2=(A_2^{(2)}-A_3^{(2)})/\beta_2^{(2)}$, and $\dot{q}_2^{(2)}(t=t_2)=\dot{q}_1^{(2)}(t=t_2)$ and so $B_2^{(2)}=\tan^{-1}(-\beta_2^{(2)}/A_2^{(2)})-(-A_2^{(2)}-A_3^{(2)})/\beta_2^{(2)}$. To summarize at the level 2 we obtain

$$q^{(2)}(t)=\begin{cases}q_1^{(2)}(t)=A_2^{(2)}\cos(\omega^{(2)}t+B_2^{(2)}) & \forall q\in[A_1^{(2)},A_2^{(2)}]\\q_2^{(2)}(t)=\beta_2^{(2)}t+A_3^{(2)} & \forall q\in[A_2^{(2)},A_3^{(2)}]\\q_3^{(2)}(t)=A_4^{(2)}\cos\left(\omega^{(2)}t\right) & \forall q\in[A_3^{(2)},A_4^{(2)}]\end{cases}$$

With respect to the level 1 we have different amplitudes, frequencies and phases, but not casual ones since they are in agreement with the fractal model inside them.

Consequently, an external observer looking at the motion of a particle under a fixed solicitation can measure an unusual behaviour with respect to a continuous support, that is obvious if the knowledge of the support behaviour is given. In this case he can make the hypothesis of an uncertainty or a stochasticity in the process (motion), while there is just an ignorance with respect to the material support on which the motion happens.

It is easy and obvious to obtain at the level 3 the following results:

$A^{(3)}=[A_1^{(3)},A_2^{(3)}]\cup[A_3^{(3)},A_4^{(3)}]\cup[A_5^{(3)},A_6^{(3)}]\cup[A_7^{(3)},A_8^{(3)}]$

and

$$f^{(3)}(q)=\begin{cases}-\varepsilon^{(3)}\alpha q & \forall q\in[A_1^{(3)},A_2^{(3)}]\\ 0 & \forall q\in[A_2^{(3)},A_3^{(3)}]\\ -\varepsilon^{(3)}\alpha q & \forall q\in[A_3^{(3)},A_4^{(3)}]\\ 0 & \forall q\in[A_4^{(3)},A_5^{(3)}]\\ -\varepsilon^{(3)}\alpha q & \forall q\in[A_5^{(3)},A_6^{(3)}]\\ 0 & \forall q\in[A_6^{(3)},A_7^{(3)}]\\ -\varepsilon^{(3)}\alpha q & \forall q\in[A_7^{(3)},A_8^{(3)}]\end{cases}$$

consequently,

$$q^{(3)}(t)=\begin{cases}q_1^{(3)}(t)=A_2^{(3)}\cos(\omega^{(3)}t+B_1^{(3)}), & \forall q\in[A_1^{(3)},A_2^{(3)}]\\ q_2^{(3)}(t)=\beta_2^{(3)}t+A_3^{(3)}, & \forall q\in[A_2^{(3)},A_3^{(3)}]\\ q_3^{(3)}(t)=A_4^{(3)}\cos\left(\omega^{(3)}t+B_3^{(3)}\right), & \forall q\in[A_3^{(3)},A_4^{(3)}]\\ q_4^{(3)}(t)=\beta_4^{(3)}t+A_5^{(3)}, & \forall q\in[A_4^{(3)},A_5^{(3)}]\\ q_5^{(3)}(t)=A_6^{(3)}\cos(\omega^{(3)}t+B_5^{(3)}), & \forall q\in[A_5^{(3)},A_6^{(3)}]\\ q_6^{(3)}(t)=\beta_6^{(3)}t+A_7^{(3)}, & \forall q\in[A_6^{(3)},A_7^{(3)}]\\ q_7^{(3)}(t)=A_8^{(3)}\cos\left(\omega^{(3)}t\right), & \forall q\in[A_7^{(3)},A_8^{(3)}]\end{cases}$$

where more in general at the level n the initial condition for the velocity are

$$\beta_z^{(n)}=-A_{z+2}^{(n)}\omega^{(n)}\sin\cos^{-1}(A_{z+1}^{(n)}/A_{z+2}^{(n)})-(A_{z+1}^{(n)}-A_z^{(n)})/\beta_z^{(n)},$$

with $z=0,\ldots,2^n-1$,

$$B_{z+1}^{(n)}=\tan^{-1}(-\beta_z^{(n)}/A_{z+1}^{(n)})-(A_{z+1}^{(n)}-A_z^{(n)})/\beta_z^{(n)}$$

Consequently, by taking into account Hausdorff $\aleph(C)$ measure the asymptotic behaviour gives us

$$\begin{aligned}f^{(\infty)} &= \lim_{n\to\infty}f^{(n)}=-\frac{\alpha q}{\aleph(C)},\\ k_\infty &= \lim_{n\to\infty}k_{n+1}=0,\\ \left(\omega^{(\infty)}\right)^2 &= \lim_{n\to\infty}\left(\omega^{(n)}\right)^2=\frac{\left(\omega^{(1)}\right)^2}{\aleph(C)}\\ \beta_z^{(\infty)} &= \lim_{n\to\infty}\beta_z^{(n)}=\beta_1^{(1)}=0\\ B_z^{(\infty)} &= \lim_{n\to\infty}\left(B_z^{(n)}\right)=0\\ A_z^{(\infty)} &\in C\end{aligned}$$

With respect to $\beta_1^{(1)}$, here we have used the initial condition $\beta_1^{(1)}=0$, but it is obvious that there are no changes if $\beta_1^{(1)}=v_0$.

Thanks to the previous relation we obtain

$$q^{(\infty)}(t)=\begin{cases}q_{odd}^{(\infty)}=A_{odd+1}^{(\infty)}\cos(\omega^{(\infty)}t+B_{odd}^{(\infty)}),\\ q_{even}^{(\infty)}=v_0t+A_{even+1}^{(\infty)}),\end{cases}$$

The same considerations can be done for a $\varepsilon^{(\infty)}$ El Naschie Cantorian space to obtain similar results.

It is interesting to note that if we have an external solicitation $F=\alpha q$, the motion equation on C becomes

$$\ddot{q}+2\omega^2q=F,$$

and so

$$\ddot{q}+\omega^2q=0,$$

that is the traditional motion equation for a massive point in an elastic force field. In other word, what we consider an external force for the support could be the classical elastic solicitation on a massive point moving on a continuous support[1]. This example is just a toy model when we deal with macroscopic systems, since the frequency of oscillation of the support could be very different with respect to the system. But this toy model could be very realistic with respect to microscopic and quantum processes and systems. For this reason in the next paragraph we consider oscillating force fields in quantum mechanics to show how the Heisenmberg uncertainty principle can be translated from the processes and systems to the support, where we have classical dynamics.

5 Dynamical Systems and genesis of $\varepsilon^{(\infty)}$ Cantorian space

To consider a more compact formulation of the oscillation on a Cantorian support it is preferable using the Lagrangian and Hamiltonian formalism than Newtonian formulation.

For a structureless massive point *(P,m)* moving on one dimension under the influence of a time-independent potential $V(q)=(1/2)kq^2$ on a continuum support, we have seen that it corresponds to $V^{(\infty)}(q)=(1/2)((kq^2)/(\aleph(C)))$ on Cantorian space *C*. With respect to Newton's equation of motion $m\ddot{q}=-\partial V/\partial q$ to study the time evolution of the trajectory, an alternative and more flexible description of the same system is obtained by using the Lagrangian functional

$$L^{(\infty)}(q,\dot{q})=T-V=\frac{1}{2}m\dot{q}^2-V^{(\infty)}(q)$$

Consequently, the solution *q(t)* can be achieved as solution of Euler-Langrage differential equation

$$\frac{d}{dt}\frac{\partial L^{(\infty)}}{\partial\dot{q}}-\frac{\partial L^{(\infty)}}{\partial q}=0.$$

If we use the asymptotic potential $V^{(\infty)}$, we easily obtain the same results as the previous paragraph.

Following the classical development of mechanics and thanks to the Legendre transformation $(q,q)\to(q,p)$ the Hamiltonian functional results

[1] To be more correct there is a change in the sign of the force, but this is linked with the subject of the problem (support or massive point). In other words, the sign minus of the elastic source becomes plus if we have it as an external source for the material support.

$$H^{(\infty)}(q,p) = p\dot{q}(p) - L^{(\infty)}(q,\dot{q}(p))$$

is introduced. Consequently, the motion is described in terms of the system

$$\begin{cases} \dot{p}^{(\infty)} = -\dfrac{\partial H^{(\infty)}}{\partial q} \\ \dot{q}^{(\infty)} = \dfrac{\partial H^{(\infty)}}{\partial p} \end{cases}$$

To consider the quantum theory of the harmonic oscillator on Cantorian space, let us introduce the Hamiltonian functional for a one-dimensional oscillator

$$\hat{H}^{(\infty)} = \frac{\hat{p}^2}{2m} + \frac{m}{2}(\omega^{(\infty)})^2 \hat{q}^2,$$

In [6] we proposed a toy model for generating E-Infinity starting from a continuum.

A more realistic model can be obtained by using a linear chain[2].
. The Lagrangian reads

$$L^{(\infty)} = T - V^{(\infty)} = \sum_{n=1}^{\infty} \frac{m}{2}\dot{q}_n^2 - \sum_{n=1}^{\infty} k\left(q_{n+1} - q_n\right)^2$$

and the equations of motion are

$$\ddot{q} + \omega^{(\infty)2}(2q_n - q_{n+1} - q_{n-1}) = 0, \qquad (36)$$

with $n \in N$.

All the relations of the previous type describe a system of coupled oscillators. The main difference with respect to a continuum is that here the q_n, q_{n+1}, q_{n-1} are linked by scaling rule as found in Sect.3.

There are two interesting limits:

- When the distance between the oscillators $l \to 0$, that means to consider the complement of Cantorian set $(C)^c$; then

$$\frac{q_{n+1} - 2q_n + q_{n-1}}{l^2} \to \frac{\partial^2}{\partial r^2} q(r,t),$$

and so the motion equation becomes the wave equation

$$\Box q(r,t) = 0,$$

with $\Box = \dfrac{\partial^2}{\partial t^2} - (v)^2 \dfrac{\partial^2}{\partial r^2}$ where $v = \left(l^2 \dfrac{k}{m}\right)^{1/2}$.

- When we consider C we obtain a linear asymptotic motion with harmonic fluctuations on C - support.

By introducing the canonical momenta $p_n = \dfrac{\partial L^{(\infty)}}{\partial \dot{q}_n} = m\dot{q}_n$, the Hamiltonian functional becomes

$$H^{(\infty)} = \sum_{n=1}^{N} \frac{p_n^2}{2m} + \sum_{n=1}^{N} k\left(q_{n+1} - q_n\right)^2.$$

[2] The 3-dimensional case is a trivial generalization.

To solve the motion equation (36) we specify the boundary condition $q_1 = q_N$. It is useful to introduce normal coordinates with a set of linearly independent basis functions

$$u_n^k = \frac{1}{\sqrt{N}} e^{ikln}$$

where k is the index of the basis set.
Consequently, we obtain the discrete Fourier decomposition

$$q_n(t) = \sum_k a_k(t) u_n^k. \qquad (37)$$

We can make the following remarks

- k plays the role of wave number;
- $k = \dfrac{2\pi}{Nl} b$ with $b \in Z$;
- $-\dfrac{N}{2} \le b \le \dfrac{N}{2}$;
- $\sum_{n=1}^{N} u_n^{k'*} u_n^k = \delta_{kk'}$ (ortonormality condition);
- $\sum_k u_n^{k'*} u_n^k = \delta_{nn'}$ (completness condition);
- $u_n^{k*} = u_n^{-k}$;
- $a_k^*(t) = a_{-k}(t)$.

By introducing (37) in (36) we obtain

$$\ddot{a}_k(t) = 2\frac{k}{m} \sum_{k'} a_{k'}(t) \sum_{n=1}^{N} u_n^{k*} (u_{n+1}^{k'} + u_{n-1}^{k'} - 2u_n^k),$$

and by taking into account that the value of u_n^k for different n differs from a phase factor, that is $u_{n\pm1}^k = e^{\pm ikl} u_n^k$ we obtain

$$\ddot{a}_k(t) = 2\frac{k}{m}\left(e^{ikl} + e^{-ikl} - 2\right) a_k(t) = -\left(\omega_k^{(\infty)}\right)^2 a_k(t), \qquad (38)$$

with

$$\omega_k^{(\infty)} = \sqrt{4\frac{k}{m}(1 - \cos kl)} = 4\omega^{(0)} \left|\sin\frac{kl}{2}\right|.$$

In other words, starting from a system of coupled oscillators (that is (36)) we obtain solutions as uncoupled oscillators (this is the reason to use normal coordinates).
The solution of (38) is

$$a_k(t) = b_k e^{-i\omega_k^{(\infty)} t} + b_{-k}^* e^{i\omega_k^{(\infty)} t},$$

and so (for $k \to -k$)

$$q_n(t) = \sum_k \left(b_k e^{-i\omega_k^{(\infty)} t} u_n^k + b_k^* e^{i\omega_k^{(\infty)} t} u_n^{k*}\right)$$

In order to consider the quantum approach, it is useful to write the canonical momenta as

$$p_n(t) = m\sum_k \left(-i\omega_k^{(\infty)}\right)\left(b_k e^{-i\omega_k^{(\infty)}t}u_n^k - b_k^* e^{i\omega_k^{(\infty)}t}u_n^{k*}\right)$$

that leads to

$$H = \sum_k 2m\left(\omega_k^{(\infty)}\right)^2 b_k^* b_k .$$

Such as above the transition from classical to quantum approach is obtained by replacing the position q_n with the linear operator q_n and the momentum with the operator p_n; consequently

$$\hat{H}^{(\infty)} = \sum_{n=1}^{N}\frac{p^2}{2m} + \sum_{n=1}^{N} k\left(q_{n+1} - q_n\right)^2 = \\ = \sum_k \hbar\omega_k^{(\infty)}\left(d_k^{(\infty)+}d^{(\infty)} + \frac{1}{2}\right)$$

where

$$d_k = \sqrt{\frac{2m\omega_k^{(\infty)}}{\hbar}}b_k$$

with

$$b_k(t) = \frac{1}{2}\sum_{n=1}^{N} u_n^{k*}\left[q_n(t) + \frac{i}{m\omega_k^{(\infty)}}p_n(t)\right],$$

$$b_k^+(t) = \frac{1}{2}\sum_{n=1}^{N} u_n^{k*}\left[q_n(t) - \frac{i}{m\omega_k^{(\infty)}}p_n(t)\right],$$

The previous considerations for a single quantum oscillator on a Cantorian space are also valid for the chain presented here in term of normal operators *d*.

Remark 5. *The creation and annihilation operators, in this case, creates and destroy holes, whose behaviour is comparable with quantum harmonic oscillators. In some sense instead of consider the motion of a system in quantum mechanics, we have also another chance, that is, we can consider the classical motion of a system but on a Cantorian support, that shows the behaviour of a harmonic quantum chain.*

Remark 6. *By considering Dirac Hole Theory [19], in our case the paradigm particle-antipartcle becomes matter-antimmater, where antimmatter means voids. In this vision we could write the Dirac Equation but not only for charged particles, but also for gravitating ones.*

6 Conclusion

In this paper we have studied the effect of a stochastic self-similar and fractal support on some physical quantities and relations. In particular, we have found an algebraic uniterative relation to find the extremes of a Cantor segmentation at any level of fragmentation. In our paper the knowledge of these points is useful to fix the initial conditions of the motion on a fractal support.

- A Cantorian space could explain some relevant stochastic and quantum processes, if the space acts as a harmonic oscillating support, such as it happens in Nature (see [20]).
- An apparent uncertainty, linked with a fractal support rather than a continuous one, can produce an uncertainty on the motion of a physical object, which is explained thanks to stochastic or quantum process. This means that a quantum process, in some cases, could be explained as a classical one, but on a non continuous and fractal support. Consequently, an external observer looking at the motion of a particle under a fixed solicitation can measure an unusual behaviour with respect to a continuous support, that is obvious with respect to the knowledge of the fractal support behaviour. In this case, he can make the hypothesis of an uncertainty or a stochasticity in the process (motion), while there is just an ignorance with respect to the support on which the motion happens.
- Heisenberg's uncertainty principle can be translated from the processes and systems to the support, where we have classical dynamics. We considered the validity of this point of view, that in principle could be more realistic, since it describes the real nature of the matter and space, which does not only exist in Euclidean space or curved one, but also in a Cantorian one. To do this we introduced the creation and annihilation operators, that here, create and destroy holes, whose behaviour is comparable with quantum harmonic oscillators.
- We have showed that an alternative to classical and quantum path is the classical possible path. We call it the classical possible path, since we consider the motion of a massive point under the effect of a force field by using classical mechanics, but we also consider a non continuous path, that is a support with nonmaterialistic forbidden voids. In this way we can introduce fluctuations and stochastic effects in the motion of a classical massive point without invoking quantum effects, but considering a more realistic fractal support. To be more precise, the support could be a material one, but it could also be a vacuum with fixed energetic selection rules; this means that the particle follows the path with the minimum energy to spend on favorable energetic line, such as in the waveguide theory of Einstein and Bohm. Consequently, E-Infinity appears as an energetic net in which we observe the evolution of dynamical systems. Fig.1 shows a possible energetic path net instead of a classical continuum. In other words, as it can be seen, there are some privileged paths, corresponding to different energies (i.e. colours).
- The present Universe can be seen as a complex oscillating system based on a collection of about 10^{80} nucleon oscillators. The cosmological consequences of the previous model for describing the Universe are interesting. For example R.Durrer and J.Laukenmann in [21] showed how an oscillating Universe can be an alternative to inflation. Moreover oscillatory universe solves the flatness or entropy problem. Fig.2 shows a

Universe based on waveguide channels (for more details see [18]).

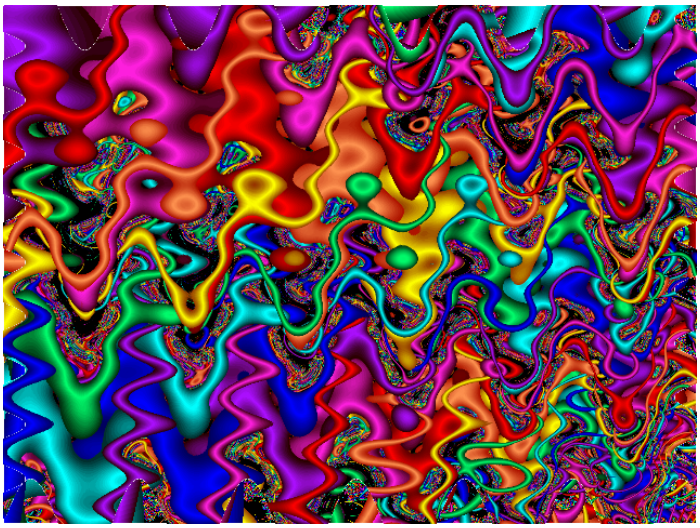

Figure 1*: Possible paths in a waveguides planar energetic scenario.*

Figure 2*: Fractal waveguide universe.*

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