On Synthesis of Asymptotic Filter Banks Based on a Generalization of the Tellegen's Principle

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Abstract: - This paper deals with structural properties of a class of asymptotic filters from the filter banks design point of view. It is shown that both, the continuous- and discrete-time lattice filter structures can be derived as a natural consequence of strict causality, minimality and asymptotic stability requirements. An abstract form of classic Tellegen's relation is introduced and used as a basic design tool expressing the signal energy conservation law for filter state space representations. It is demonstrated that in discrete-time version the resulting asymptotic filter bank structure contains the well known direct FIR filter structure as special case.

Key-Words: - Asymptotic filter, continuous filter, digital filter, energy function, filter bank, Lattice structure, Tellegen's theorem, wavelets filters.

1 Introduction

Many various structures for implementing filters have been developed. There are basically two, both intrinsically linear, but fundamentally different approaches to the signal filtering. The first one leads to frequency filters. In few last decades the state space filter representations have played the growing role, mainly in context of the stochastic optimal Wiener-Kalman-Bucy filtering. It has been shown in [3] that both the stochastic and frequency filtering techniques have some common roots and can be exposed as special cases of the so called asymptotic filtering design philosophy. In this contribution some fundamental properties and structural features of both the continuous-time and discrete-time asymptotic filter state space representations are studied and used for filter banks design. The goal of the contribution is to show that asymptotic filter banks based on generalized Tellegen's principle [1] can be derived by a straightforward way from natural requirements of strict causality, energy conservation, asymptotic stability and state minimality.

2 Continuous and discrete-time generalization of Tellegen's principle

The new concept of asymptotic filtering has been introduced in [3], [4], [5] and [6]. Certainly, any realizable filter has to fulfil some causality and energy conservation requirements. The Tellegen's theorem is known to be one of the most powerful tools *of system analysis and synthesis* in electrical network theory. It asserts that *Kirchhoff's laws are sufficient for energy conservation* in an electrical network. Let us briefly summarize the *essential features* of the *original version* of Tellegen's theorem [12]. Assume that an *arbitrary connected electrical network* of b components is given. Let us *disregard* the specific nature of the *network components* and represent the *network structure* by an *oriented graph* with n *vertices and* b *branches*. Let the *set of Kirchhoff law constraints* be given in a form

$$Ai = 0 \qquad Bv = 0 \tag{1}$$

where A is a *node incidence matrix*, B is *loop incidence matrix*, and vectors i and v are defined

$$i = [i_1, i_2, \dots, i_n]^T \quad v = [v_1, v_2, \dots v_n]^T \tag{2}$$

Let J be the set of all vectors i and V be the set of all vectors v such that i and v satisfy (1). Both the vectors of currents and voltages are elements of a b-dimensional *vector space with the inner product.* Then the Tellegen's principle follows from:

Theorem 1. (Classical Tellegen's theorem - CTT)
If $i \in J$ and $v \in V$ then it holds

$$\forall t: \quad \langle i(t), v(t) \rangle = 0 \tag{3}$$

That is to say J and V are *orthogonal subspaces* of the Euclidean space E_b. Furthermore J and V together *span* E_b. Unfortunately, since *digital filter networks are not subject to Kirchhoff's laws, Tellegen's theorem in its original form does not apply in digital signal processing.* It has motivated further research work with the goal to modify *it for discrete-time systems* [5], [14].

Theorem 2. (Tellegen's theorem in difference form)
Consider *two signal-flow graphs* with *the same topology*. Let N denote the number of network nodes.

The network node variables, branch outputs and source node values in the first network are denoted by w_k, v_{jk}, and x_j, respectively and in the second network by w'_k, v'_{jk} and x'_j. Then [2]

$$\sum_{k=1}^{N}\sum_{j=1}^{N}(w_k v'_{jk} - w'_k v_{jk}) + \sum_{k=1}^{N}(w_k x'_k - w'_k x_k) = 0 \qquad (4)$$

It is obvious fact, following directly from the *definition of inner product*, that relation (10) *is just a form of constant energy statement* for a class of representations in which *elements* of a *set* of *voltages and currents* have been *chosen* as *state variables*, as well as components of a *gradient vector of a scalar field in the state space.* It is of crucial importance to realize that voltages $v_1, v_2,\ldots, v_b$ and currents $i_1, i_2,\ldots, i_b$ are picked *arbitrarily* subject only the Kirchhoff current and voltage law constraints. The *arbitrariness* motivates introducing a *group of state- and feedback-transformations* on which the proposed *generalization of classical Tellegen's principle* has been issued in [12].

$$\begin{aligned} &\exists\varphi,\ \exists T,\ T^{-1}:\ \bar{x} = T(x),\ \ \bar{u} = \varphi(u,\bar{x}): \\ &\left\langle f,(grad\ \ E)^T \right\rangle = 0 \Leftrightarrow \left\langle \dot{\bar{x}},\bar{x} \right\rangle = 0 \\ &\Leftrightarrow \forall t:\ \bar{E}[\bar{x}(t)] = E[x(t)] \end{aligned} \qquad (5)$$

Let's now consider a class of discrete-time finite dimensional *internal system representations*

$$\begin{aligned} &x(k+1) = f\left[x(k)\right] + w(k), \\ &w(k) = Bu(k),\ \ y(k) = Cx(k) \end{aligned} \qquad (6)$$

induced by an *external digital filter description*. Similarly as in the case of continuous-time systems, a new *discrete-time generalization of Tellegen's principle* has been formulated. If any input *u(k)* and any state value *x(k)* will be chosen then the next state value *x(k+1)* is given, and the *state difference vector* $\Delta x(k)$ can be defined as

$$\Delta x(k) = x(k+1) - x(k) \equiv \Delta x_k,\ \ k \in \{0,1,2,..\} \qquad (7)$$

together with a row *vector η(k)* defined by:

$$\eta(k) = \frac{1}{2}[x(k+1)+x(k)]^T \equiv \eta_k,\ k \in \{\ 0,1,2,\ldots\ \} \qquad (8)$$

Interpretation of the vector η_k as a natural *discrete-time energy function gradient vector* is obvious, and the *discrete-time generalization of Tellegen's principle* is then given by the *inner product*:

$$\begin{aligned} &\Delta t_k \equiv t_{k+1} - t_k:\ \ \forall \Delta t_{k,} \\ &\forall t \equiv k, k \in \{0,1,2,\ldots\}: \left\langle \Delta x_k\ ,\ \ \eta_k{}^T \right\rangle = 0 \\ &\Leftrightarrow E(t)\big|_{t=k} = E(t)\big|_{t=0} = \text{const.} \end{aligned} \qquad (9)$$

For deeper understanding the *geometric interpretation* of both the continuous and discrete-time versions of the generalized Tellegen's principle is visualized at the Fig.1.

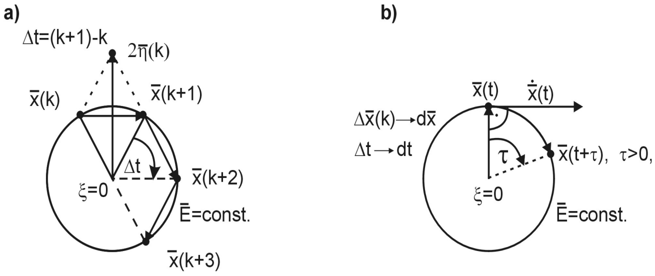

Fig.1. Geometric interpretation of a) discrete-time b) continuous-time generalized Tellegen's principle (for *n*=2)

3 Lattice-ladder structure of continuous-time asymptotic filters

In fact, it follows from the previous analysis that *it is not the physical energy by itself*, but only a *measure of distance from the system equilibrium to the actual state x(t)*, what is needed. Thus, instead of the physical energy a *metric* $\rho[\mathrm{x(t)}, \mathrm{x}^*]$ will be defined in a proper way, and for an abstract energy E(x) we then put formally:

$$\mathrm{E(x)} \triangleq \frac{1}{2}\rho^2\left[\mathrm{x(t)},\ \mathrm{x}^*\right] = \frac{1}{2}\| x(t) - x^* \|^2 \qquad (10)$$

We start with a *natural assumption that every real signal must be generated by a realizable system*. Let such a system, called *signal generating system* (SGS), be given in the form:

$$\begin{aligned} &\Re\{S\}: \dot{x}(t) = A\cdot x(t) + B\cdot u(t),\ x(t_0) = x^0, \\ &y(t) = C\cdot x(t), \end{aligned} \qquad (11)$$

where the *matrices A, B, C* are assumed to be *known*, the *input and output signals* are supposed to be *measured* on some given *observation time interval*, (perhaps *with some uncertainty*), and the *initial state* is assumed to be *completely unknown*. Notice that the *state space representation* $\Re\{S\}$ under consideration has *the strict causality property*. The resulting state equivalent asymptotic filter representation is then specified by the triplex ($\widetilde{A}, \widetilde{B}, \widetilde{C}$) as follows:

$$\tilde{A} = \begin{pmatrix} -\alpha_1, & \alpha_2, & 0, & 0, & & \cdots, & 0, & 0 \\ -\alpha_2, & 0, & \alpha_3, & 0, & & \cdots, & 0, & 0 \\ 0, & -\alpha_3, & 0, & \alpha_4 & & \cdots, & 0, & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots, & \vdots & \vdots \\ 0, & 0, & 0, & 0, & \cdots & -\alpha_{n-1}, & 0, & \alpha_n \\ 0, & 0, & 0, & 0, & & \cdots 0, & -\alpha_n & 0 \end{pmatrix}$$

$$\tilde{C}^T = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\quad \tilde{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \qquad (12)$$

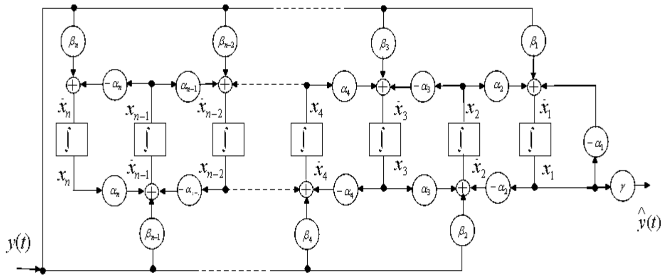

Fig. 2. Lattice structure of continuous-time asymptotic filter in the dissipation normal form

It is easy to show that the set of *real basic design parameters* α_i, γ, β_i must satisfy the following fundamental *consistency conditions:*

1. $\forall i,\quad i \in \{1,2,...,n\}: 0 < \alpha_i < \infty \Leftrightarrow$

structural asymptotic stability of the asymptotic filter

2. $\forall i, i \in \{2,3,...,n\}: 0 \neq \alpha_i, \gamma \neq 0, \exists i: \beta_i \neq 0 \Leftrightarrow$

structural minimality of the asymptotic filter

The block diagram of continuous asymptotic filter is shown in Fig. 2, [11].

4 Lattice-ladder structure of discrete-time asymptotic filters

In discrete-time case we proceed conceptually *by exactly the same way* as before. The *signal generating system* (SGS) is now represented by:

$$\Re\{S\}: x(k+1) = A \cdot x(k) + B \cdot u(k), \qquad (13)$$
$$x(k_0) = x^0, \quad y(k) = C \cdot x(k),$$

The resulting state equivalent asymptotic filter is specified by the triplex ($\widetilde{A}, \widetilde{B}, \widetilde{C}$) as follows:

$$\tilde{A} = \begin{bmatrix} -\Delta_{n-1}\cdot\Delta_n & \delta_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ -\Delta_{n-2}\cdot\delta_{n-1}\cdot\Delta_n & -\Delta_{n-2}\cdot\Delta_{n-1} & \delta_{n-2} & \vdots & \vdots & \vdots & \vdots \\ -\Delta_{n-3}\cdot\delta_{n-2}\cdot\delta_{n-4}\cdot\Delta_3 & \vdots & \vdots & \ddots & \delta_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & -\Delta_2\cdot\Delta_3 & \delta_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & -\Delta_1\cdot\delta_2\cdot\Delta_3 & -\Delta_1\Delta_2 & \delta_1 \\ \delta_1\cdot\delta_2\cdots\delta_{n-1}\cdot\Delta_n & \cdots & \cdots & \cdots & \delta_1\cdot\delta_2\cdot\Delta_3 & \delta_1\cdot\Delta_2 & \Delta_1 \end{bmatrix},$$

$$\tilde{C}^T = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \qquad (14)$$

It is easy to show that the set of *real basic* (direct) *design parameters* δ_i and the set of *real complementary* (feed-back) *parameters* Δ_i must satisfy the following *consistency conditions*:

$$0 < \delta_i \leq 1, \delta_i^2 + \Delta_i^2 = 1,\ i \in \{1,2,...,n\},\ \delta_n = \gamma, \qquad (15)$$

having important *consequences*:

1. $\forall i,\quad i \in \{1,2,...,n\}: \ |\Delta_i| < 1 \quad \Leftrightarrow$ *structural asymptotic stability of IIR asymptotic filter*

2. $\forall i: 0 < \delta_i \leq 1,\ \gamma \neq 0,\ \beta_n \neq 0 \quad \Leftrightarrow$ *structural minimality* (observability & controllability) *of the asymptotic IIR for* $0 < \delta_i < 1 \Leftrightarrow (\Delta_i \neq 0)$, and of the *standard FIR filters for* $\delta_i = 1 \Leftrightarrow (\Delta_i = 0)$

The derived *lattice structure of the discrete-time asymptotic filter in dissipation normal form* corresponding to the Eqns. (14) is shown at the Fig. 3. It can be seen that it is exactly the same as the well known *dual lattice realization* [8] *of standard IIR digital filters*. An important *special case* arises if we set $\delta_i = 1$ then all the *complementary parameters* Δ_i *vanish* and the *structure of asymptotic filter reduces* to the *standard dual transversal FIR filter structure* [14].

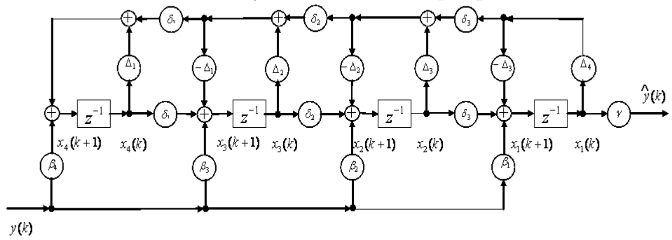

Fig. 3. Lattice structure of the 4^{th} order discrete-time asymptotic filter in dissipation normal form

5 Experimental results

The *physical interpretation of the internal interaction matrix A* and the *observation matrix C* can be useful, as the decompositions visualized at the Fig. 4. shows.

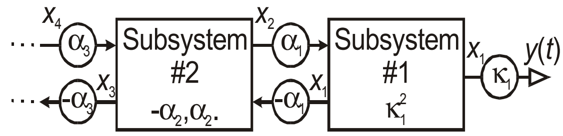

Fig. 4. Physical interpretation of the system interactions.

On the chain structure different classes of physically realizable filters, called asymptotic filters are based. The recursively given normalized optimal analog filter transfer function has form:

$$F(s) = \frac{1}{P_n(s)}, \text{ where } P_0(s) = 1,\ P_1(s) = s + \omega_0, \qquad (16)$$
$$P_k(s) = sP_{k-1}(s) + \omega_0^2 P_{k-2}(s),\ for\ k \in \{2,...,n\}$$

Note that the positive design parameter ω_0 has the meaning of time scale transformation and can be used to adjust the required bandwidth of the filter. The integer *n* has been defined as the order of filter. When this filter is build as passive LC analog filter, in the low-pass prototype, all inductors have the same value, and all capacitors have the same value, see Fig. 5. This minimum number of circuits elements provides design simplicity and reduces cost. Fig. 5 shows the physical structure of a eight-pole low-pass filter prototype.

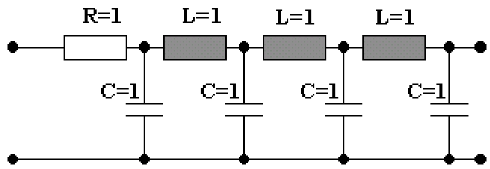

Fig. 5. Example of equal-element 8 pole low-pass asymptotic filter

Fig. 6. Frequency response of IIR filter realized by bilinear transformation method.

5.1 Wavelet construction principles

In practical applications of the wavelet transform we only use the expansion coefficients of the signals and thus made the discrete wavelet transform (DWT). The scaling function and the wavelets themselves are not needed. There is also the fact that in most cases we don't start from given scaling functions and wavelets and determine the filter coefficients $h_0(n)$ and $h_1(n)$ from there. More often we start with suitable set of coefficients $h_0(n)$ and $h_1(n)$ and use them to calculate the DWT. The coefficient sets or filter impulse responses $h_0(n)$ and $h_1(n)$ must fulfill the following conditions to be the expansion coefficients of scaling functions and wavelets [7], [8], [9], [10]:

- The filters $h_0(n)$ and $h_1(n)$ must set up a filter bank with perfect reconstruction and unambiguous projection.
- The scaling coefficients $h_0(n)$ must fulfil the scaling condition:

$$\sum_{n=0}^{N} h_0(n) = \sqrt{2} \qquad (17)$$

- The transfer function $H_0(z)$ must be regular.

In practical wavelets construction the low-pass impulse response is cut off at sufficient value of N, in order to be able to derive an inverted-time high-pass filter impulse response $h_1(n)$:

$$h_1(n) = (-1)^{(N-1-n)} h_0(N-1-n) \qquad (18)$$

From low-pass and high-pass impulse response the scaling and wavelets coefficients can be computed.

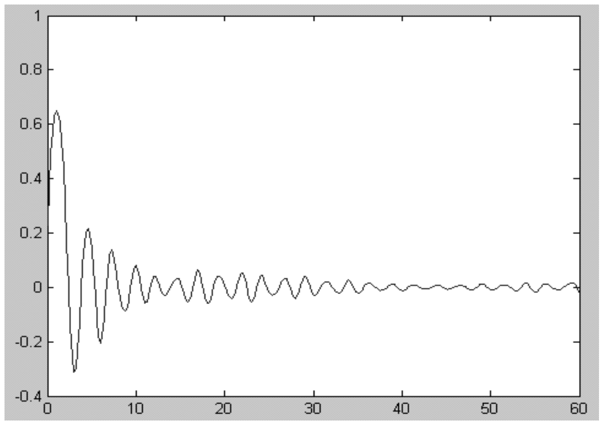

Fig. 7. Scaling function (Left). The 7-th order low-pass asymptotic filter impulse response, cut off at N=60.

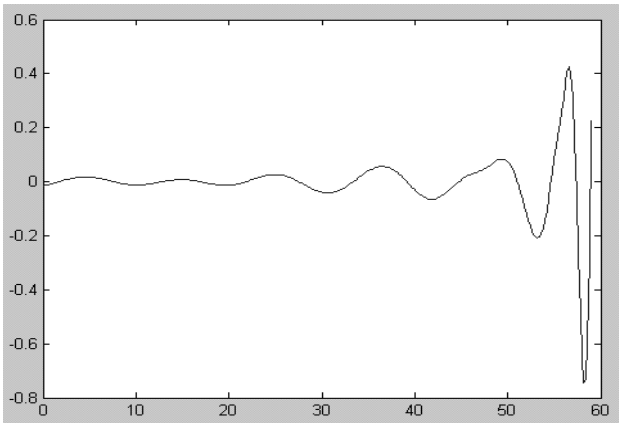

Fig. 8. Wavelet function. The high-pass impulse response derived from low-pass impulse response, using Eqn. (18).

5.2 Asymptotic filters discretization

In this part example of discretization technique (bilinear transformation) is presented [11]. The following example of 4-th order continuous-time asymptotic low-pass filter given by the Eqn. (16) for $n=4$ and $\omega_0=1$ has been considered. The prototype continuous filter transfer function is given by (19):

$$F(s) = 1/(1 + 2s + 3s^2 + s^3 + s^4) \qquad (19)$$

The bilinear transformation from the s-plane to z-plane is known to be given by:

$$s = (2/T_S).(1 - z^{-1})/(1 + z^{-1}) \qquad (20)$$

and we get:

$$H_0(z) = \frac{0.012(1 + 2z^{-1} + z^{-2})^2}{(1 - 1.36z^{-1} + 0.548z^{-2})(1 - 0.86z^{-1} + 0.886z^{-2})} \qquad (21)$$

The frequency response of IIR filter realized by bilinear transformation method is shown in Fig. 6.

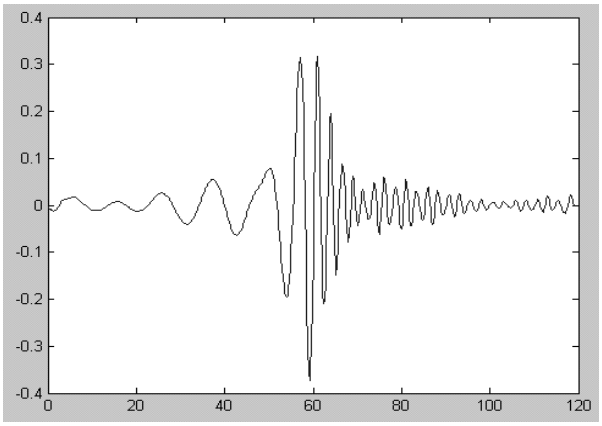

Fig. 9. The impulse response of the wavelet band-pass filter derived from low-pass and high-pass impulse response of 7-th order asymptotic filter.

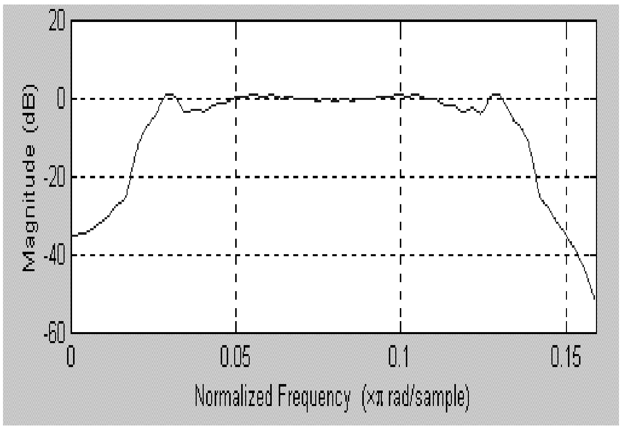

Fig. 10. Frequency spectrum of the resulting band-pass wavelet

5.3 Example of wavelet construction

Example of wavelet construction, based on 7-th order asymptotic filter is described [11]. The low-pass impulse response $h_0(n)$, computed from (21), (Fig. 7), was cut off at N = 60, in order to able to derive an inverted time high-pass filter by Eqn. (18). The high-pass impulse response $h_1(n)$ is shown in Fig. 8. Impulse response, computed from low-pass and high-pass impulse responses is shown in Fig. 9, corresponding spectrum of the finite impulse response band-pass filter is shown in Fig. 10.

5.4 Two channel filter bank design

A 2-channel filter bank decomposes a signal into two frequency bands enabling to process each signal separately. This decomposition is useful in the areas of image processing, speech coding and also in adaptive filtering. The block diagram of 2-channel filter bank, also called as a quadrature mirror filter bank (QMF) is shown in Fig. 11. The input signal $x(n)$ is decomposed in two frequency bands by means of analysis bank $H_0(e^{j\omega})$ (low-pass filter, order N-1) and $H_1(e^{j\omega})$ (complementary high-pass, overlapping filter). The output signals from decomposition filters are decimated. In the synthesis part, the signals are interpolated, filtered by the synthesis filters $G_0(e^{j\omega})$ and $G_1(e^{j\omega})$, and recombined to gain a reconstructed signal. The overlapping feature of the analysis filters enable to use low-order filters at the expense of introducing aliasing. It was shown in [9] that aliasing can be cancelled by proper design of the synthesis filters. The analog filter bank frequency response is shown in Fig. 12, digital filter bank impulse responses in Fig. 13 and digital filter bank frequency responses in Fig. 14, [13].

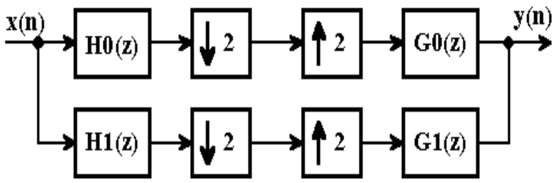

Figure 11. Two-Channel filter bank block diagram.

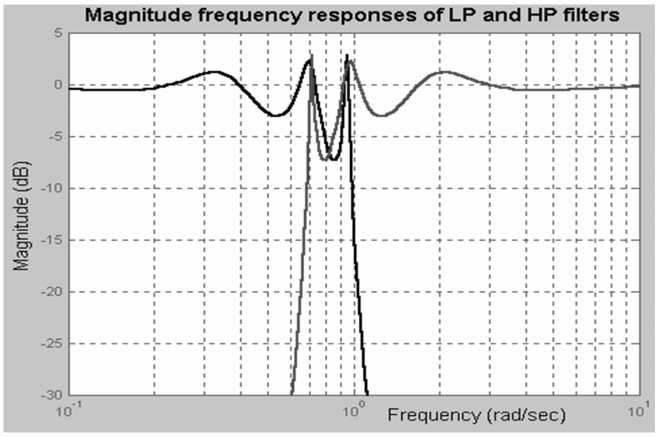

Fig. 12. Analog filter bank frequency response

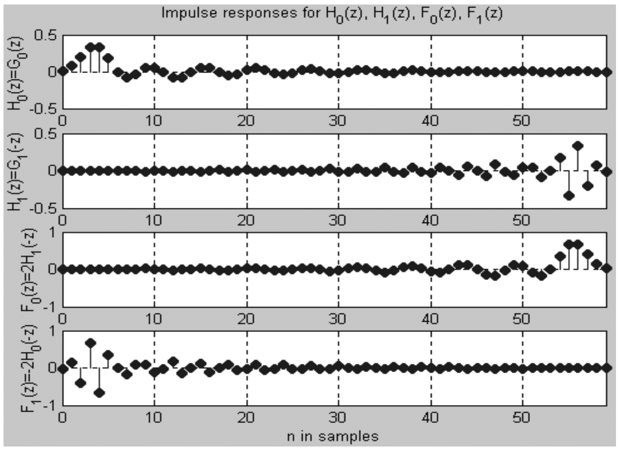

Fig. 13. Digital 2-channel filter bank impulse responses of $H_0(e^{j\omega})$, $H_1(e^{j\omega})$, $G_0(e^{j\omega})$ and $G_1(e^{j\omega})$

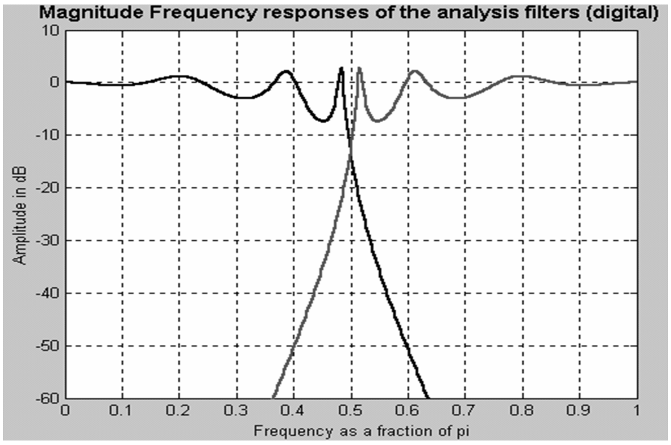

Fig. 14. Digital filter bank frequency response, low-pass and high-pass filters.

6 Conclusion

In the contribution a technique of continuous asymptotic filters and filter bank construction using a special class of IIR filters called asymptotic filters has been proposed. Only continuous-time asymptotic filters with minimal number of natural design parameters, (two-dimensional parameter space for any finite filter order *n*), resulting from the filtering error signal energy minimization have been considered [4], [5], [13]. Simulation experiments confirm the expectation that even for low number of filter parameters the excellent convergence properties of asymptotic filters will give good approximations with the impulse response cut down to the finite length (of reasonable value). The filter banks are described more detailed in [15].

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