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# Synthesis of Σ-Δ Modulators with Elliptic Noise-Shaping Functions

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*Abstract:* - This paper presents a method for synthesizing sigma-delta (Σ-Δ) modulators. The synthesis method is derived from a new noise-shaping equation which interprets the modulator as a device performing frequency-weighted minimization of the modulation error. In the equation, a noise-shaping function which spectrally shapes the modulation error is chosen to be an elliptic filter. The resulting elliptic noise-shaping function, featuring a high attenuation of in-band noise, satisfies the modulator's stability requirement. A four-bit 6MHz wideband modulator and a single-bit 10.7MHz bandpass modulator are designed to demonstrate the superiority of this simple synthesis method.

*Key-Words:* - Sigma-delta modulation, noise shaping, frequency-weighted minimization, elliptic filter, feedback coding, over-sampling A/D converter.

## 1 Introduction

Sigma-delta modulation uses negative feedback to quantize a signal in attempting to achieve higher resolution within the signal bandwidth [1]. However, the nonlinear quantizer in a feedback loop makes the dynamics extremely complex and difficult to analyze. Despite the lack of an exact theory to stabilize the modulator, engineers still employ without hesitation the Σ-Δ modulation technique in various applications, such as data converters [2], class-D amplifiers [3], digital microphones [4] and many more.

A well-known rule of thumb for designing a stable modulator was introduced by Lee [5], which constrained the out- of-band gain of the noise transfer function to prevent the accumulation of quantization error. Also, based on Lee's concept, Schreier introduced a method for designing a noise transfer function and wrote a popular MATLAB function "synthesizeNTF" [6]. Another popular method introduced by Engelen and Plassche [7] was based on the describing function method, which modeled the quantizer as a single gain with phase uncertainty and synthesized the modulator using the root locus method. Although these methods or rules of thumb provide insights into the modulator design, but they still lack an exact theory to support and verify the designs.

In a new book, *Unsolved Problems in Mathematical Systems and Control Theory*, sigma-delta modulator synthesis is included as one of the open problems [8]. The question of how to synthesize a high-order high-resolution stable modulator still baffles most of the practicing engineers and researchers who design or study sigma-delta modulation. However, some progress has been reported in author's recent papers [9, 10], which present a synthesis method for a single-bit Σ-Δ modulator based on the theory of sliding modes. From this sliding mode aspect, the modulator can be thought of as a device performing constrained frequency-weighted minimization of the modulation error. This constrained minimization interpretation not only advances the understanding of sigma-delta modulation, but also helps develop new types of feedback quantization schemes, see [11, 12].

This paper attempts to provide a modulator synthesis method which is generally applicable to both single-bit and multi-bit cases. First, Section 2, following the work of [11, 12], presents a constructive derivation of sigma-delta modulation and its noise-shaping equation. Section 3 gives the stability conditions for the modulator. Finally, based on the noise-shaping equation and the stability condition, Sec. 4 provides a method for designing a high-performance modulator.

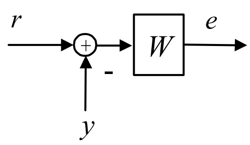


Fig. 1. Constrained frequency-weighted minimization of the modulation error

## 2 A Pathway to Σ-Δ Modulation

Taking the viewpoint of [10, 11, 12], this section provides a constructive derivation of sigma-delta modulation. The derivation helps us to develop a new modulator synthesis method. The derivation begins with the formulation of a noise-shaping modulation problem as a constrained frequency-weighted minimization problem shown in Fig. 1. Given a filter $W$ and an input signal $r$, $y$ is selected from a finite set of quantization levels at each sampling time, to minimize the output level of a linear filter $W$.

The idea behind this minimization is simple. Filter $W$ plays a role of a frequency-weighted function, specifying the frequency range over which the minimization is emphasized. The filter output $e$ in Fig. 1 is a frequency-weighted modulation error, which can be written in the following $z$-domain representation,

$$E(z) = W(z)[R(z) - Y(z)], \tag{1}$$

where $E$, $R$, and $Y$ are the $z$-transforms of $e$, $r$, and $y$, respectively. The magnitude minimization of $E$ will be more emphasized in the frequency band where $W$ has larger magnitude. If the output level of $W$ to $r$-$y$ was made small, then most of the frequency contents of the modulation error would be in the stopband of $W$. Equation (1) can be alternatively written as,

$$Y(z) = R(z) - W^{-1}(z)E(z). \tag{2}$$

Equation (2) is called a noise-shaping equation, see also [10, 12]. The quantized signal $Y$ contains the desired signal $R$ and unwanted noise $W^{-1}E$. Transfer function $W^{-1}$, which spectrally shapes the residual minimization error $E$, is referred to as a noise-shaping function in this paper.

Assume that $W(z)$ is an $n$th-order sampled-data filter of relative degree zero, having no common factor between its numerator and denominator. Without loss of generality, the gain of $W$ is assumed to be normalized such that $W(\infty) = 1$ (see Remark 5 in Sec. 3). In this case, $W$ has the state-space representation [$A$, $B$, $C$, $D$] with $D$=1 (i.e., $C(zI - A)^{-1}B + 1 = W(z)$). The output $e$ of $W$ can be described by,

$$\begin{cases} x(k+1) = Ax(k) + B\big[r(k) - y(k)\big] \\ e(k) = Cx(k) + r(k) - y(k) \end{cases} \tag{4}$$

Taking an $N$-bit normalized uniform quantization with quantization step $\Delta = 2/(2^N - 1)$ gives a candidate set of quantization levels, **S**={-1, $-1+\Delta$, $-1+2\Delta$, …, $1-\Delta$, 1}. A simple quantization scheme is to choose $y(k)$ from set **S** to minimize the level of $e(k)$ in the second equation of (4) at each sampling time, yielding the following optimal solution (the horizon-one optimal solution of [11]),

$$y(k) = Q\big(u(k)\big) \tag{5}$$

with

$$u(k) = Cx(k) + r(k) \tag{6}$$

where $Q(u(k))$ quantizes $u(k)$ to the nearest value in **S**. The resulting block diagram is shown in Fig. 2. Notably, the resulting feedback modulator is exactly a sigma-delta modulator, performing noise shaping via feedback connection of a linear filter with a quantizer. It is interesting to note that the modulator in Fig. 2 is exactly a generalization of the topology proposed by Silva et al. [13], which excels in the low-distortion property.

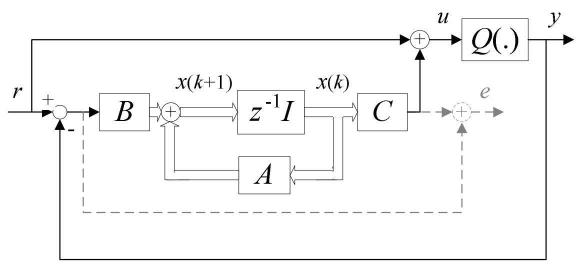


Fig. 2. Optimal modulation scheme that minimizes $|e(k)|$.

## 3 Stability Analysis

This section analyzes the stability of the sigma-delta modulator of Fig. 2 by treating the modulator output $y$ as a quantized state feedback control. From (4)-(6), the dynamics of the modulator can be rewritten as,

$$x(k+1) = (A - BC)x(k) + Be(k), \tag{7}$$

where signal $e$, the output of filter $W$, is also the quantization error,

$$e(k) = u(k) - Q(u(k)). \tag{8}$$

The governing equation (7) is a nominal system subject to a perturbation caused by the quantization error. The nominal system in (7) is stable if matrix $A$-$BC$ is Hurwitz. Quantizer input $u$ can be represented in the z-domain as,

$$\begin{aligned} U(z) &= R(z) + C[zI - (A - BC)]^{-1}BE(z) \\ &= R(z) + [1 - W^{-1}(z)]E(z) \end{aligned} \tag{9}$$

As a result, the block diagram of the modulator can be equivalently redrawn in an error feedback configuration displayed in Fig. 3. Note that, since $W(\infty)=1$ , loop filter $1\text{-}W^{-1}$ is strictly proper, meaning that the error feedback topology in Fig. 3 is realizable. The poles of $1\text{-}W^{-1}$, namely the zeros of $W$, are the characteristic roots of the nominal system. Designing a stable nominal system plus the prevention of quantizer overload suffices to guarantee the stability of the modulator.

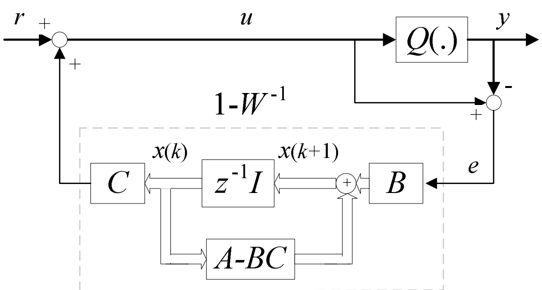


Fig. 3. An equivalent error-feedback topology.

**Proposition 1** (*Stability in the small*) Let $\|.\|_1$ denote the $\boldsymbol{l}_1$ norm of a transfer function. For the modulator of Fig. 2 or 3 with zero initial condition, if the following conditions are satisfied,

(i) Matrix $A\text{-}BC$ is Hurwitz,
(ii) $|r(k)| \le 1-\frac{\Delta}{2}\left(\| 1\text{-}W^{-1} \|_1 -1\right) \ \ \forall k$ ,

then the modulator is stable and the quantization error $e$ is bounded by,

$$|e| \le \Delta/2 . \tag{10}$$

**Proof.** The zero initial condition $x(0)=0$ and the input level constraint (ii) ascertain that the quantizer input level is initially within the quantization range $|u| \le 1+\Delta/2$ and accordingly $|e| \le \Delta/2$ . Also, condition (i) guarantees the stability of $1\text{-}W^{-1}$, and thus according to the linear system theory,

$$|u(k)| \le \|r\|_\infty + \|1-W^{-1}\|_1 \|e\|_\infty , \tag{11}$$

where $\|r\|_\infty$ denotes the peak absolute value of signal $r$. Inequality (11) together with input level constraint (ii) and $|e| \le \Delta/2$ prevents the quantizer input level from exceeding the quantization range and thereby $|e| \le \Delta/2$ , which once again making $|u| \le 1+\Delta/2$ . Therefore, $u$ and $e$ remain bounded all the time, and so are the state variables $x$. ■

Some remarks on Proposition 1 are given below.

**Remark 1.** Requiring the modulator to have zero initial state is not always necessary; however, a nonzero initial state may reduce the allowed modulator input level, or even worse to cause instability. In practice, a zero initial state can be achieved by resetting all integrators of the modulator's loop filter when powered on.

**Remark 2.** According to condition (i) in Proposition 1, the poles of the noise-shaping function $W^{-1}$ must be stable, but nothing about its zeros is mentioned. In fact, the zeros of $W^{-1}$ affect the stability margin as well as the performance. Moving the zeros outside the unit circle will help reduce the harmonically-related tones at the cost of a lower stability margin (i.e., lower allowable input level) and higher noise floor [14, 15]. Placing the zeros optimally spread along the unit circle within the signal bandwidth theoretically achieves the highest signal-to-noise ratio (SNR) [16]. On the other hand, moving the zeros inside the unit circle will trade noise and tones for a wider input range; when all zeros are very close to the origin, the modulators may even become globally stable at the sacrifice of its noise-shaping ability.

**Remark 3.** Constraint (ii) in Proposition 1 specifies the maximum allowable input level of the modulator. However, experience shows that the constraint (ii) is over-restrictive; its estimate of the stable input level is often too conservative to have any practical use. Today, a reliable estimate of stable input level must still rely on the numerical simulation.

**Remark 4.** The input level constraint (ii) implies that employing more quantization levels and smaller quantization step $\Delta$ will increase the allowed input level, and thereby allow the noise-shaping function to have more dramatic attenuation in the wider frequency band.

**Remark 5.** The gain of $W$ does not affect the stability and performance of the modulator. This fact can be shown by substituting $kW(z)$ for $W(z)$ in the derivation of Sec. 2. This substitution yields an identical feedback modulation with the same output $y$. Also, the magnitude of $E$ will change in the same proportion of the change of the gain of $W$, thus making the noise term in (2) unchanged.

# 4 Modulator Synthesis

The noise-shaping function $W^{-1}$ is critical to both the stability and performance of the modulator, directly influencing the allowed maximum input level and achievable peak signal-to-noise-plus-distortion ratio

(SNDR) of the modulator. According to the previous analysis, noise-shaping function $W^{-1}$ is chosen to satisfy the following conditions,

- The poles of $W^{-1}$ are placed within the unit disk $|z|<1$ for stability (condition (i) in Proposition 1).
- The zeros of $W^{-1}$ are optimally spread throughout the signal band to achieve maximum in-band noise attenuation.

A simple choice for $W^{-1}$ is an elliptic filter. Since the elliptic filter is stable and has its zeros optimally placed on the unit circle to achieve a steep transition and high stopband attenuation, the above two design requirements are readily met. As shown in Fig. 4, an elliptic filter is specified by its cutoff frequency fc (or two cutoff frequencies for a bandpass or bandstop filter), passband ripple Rp, and stopband attenuation Rs. The cutoff frequency is at the edge of the passband. Between the passband and stopband is the transition band. In order to achieve uniform attenuation within the modulator bandwidth, the cutoff frequency must be chosen large enough so that the filter stopband will cover the entire signal band. For an elliptic filter with larger Rs and smaller Rp, the cutoff frequency should be chosen larger to attain the same stopband, because the transition band will be wider with a larger value of stopband attenuation and a smaller value of passband ripple.

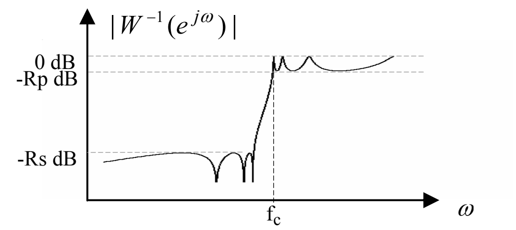


Fig. 4. Characteristic of a highpass elliptic filter.

The following examples provide an illustration of the design method. If the noise-shaping function is designed, the modulator can be simulated or implemented in the topology of Fig. 2 or 3.

**Example 1** (*Wideband 4-Bit Modulator Design*) A 4-bit lowpass sigma-delta modulator that samples at 96 MHz is designed to convert signals of frequencies up to 6 MHz. The noise-shaping function $W^{-1}$ is chosen as a seventh-order highpass elliptic filter of parameters fc=0.2031, Rp=1, and Rs=90. The numerator and denominator of the elliptic filter are obtained by the following MATLAB codes,

```
% Let Num & Den be the coefficient vectors of
% the numerator and denominator of W^-1(z).
 [Num,Den]=ellip(7,Rp,Rs,fc,'high');

% Set the leading coefficient of Num equal to 1.
Num=Num/Num(1);
```

The resulting modulator is simulated with a 100kHz sinewave as a test input. According to simulations, the maximum stable input amplitude is 0.55. Figure 5 displays the output spectrum of the modulator at the input amplitude of 0.55, and also a plot of the magnitude response of the noise-shaping function $W^{-1}$. The output spectrum reports an isolated tone at 100 kHz and the noise spectrum following the shape of the magnitude response of $W^{-1}$. An attenuation of more than 90 dB within the signal bandwidth is achieved, resulting in an SNDR is 106 dB and an effective resolution of 17 bits. The characteristics of the modulator are summarized in Table 1.

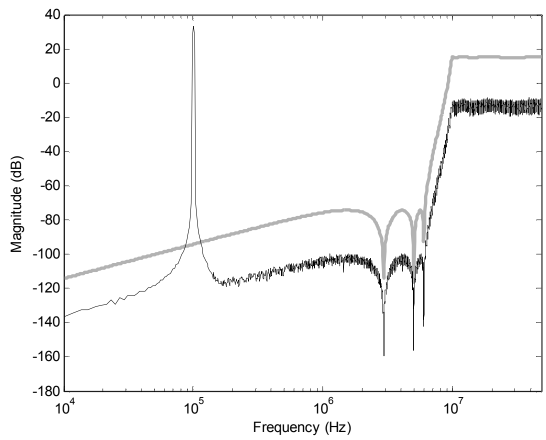


Fig. 5. Output spectrum (black) and $|W^{-1}|$ (gray).

Table 1 Seventh-order lowpass modulator

| | |
|---|---|
| Sampling rate | 96 MHz |
| Bandwidth | 6 MHz |
| Test input frequency | 100 kHz |
| Peak SNDR | 106 dB |
| Maximum stable input level | 0.55 |

**Example 2** (*Bandpass 1-Bit Modulator Design*) This example designs a single-bit bandpass sigma-delta modulator for the conversion of 10.7MHz signals with a 200kHz bandwidth, at a sampling rate of 40 MHz. The noise-shaping function of the modulator is designed to be a sixth-order bandstop elliptic filter with parameters f1=0.475, f2=0.594, Rp=1, and Rs=80, having the numerator and denominator given by,

```
[Num,Den]=ellip(3,Rp,Rs,[f1 f2],'stop');
Num=Num/Num(1);
```

The modulator is simulated with a 10.7MHz sinewave as a test input. Simulations reveal that the modulator achieves a maximum SNDR of 101.7 dB and the maximum stable input amplitude is 0.67. Figure 6 plots the output spectrum of the modulator. Again, the noise spectrum follows the shape of $|W^{-1}|$ as expected. Table 2 lists the characteristics and performance.

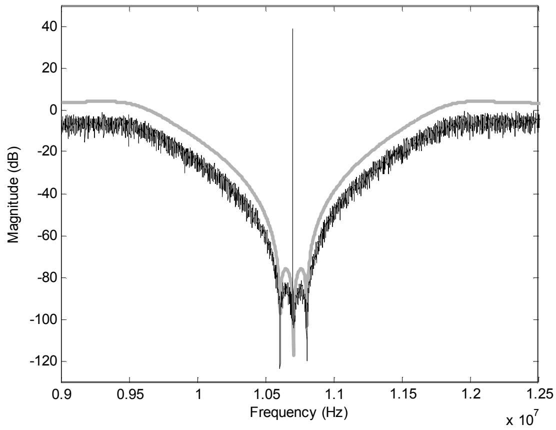


Fig. 6. Output spectrum (black) and $|W^{-1}|$ (gray).

Table 2 Sixth-order bandpass modulator

| | |
|---|---|
| Sampling rate | 40 MHz |
| Center frequency | 10.7 MHz |
| Bandwidth | 200 kHz |
| Peak SNDR | 101.7 dB |
| Maximum stable input level | 0.67 |

## 5 Conclusion

Sigma-delta modulators are synthesized with elliptic noise-shaping functions characterized by high in-band noise attenuation. The synthesis method is generally applicable to lowpass and bandpass sigma-delta modulators of arbitrary order. It is shown that this simple and effective synthesis method together with the exact stability and performance analysis takes the modulator performance toward its limit.

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