A Frequency Estimation Comparison between the Modified Covariance Method and the Fourier Transform

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Abstract: - Fourier methods give excellent frequency estimation performance with either single complex exponentials or long data lengths. For multiple exponentials parametric techniques offer superior resolution. This work examines the signal conditions which determine the choice of a particular method. Equations are developed that describe the Fourier performance with data length, SNR and relative frequencies and amplitudes. Optimal methods are identified for a spectrum of conditions.

Key-Words: - Bias, Cramér-Rao bound, Fourier transform, Frequency estimation, Modified covariance, Variance

1 Introduction

Extension of various frequency estimation methods continues to occupy a large section of the signal processing literature.

Methods based on the Fourier transform are well documented, easy and quick to implement and offer excellent performance in noisy conditions. However, these methods suffer from spectral leakage. This is due to the inherent mismatch between the assumption of infinite time domain extension outside the data section of interest and the real-world necessity of windowing the data, or of analyzing a non-infinite vector size [1,2]. Despite spectral broadening arising from windowing, Fourier techniques are maximum likelihood (ML) for a single complex exponential in noise, or for multiple signals given sufficient data. Each spectral peak reduces in bandwidth with increasing time and the multiple peak problem reduces to multiple single peak equivalents.

However, in practice sufficient data is not available, and the use of sidelobe reducing windows is employed at the expense of estimation variance. Interest then turns to parametric techniques which require no restrictive data length assumptions. Autoregressive (AR) techniques estimate the linear prediction coefficients and use these to gain information on signal frequencies [3,4]. With multiple signals, frequency resolution is much improved over Fourier methods, spectral interference is reduced and the frequency estimate approaches the Cramér-Rao bound (CRB) [5,6].

However, not only do parametric methods lack the simplicity and consequent processing speed of Fourier methods, often the CRB is only approached and not achieved. This work examines the trade-off involved between the performance of the modified covariance method and the maximization of the Fourier spectrum. Equations are developed that aid understanding of the Fourier bias problem and the two methods are compared over a range of signal parameters. It is shown that when the frequency estimation problem concerns a primary component and additional signals are unwanted (can be considered noise) Fourier techniques are to be preferred over a range of SNR that depends on relative component amplitudes.

2 Frequency estimation methods

The system under consideration is the sum of multiple complex exponentials:

$$y(t) = \sum_{k=1}^{K} A_k e^{j(2\pi f_k t + \varphi_k)} + w(t), \quad 0 < t < t_{\mathrm{m}}, \quad (1)$$

where A_k, f_k and φ_k are the amplitude, frequency and phase of the k^{th} component, and $w(t)$ is complex white Gaussian noise of variance σ^2. This work uses $k = 2$.

2.1 The modified covariance method

This autoregressive model assumes the data set can be entirely described by a linear combination of previous outputs and driving noise. The modified covariance method estimates the P coefficients, where P is the model order, by minimizing the forward and backward prediction errors in the least squares sense:

$$\hat{\rho} = \frac{1}{2}(\hat{\rho}^{\mathrm{f}} + \hat{\rho}^{\mathrm{b}}), \tag{2}$$

where:

$$\hat{\rho}^{\mathrm{f}} = \frac{1}{N-P}\sum_{n=0}^{N-1-P}\left| y_n + \sum_{k=1}^{P} a_k y_{n-k} \right|^2, \tag{3}$$

and similarly for $\hat{\rho}^{\mathrm{b}}$, where N is the data length and a_k is the k^{th} AR coefficient.

2.2 The Fourier transform

The amplitude of the Fourier transform of the windowed data:

$$Y(f) = \int_{-\infty}^{\infty} w_{\mathrm{win}}(t) y(t) e^{-j2\pi ft}\, dt, \tag{4}$$

is maximized to give an estimate of the frequency components. A two stage process is common: 1) a coarse search to give an approximation of the spectral peaks followed by 2) a fine search using a smaller bin width to improve the estimation accuracy. In discrete notation:

$$\hat{f} = \arg_f \left| \frac{1}{N} \sum_{n=0}^{N} w_{\mathrm{win}}(n) y(n) e^{-j2\pi fn/N} \right|^2, \tag{5}$$

where estimate resolution (as distinct from resolution between *components*) can be increased either through zero-padding or, equivalently, frequency domain interpolation.

2.3 Spectral comparison

The figure-of-merit that compares the resolution ability between components is bandwidth [7]:

$$\begin{aligned} BW_{\mathrm{mc}} &\approx \frac{6\sigma^2}{\pi P(P+1)A^2} \\ BW_{\mathrm{F}} &\approx \frac{f\mathrm{s}}{\mathrm{N}}, \end{aligned} \tag{6}$$

hence the parametric technique performs rather well at high SNR and the Fourier method requires a large data set. A comparison is given in Fig.1.

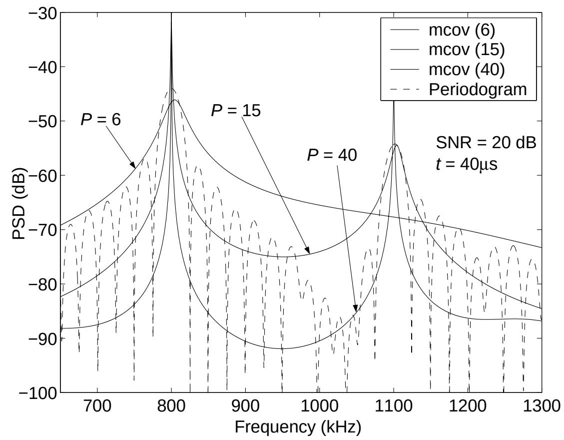

Fig. 1 - For short data records and high SNR, the modified covariance method has better resolution.

2.4 Single complex exponential performance

It is well known that the maximization given in eqn.5 leads to the ML estimate for a single complex exponential hence the Cramér-Rao bound is achieved:

$$\sigma_{\mathrm{f}}{}^2 = \frac{6 f_{\mathrm{s}}{}^2}{4\pi^2 \rho N(N^2-1)}, \tag{7}$$

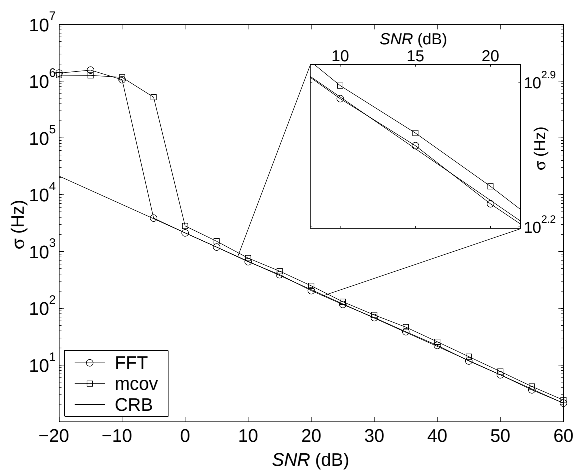

Fig. 2 - Fourier methods are ML past cut-off for a single exponential; the covariance method only approaches the CRB.

where ρ is the SNR. The performance of the modified covariance method is a function of order. With $P \approx N/4$ the CRB is almost reached. At low values of SNR the methods are not efficient. It is possible to calculate this cut-off point by estimating the probability that spurious noise-induced spectral bin magnitudes exceed the magnitude of the true spectral peak [8].

3 Bias in the Fourier transform

Consider the spectrum of a single complex exponential windowed by a boxcar function:

$$Y = \frac{A\sin[\pi\tau(f - f_1)]}{\pi f}. \tag{8}$$

The spectral location of interest is the bias-inducing section of the second component, and the important aspect is the slope at $f = f_1$:

$$\frac{dY}{df} = \frac{A_2\pi^2\tau f\cos[\pi\tau(f - f_2)]}{\pi^2(f - f_2)} \cdots - \frac{A_2\pi\sin[\pi\tau(f - f_2)]}{\pi^2(f - f_2)}. \tag{9}$$

The frequency estimate derived from the spectrum of component 1 can be given by $\hat{f} = -c_2/2c_1$ where each c_i, where i = 1, 2, 3, is found from the matrix equation $\mathbf{C} = \mathbf{X}^{-1}\mathbf{Y}$, where $\mathbf{Y}$ consists of the three peak magnitudes and $\mathbf{X}$ has rows of bin frequencies arranged quadratically.

The modified spectrum consists of the zero-phase sum of the preceding approximation and the linear approximation of the interfering spectrum at f_1. An expression for bias is obtained by calculating the influence of the linear interpolation. It is seen that the slope of the linear interpolation is added to c_2, hence the new peak location is given by:

$$\hat{f} = -\frac{1}{2c_1}\left(c_2 - \left.\frac{dY}{df}\right|_{f=f_1}\right). \tag{10}$$

This can be considered representation of the maximum bias under the worst phase conditions. The true bias will be less and depend on $\phi = \phi_1 + \phi_2$ where $\phi_1 = \varphi_2 - \varphi_1$ and $\phi_2 = \pi\tau\Delta f_{21}$. From a consideration of how spectra add [9]:

$$b(f) = \frac{\cos(\varphi_2 - \varphi_1 + \pi\tau\Delta f_{21})}{2c_1}\left(\left.\frac{dY}{df}\right|_{f=f_1}\right). \tag{11}$$

Fig.3 shows the bias as relative frequency is varied, showing good agreement between eqn.9 and simulation.

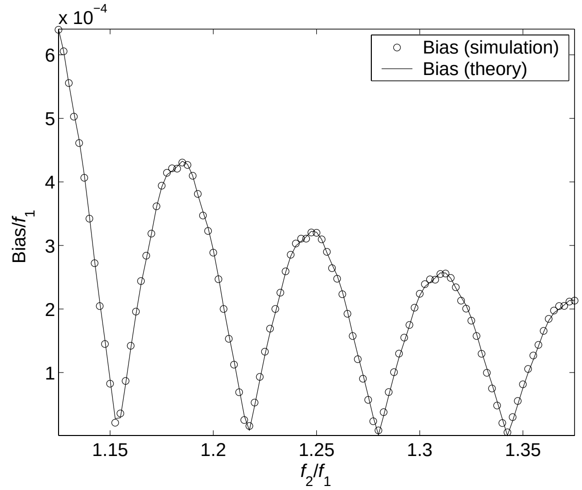

Fig. 3 - Bias is dictated by the differential of the second component spectrum.

4 Error comparison

The total error comprises noise-induced variance according to:

$$MSE = \sigma_f^2 + b^2(f), \tag{12}$$

hence the Fourier performance can be written:

$$E[\hat{f} - f] = \frac{6f_s^2}{4\pi^2\rho N(N^2 - 1)} + B^2, \tag{13}$$

where B is calculated for a uniform distribution of phase: $0 < \varphi < 2\pi$ and frequency separation: $\Delta f_{\min} < \Delta f < \Delta f_{\max}$:

$$B = \frac{1}{2\pi\Delta f}\int_0^{2\pi}\int_{\Delta f_{\min}}^{\Delta f_{\max}} \frac{\cos^2(\phi)}{2c_1}\left(\left.\frac{dY}{df}\right|_{f=f_1}\right)^2 df\,d\phi, \tag{14}$$

an integral that is best performed numerically. Fig.4 compares the modified covariance performance with Fourier for k = 2, A_2/A_1 = 0.3, 1.9 < $\Delta f/f_1$ < 2.5, uniformly distributed and random phase. It is seen that the Fourier method closely follows the lines predicted by eqns. 11 & 12, which show that as bias becomes significant compared to noise-induced variance, a floor is reached in performance. At

SNRs less than the cut-off, the modified covariance method performs less well.

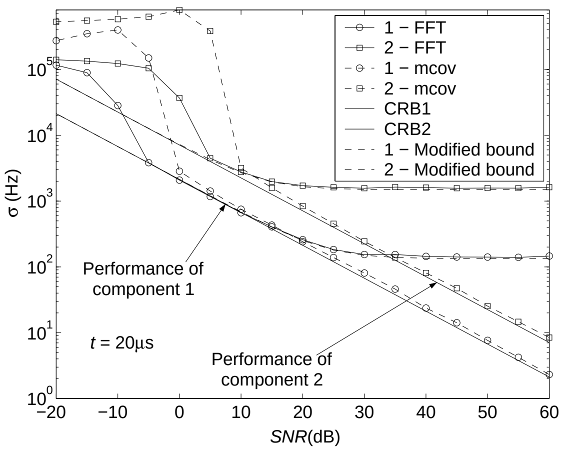

Fig. 4 - The covariance method does not suffer from spectral interference.

Fig.5 compares performance over SNR and frequency separation. At low SNR the Fourier method achieves the CRB. However, deviation quickly occurs at lower SNR, though the cut-off is smaller than for the covariance method. When clean signals are used the bias becomes significant and the covariance method performs better. The cross-over point is a function of frequency separation and it is seen that the SNR range over which Fourier is preferred increases with Δf.

It is noted that the superior performance will not be achieved with non-rectangular window functions. The Fourier method will perform better at high SNR due to sidelobe repression but advantage at low SNR is lost from decreased variance performance [10,11].

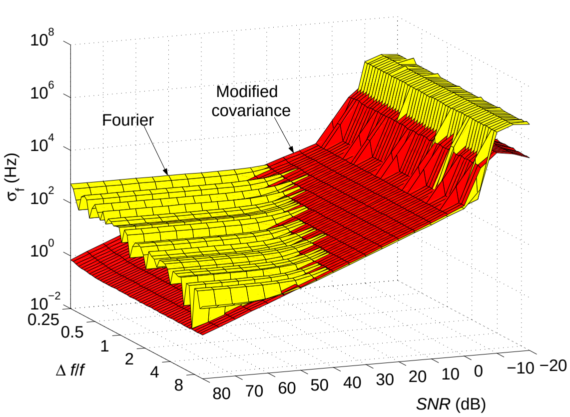

Fig. 5 - Relative performance in 3d-space. Fourier is preferable at large Δf and low SNR.

5 Conclusion

The modified covariance method has been compared to the Fourier technique. Despite the limitations of the latter method in regard to spectral interference, it has been shown that it is possible to take advantage of the desirable qualities under certain circumstances. Given sufficient spectral separation and relative amplitudes and an interest only in the frequency location of the component of greatest magnitude, Fourier performance is better than the modified covariance over a wide range of SNR. The CRB is very nearly achieved at a lower SNR than the AR method.

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