Modified Theories of Laminar Flow Around a Rigid Cylinder and Flow Outside and Inside a Liquid Cylinder in Uniform and Counterflow Gaseous Jets

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Abstract: - The scale-invariant forms of conservation equations in reactive fields are described. The modified form of the *Helmholtz* vorticity equation is solved to determine laminar flow outside a rigid cylinder and flow inside and outside of a cylindrical liquid body in a uniform gaseous stream or at the stagnation-point of two symmetric gaseous planar counterflow jets. For the former problem, a modified solution for flow around rigid cylinder is presented that resolves the *Stokes* paradox and is harmonious with the *Oseen*'s classical solution. For the latter problem, parallel to the classical *Hill* spherical vortex, the solution describing two cylindrical vortex lines is presented. Also, the stream functions representing flow within two concentric immiscible liquid cylinders in uniform or planar counterflow gaseous streams are presented.

Key –Words: - Theory of laminar flow across a cylinder. Cylindrical vortices. Stokes paradox.

1 Introduction

The universality of turbulent phenomena from stochastic quantum fields to classical hydrodynamic fields resulted in recent introduction of a scale-invariant model of statistical mechanics and its application to the field of thermodynamics [4]. The implications of the model to the study of transport phenomena and invariant forms of conservation equations have also been addressed [5, 6]. In the present study, the modified form of the *Helmholtz* vorticity equation is solved for the problems of flow across a rigid or liquid cylinder located in a uniform stream or flow within a liquid cylinder at the stagnation-line of planar gaseous counterflow jets.

2 A Scale-Invariant Model of Statistical Mechanics

Following the classical methods [1-3], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [4, 5]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \tag{2}$$

The invariant definitions of the peculiar and the diffusion velocities have been introduced as [4]

$$\mathbf{V'}_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta = \mathbf{V'}_{\beta+1} \tag{3}$$

3 Invariant Forms of the Conservation Equations for Chemically Reactive Flow Fields

Following the classical methods [1-3], the scale-invariant forms of mass, thermal energy, linear and angular momentum conservation equations [5, 6] at scale β are given as

$$\frac{\partial \rho_\beta}{\partial t} + \nabla . (\rho_\beta \mathbf{v}_\beta) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla . (\varepsilon_\beta \mathbf{v}_\beta) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla . (\mathbf{p}_\beta \mathbf{v}_\beta) = 0 \tag{6}$$

$$\frac{\partial \boldsymbol{\pi}_\beta}{\partial t} + \nabla . (\boldsymbol{\pi}_\beta \mathbf{v}_\beta) = 0 \tag{7}$$

where $\varepsilon_\beta = \rho_\beta h_\beta$, $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$, and $\boldsymbol{\pi}_\beta = \rho_\beta \boldsymbol{\omega}_\beta$ are the *volumetric density* of thermal energy, linear and angular momentum of the field, respectively and

$\boldsymbol{\omega}_\beta = \nabla \times \mathbf{v}_\beta$ is the vorticity. Also, Ω_β is the chemical reaction rate and h_β is the absolute enthalpy [5].

The local velocity $\mathbf{v}_\beta$ in (8)-(11) is expressed in terms of the convective $\mathbf{w}_\beta = <\mathbf{v}_\beta>$ and the diffusive velocities [5]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \qquad (8a)$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \qquad (8a)$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \qquad (8c)$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta rhg} \quad , \quad \mathbf{V}_{\beta rhg} = -\nu_\beta \nabla \ln(\boldsymbol{\pi}_\beta) \qquad (8d)$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$, $\mathbf{V}_{\beta rhg}$) are respectively the diffusive, the thermo-diffusive, the linear hydro-diffusive, and the angular hydro-diffusive velocities. For unity Schmidt and Prandtl numbers $Sc_\beta = Pr_\beta = \nu_\beta/D_\beta = \nu_\beta/\alpha_\beta = 1$, one may express

$$\mathbf{V}_{\beta tg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta t} \qquad (9a)$$

$$\mathbf{V}_{\beta hg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta h} \qquad (9b)$$

$$\mathbf{V}_{\beta rhg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta rh} \qquad (9a)$$

that involve the thermal $\mathbf{V}_{\beta t}$, the linear (translational) hydrodynamic $\mathbf{V}_{\beta h}$ and the angular (rotational) hydrodynamic $\mathbf{V}_{\beta rh}$ diffusion velocities defined as [6]

$$\mathbf{V}_{\beta t} = -\alpha_\beta \nabla \ln(h_\beta) \qquad (10a)$$

$$\mathbf{V}_{\beta h} = -\nu_\beta \nabla \ln(\mathbf{v}_\beta) \qquad (10b)$$

$$\mathbf{V}_{\beta rh} = -\nu_\beta \nabla \ln(\boldsymbol{\omega}_\beta) \qquad (10c)$$

Since for an ideal gas $h_\beta = c_{p\beta} T_\beta$, when $c_{p\beta}$ is constant and $T = T_\beta$, Eq.(3.6a) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_\beta = \rho_\beta h_\beta \mathbf{V}_{\beta t} = -\kappa_\beta \nabla T \qquad (11)$$

where κ_β and $\alpha_\beta = \kappa_\beta/(\rho_\beta c_{p\beta})$ are the thermal conductivity and diffusivity. Similarly, (10b) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [5]

$$\boldsymbol{\tau}_{ij\beta} = \rho_\beta \mathbf{v}_{j\beta} \mathbf{V}_{ij\beta h} = -\mu_\beta \partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_i \qquad (12)$$

Finally, (10c) may be identified as the torsional stress induced by diffusional flux of angular momentum and expressed as

$$\boldsymbol{\tau}_{ijr\beta} = \rho_\beta \boldsymbol{\omega}_{j\beta} \mathbf{V}_{ij\beta rh} = -\mu_\beta \partial \boldsymbol{\omega}_{j\beta} / \partial \mathbf{x}_i \qquad (13)$$

Substitutions from (8a)-(8d) into (4)-(7), neglecting cross-diffusion terms and assuming constant transport coefficients with unity *Prandtl* and *Schmidt* numbers $Sc_\beta = Pr_\beta = 1$, result in [6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \qquad (14)$$

$$h_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial h_\beta}{\partial t} + \mathbf{w}_\beta . \nabla h_\beta - \alpha_\beta \nabla^2 h_\beta \right] = 0 \qquad (15)$$

$$\mathbf{v}_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta \right] = 0 \qquad (16)$$

$$\boldsymbol{\omega}_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial \boldsymbol{\omega}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\beta . \nabla \mathbf{w}_\beta - \nu_\beta \nabla^2 \boldsymbol{\omega}_\beta \right] = 0 \qquad (17)$$

The above forms of the conservation equations perhaps help to better reveal the coupling between the gravitational versus the inertial contributions to total energy and momentum densities of the field.

Substitutions from (14) into (15)-(17) result in scale-invariant forms of conservation equations in chemically reactive fields [6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \qquad (18)$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta . \nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -h_\beta \Omega_\beta / (\rho_\beta c_{p\beta}) \qquad (19)$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\mathbf{v}_\beta \Omega_\beta / \rho_\beta \qquad (20)$$

$$\frac{\partial \boldsymbol{\omega}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \boldsymbol{\omega}_\beta - \nu_\beta \nabla^2 \boldsymbol{\omega}_\beta = -\boldsymbol{\omega}_\beta . \nabla \mathbf{w}_\beta - \frac{\boldsymbol{\omega}_\beta \Omega_\beta}{\rho_\beta} \qquad (21)$$

Equation (21) is the modified form of the *Helmholtz* vorticity equation for chemically reactive flow fields. The last two terms of (21) respectively correspond to vorticity generation by vortex-stretching and chemical reactions. Hence, $(-\boldsymbol{\omega}_\beta \Omega_\beta/\rho_\beta)$ represents generation $\Omega_\beta < 0$

(annihilation $\Omega_\beta > 0$) of angular momentum accompanied by release (absorption) of thermal energy associated with exothermic (endothermic) chemical reactions. As an example, the latter source term may be used to describe the change of angular momentum of a ballet dancer. In this case, the loss of mass due to chemical reactions in the body of a spinning dancer that brings the arms inward, thus doing work against centrifugal forces, results in an increase in the dancer's angular momentum.

4 Solution of the Modified Helmholtz Vorticity Equation for Flow Inside a Liquid Cylinder in a Uniform Gaseous Stream

For flow within a liquid cylinder located in a gaseous stream the non-dimensional steady forms of (18)-(21) in cylindrical coordinate and in the absence of reactions $\Omega = 0$ reduce to

$$w_r \frac{\partial \boldsymbol{\omega}_z}{\partial r} + \frac{w_\theta}{r}\frac{\partial \boldsymbol{\omega}_z}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial}\left(r\frac{\partial \boldsymbol{\omega}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \boldsymbol{\omega}_z}{\partial \theta^2} \tag{22}$$

$$w_r \frac{\partial \mathbf{v}_r}{\partial r} + \frac{w_\theta}{r}\frac{\partial \mathbf{v}_r}{\partial \theta} - \frac{w_\theta \mathbf{v}_\theta}{r} = \frac{\partial^2 \mathbf{v}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{v}_r}{\partial r} - \frac{\mathbf{v}_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 \mathbf{v}_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \mathbf{v}_\theta}{\partial \theta} \tag{23}$$

$$w_r \frac{\partial \mathbf{v}_\theta}{\partial r} + \frac{w_\theta}{r}\frac{\partial \mathbf{v}_\theta}{\partial \theta} + \frac{w_\theta \mathbf{v}_r}{r} = \frac{\partial^2 \mathbf{v}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \mathbf{v}_\theta}{\partial r} - \frac{\mathbf{v}_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 \mathbf{v}_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial \mathbf{v}_r}{\partial \theta} \tag{24}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{25}$$

with (23)-(25) subject to the boundary conditions

$$r = 0 \qquad \frac{\partial v_r}{\partial r} = \frac{\partial v_\theta}{\partial r} = 0 \tag{26a}$$

$$r = R \qquad v_r = v_\theta + 2\sin\theta = 0 \tag{26b}$$

where $(v_r, v_\theta, w_r, w_\theta) = (v'_r, v'_\theta, w'_r, w'_\theta)/U$, $r = r'/(\nu/U)$, $R = R'/(\nu/U)$, R' is the cylinder radius and U is the uniform stream velocity. The dimensionless uniform far field *convective* velocity components are given by

$$w_r = \cos\theta \quad , \quad w_\theta = -\sin\theta \tag{27}$$

An exact solution of (22) and (26) may be expressed by the dimensionless vorticity

$$\omega_z = -8r\sin\theta / R^2 \tag{28}$$

corresponding to the stream function

$$\Psi_i = r[(r/R)^2 - 1]\sin\theta \tag{29}$$

where $\Psi_i = \Psi'_i/\nu$, and $\omega_z = \omega'_z(\nu/U^2)$. The radial and angular velocity components within the cylinder are given by

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = [(r/R)^2 - 1]\cos\theta \tag{30}$$

$$v_\theta = -\frac{\partial \Psi}{\partial r} = -[3(r/R)^2 - 1]\sin\theta \tag{31}$$

Some of the streamlines calculated from (33) using Mathematica [9] are shown in Fig.1.

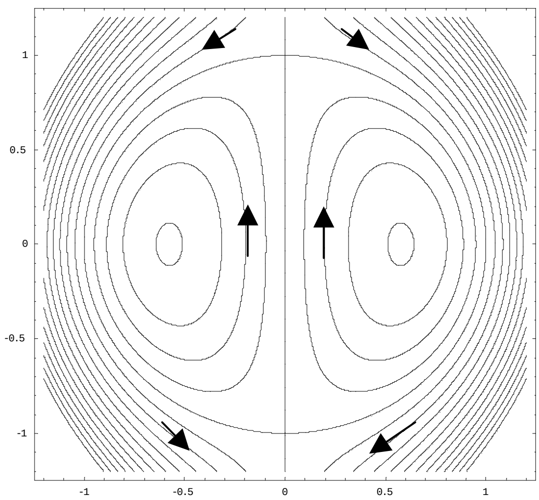

Fig.1 Streamlines for flow within a liquid cylinder in a uniform gaseous stream from (33).

If one introduces the Cartesian coordinate system

$$y = -r\cos\theta \quad , \quad x = r\sin\theta \tag{32}$$

the stream function (29) will assume the form

$$\Psi_i = x[(x/R)^2 + (y/R)^2 - 1] \tag{33}$$

leading to the velocities

$$v_y = 1 - 3(x/R)^2 - (y/R)^2 \tag{34}$$

$$v_x = 2xy/R^2 \tag{35}$$

and the vorticity

$$\omega_z = -8x / R^2 \tag{36}$$

Near the center of the spherical flow (Fig.1) i.e. for small $r \approx 0$, the velocity field (30)-(31) reduces to

$$v_{r\beta} = w_{r\beta-1} = -\cos\theta \tag{37a}$$

$$v_{\theta\beta} = w_{\theta\beta-1} = \sin\theta \tag{37b}$$

where the subscripts (β, $\beta-1$) refer to the larger and smaller adjacent scales [6]. The local velocity (37) is similar to the outer convective velocity field (27) except that it is in the opposite direction (Fig.1). Therefore, in view of the scale-invariant form of (22)-(25), one arrives at a cascade of concentric cylindrical vortices that are embedded within each other with alternating sense of rotation. This is because when the inner cylindrical vortex is small enough, it will experience a locally uniform external flow field (37) that is produced by the outer cylindrical vortex.

In view of the linearity of the governing equations, one can show that the streamline for two embedded concentric liquid cylinders may be presented as product solutions. To show this, first the modified *Helmholtz* vorticity equation (22) is written as

$$\mathsf{L}(\omega_z) = 0 \tag{38}$$

where the linear operator L is defined as

$$\mathsf{L} = w_r \frac{\partial}{\partial r} + \frac{w_\theta}{r}\frac{\partial}{\partial \theta} - \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \tag{39}$$

Next, the axial vorticity is expressed in terms of the stream function as

$$\omega_z = -\left[\frac{\partial^2 \Psi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi_i}{\partial \theta^2}\right] = \mathsf{J}(\Psi_i) \tag{40}$$

where a second linear operator J is defined as

$$\mathsf{J} = -\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \tag{41}$$

such that (38) becomes

$$\mathsf{L}\,[\mathsf{J}\,(\Psi_i)\,] = 0 \tag{42}$$

Let us now consider the flow field within two concentric liquid cylinders that are located in a uniform gaseous stream. The liquid cylinders are supposed to be composed of different immiscible fluids. For the outer cylinder of radius R_1 and the inner cylinder of radius R_2, the stream functions from (33) are

$$\Psi_{i1} = x[(x/R_1)^2 + (y/R_1)^2 - 1] \quad , \quad \Psi_{i2} = x[(x/R_2)^2 + (y/R_2)^2 - 1] \tag{43}$$

Although (37) and (27) are different, since the convective terms in (39) with (36) cancel identically

$$w_r \frac{\partial \omega_z}{\partial r} + \frac{w_\theta}{r}\frac{\partial \omega_z}{\partial \theta} = 0 \tag{44}$$

the operator L in (39) becomes identical for the outer and the inner flow fields. Hence, applying the vorticity equation (38) to the stream functions in (43) gives

$$\mathsf{L}\,[\mathsf{J}\,(\Psi_1)] = \mathsf{L}\,[\mathsf{J}\,(\Psi_2)\,] = 0 \tag{45}$$

that in view of the linearity of the operators leads to the product solution

$$\mathsf{L}\,[\mathsf{J}\,(\Psi_1\Psi_2)] = \mathsf{L}\,[\mathsf{J}\,(\Psi_3)\,] = 0 \tag{46}$$

Therefore, for flow within two concentric liquid cylinders composed of different and immiscible fluids located in a uniform flow the stream function is expressed by (43) and (46) as

$$\Psi_{i3} = \Psi_{i1}\Psi_{i2} = (x/R_1R_2)^2(x^2+y^2-R_1^2)(x^2+y^2-R_2^2) \tag{47}$$

Some of the streamlines for flow within two concentric liquid cylinders calculated from (47) are shown in Fig.2.

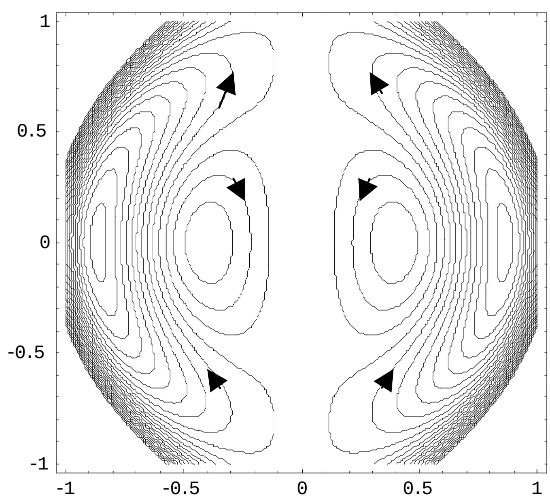

Fig.2 Streamlines in two concentric liquid cylinders in uniform gaseous stream from (47).

It is noted that as the radius of the outer cylinder R_1 is increased, the streamlines within the outer cylinder (Fig.2) become increasingly similar to the streamlines for external flow over a liquid cylinder (Fig.5) that will be considered in the Sec.6.

5 Solution of the Modified Helmholtz Vorticity Equation for Flow Inside a Liquid Cylinder at Stagnation-Point of Planar Gaseous Counterflow Jets

Following the classical solution of *Hill* [7, 8], the flow generated in a small cylindrical body of liquid that is located at the stagnation point of a planar gaseous counterflow is considered [10]. The convective velocity of the gaseous counterflow outside of the cylinder is given by [2]

$$w'_y = -\Gamma y' \quad , \qquad w'_x = \Gamma x' \tag{48}$$

where Γ is the counterflow velocity gradient. With the definitions of dimensionless velocity and coordinates

$$(\mathbf{v}_x, \mathbf{v}_y, \mathbf{w}_x, \mathbf{w}_y) = (\mathbf{v}'_x, \mathbf{v}'_y, \mathbf{w}'_x, \mathbf{w}'_y)/\sqrt{\nu\Gamma}$$
$$x = x'/\sqrt{\nu/\Gamma} \quad , \quad y = y'/\sqrt{\nu/\Gamma} \tag{49}$$

the dimensionless axial vorticity and stream function satisfying (22) and (26) are

$$\omega_z = -12(r/R)^2 \sin\theta\cos\theta$$
$$\Psi_i = r^2[(r/R)^2 - 1)\sin\theta\cos\theta \tag{50}$$

where $\omega = \omega'/\Gamma$, $R = R'/\sqrt{\nu/\Gamma}$, $\Psi = \Psi'/(\nu U/\Gamma)$ and R' is the cylinder radius. In Cartesian coordinates (50) becomes

$$\omega_z = -12xy/R^2$$
$$\Psi_i = xy[(x/R)^2 + (y/R)^2 - 1] \tag{51}$$

leading to the velocity components

$$v_y = y[1 - 3(x/R)^2 - (y/R)^2],$$
$$v_x = -x[1 - (x/R)^2 - 3(y/R)^2] \tag{52}$$

Some of the streamlines calculated from (51) are shown in Fig.3.

It is interesting to note that even if there were no liquid cylinder at the stagnation point, it is expected that a small cylindrical region of flow recirculation like that shown in Fig.3 (or like an ellipsoidal cylindrical body) will form around the stagnation point. Therefore, for fluids with finite viscosity, the critical singularity located at the stagnation point will be avoided by the global flow through the formation of such a closed region of secondary flow. The radius of such a secondary flow region is given by $R^* = \sqrt{\nu/\Gamma}$ and hence depends on the viscosity and the rate of strain.

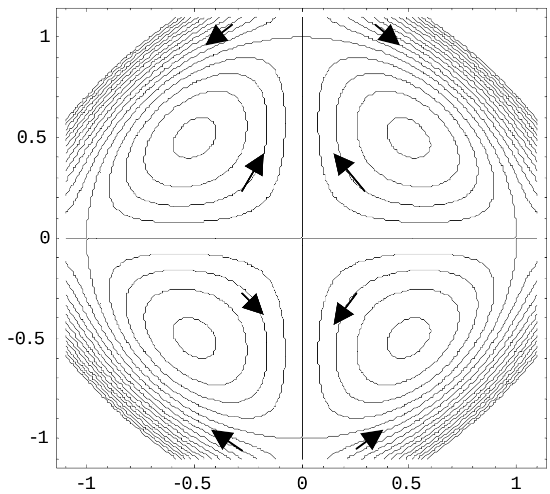

Fig.3 Streamlines in liquid cylinder at the stagnation point of planar counterflow from (51).

In the vicinity of the stagnation point, $x \approx 0$ and $y \approx 0$, the local velocity field (52) reduces to

$$v_{y\beta} = w_{y\beta-1} = y \quad , \qquad v_{x\beta} = w_{x\beta-1} = -x \tag{53}$$

that except for its opposite direction is similar to the outer convective velocity field in (48). Therefore, as was noted earlier [10-11], because of the scale-invariant nature of the conservation equations, one expects a cascade of embedded concentric cylindrical flows at ever-smaller scales to form around the stagnation point. Following the reasoning and the procedures similar to those described in (38)-(47), it can be shown that for two concentric cylinders located at the stagnation-point of a planar counterflow with the radii R_1 and R_2 and with the respective stream functions obtained from (51) as

$$\Psi_{i4} = xy[(x/R_1)^2 + (y/R_1)^2 - 1]$$
$$\Psi_{i5} = xy[(x/R_2)^2 + (y/R_2)^2 - 1] \tag{54}$$

one arrives at the product solution given by

$$\Psi_{i6} = \Psi_{i4}\Psi_{i5} =$$
$$(xy/R_1R_2)^2(x^2 + y^2 - R_1^2)(x^2 + y^2 - R_2^2) \tag{55}$$

Some of the streamlines calculated from Eq.(55) for flow within two concentric cylinders that are located at the stagnation line of two symmetric planar gaseous counterflow jets are shown in Fig.4.

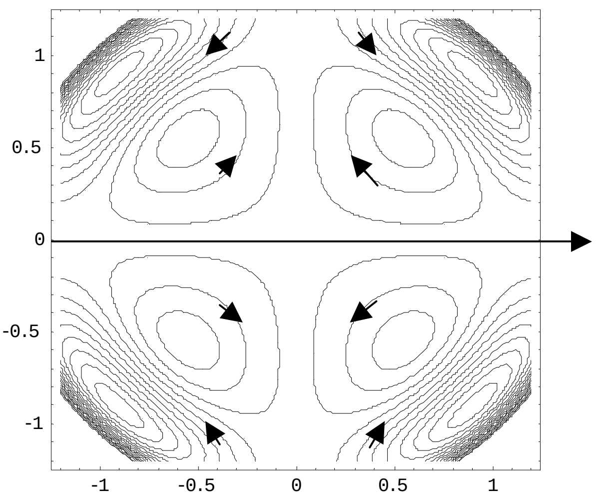

Fig.4 Streamlines in two immiscible concentric liquid cylinders calculated from (55).

Examination of Fig.4 shows that generation of many concentric cylindrical flows is accompanied by the formation of many new local planar counterflow regions. It is expected that in the vicinity of each new stagnation point smaller secondary cylindrical flows will be generated. For example, one would expect small cylindrical flows to form in the vicinity of the poles of the inner cylinder in Fig.4. Since strained flow fields are common features of turbulent flows, the generation of cascades of cylindrical line vortices at each stagnation point is expected to play an important role in turbulent dissipation process. The application of the results to the classical dynamo problem [13] requires future consideration.

6 Modified Theory of Flow Outside of a Rigid Cylinder in Uniform Stream

It is well known that the solution of the problem of viscous flow outside of a rigid cylinder in uniform stream encounters difficulties leading to what is known as the *Stokes* paradox [8]. The difficulty is that thus far a solution that simultaneously satisfies both the far field uniform velocity boundary conditions as well as the no-slip boundary conditions on the surface of the cylinder has not been possible. The problem was partially resolved by the classical solution of *Oseen* [14-16] that assumed a constant velocity convective term. It is interesting to note that the equation considered by *Oseen* [14] is indeed similar to the modified form of the *Helmholtz* vorticity equation (22).

Before discussing flow around a rigid cylinder, the solution of the simpler problem of flow outside of a *liquid cylinder* in a uniform gaseous stream is considered. For this problem with the far field uniform convective velocity

$$w_r = \cos\theta \quad , \quad w_\theta = -\sin\theta \tag{56}$$

the solution of (22) and (26) is similar to the classical result [8] and given by the stream function

$$\Psi_{of} = r[1-(R/r)^2]\sin\theta \tag{57}$$

with the velocity components

$$v_r = [1-(R/r)^2]\cos\theta \tag{58}$$

$$v_\theta = -[1+(R/r)^2]\sin\theta \tag{59}$$

It is noted that while the radial velocity on the cylinder surface vanishes, the angular velocity assumes the finite value of

$$v_\theta(R) = -2\sin\theta \tag{60}$$

that matches (31) at r = R as required.

As opposed to the internal flow that has a finite axial vorticity given in (28), the axial vorticity in the external flow vanishes identically

$$\omega_z = 0 \tag{61}$$

The result (57) expressed in Cartesian coordinates assumes the form

$$\Psi_{of} = \frac{x(x^2+y^2-R^2)}{x^2+y^2} \tag{62}$$

Some of the streamlines for flow external to a liquid cylinder calculated from (62) with $R=\sqrt{2}$ are shown in Fig.5.

For the problem of flow around a rigid cylinder, we consider superposition of a uniform flow, a doublet, and a special rectilinear vortex that has angle-dependent angular velocity and introduce the dimensionless complex potential

$$F = z + \frac{R^2}{z} - 2iRSin\alpha\ln(z/R) \tag{63}$$

where $F = F'/\nu$, $z = z'/(\nu/U)$, $r = r'/(\nu/U)$, $R = R'/(\nu/U)$ and

$$z = re^{i\theta} \tag{64}$$

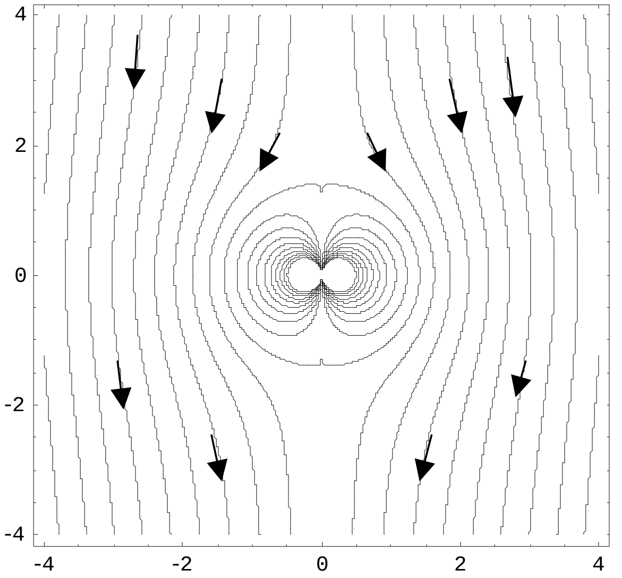

Fig.5 Streamlines outside of liquid cylinder in uniform gaseous flow calculated from (62).

It is noted that the angle dependence of the line vortex in the last term of the potential (63), unlike θ, is considered to be constant. Separating the real and the imaginary parts of the complex potential (63) leads to the velocity potential

$$\Phi = r(1+\frac{R^2}{r^2})\cos\theta + 2R\theta\sin\alpha \qquad (65)$$

and the stream function

$$\Psi_{or} = r(1-\frac{R^2}{r^2})\sin\theta - 2R\sin\alpha\ln(r/R) \qquad (66)$$

The stream function (66) gives the velocity components

$$v_r = (1-\frac{R^2}{r^2})\cos\theta \qquad (67)$$

$$v_\theta = -(1+\frac{R^2}{r^2})\sin\theta + \frac{2R}{r}\sin\alpha \qquad (68)$$

The angular dependence of the line vortex is now chosen to be $\alpha = \theta$ such that (68) becomes

$$v_\theta = -[1+\frac{R^2}{r^2}-\frac{2R}{r}]\sin\theta \qquad (69)$$

The angular velocity in (69) has the desired property that it vanishes on the surface of the rigid cylinder while simultaneously satisfying the far field convective velocity boundary condition (56) thereby resolving the *Stokes* paradox. Also, for the choice of angular dependence of vortex velocity $\alpha = \theta$, one obtains from (66) the stream function

$$\Psi_{or} = x(1-\frac{R^2}{x^2+y^2}) - \frac{2Rx}{(x^2+y^2)^{1/2}}\ln[(x^2+y^2)^{1/2}/R] \qquad (70)$$

Some of the streamlines calculated from (70) for flow over a rigid cylinder are shown in Fig.6. As is to be expected, the streamlines for the flow over a rigid cylinder shown in Fig.6 from (70) are different from those for the flow over a liquid cylinder shown in Fig.5 from (62). Therefore, for example in theoretical combustion models [3], the former flow is relevant to coal particle combustion while the latter flow applies to combustion of liquid sprays. The direct comparisons of the predicted velocity profiles with experimental observations require further future considerations.

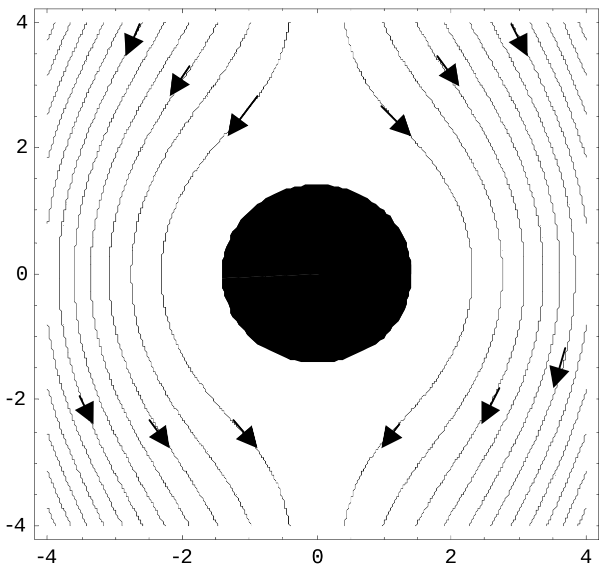

Fig.6 Streamlines for flow outside of a rigid cylinder in uniform stream calculated from (70).

7 Concluding Remarks

The modified form of the *Helmholtz* vorticity equation was solved for the classical problem of flow over a rigid cylinder. The new solution resolves the classical *Stokes* paradox for flow over a cylinder. The solutions of the modified form of the *Helmholtz* vorticity equation were also determined for flow inside and outside of liquid cylinder located in uniform gaseous flow or at the stagnation-line of planar gaseous counterflow jets. Finally, the velocity fields within two concentric liquid cylinders made of different immiscible fluids in uniform or planar counterflow gaseous streams were determined. The results may help the understanding of vortex dynamics in turbulent fields and the understanding of evaporation/combustion of cylindrical regions of liquid/solid fuels that may be encountered in turbulent spray combustion. The generation of cascades of embedded cylindrical vortices within locally strained flows (Fig.4) was identified as one possible mechanism of turbulent dissipation.

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References:

[1] de Groot, R. S., and Mazur, P., *Nonequilibrium Thermodynamics*, North-Holland, 1962.

[2] Schlichting, H., *Boundary-Layer Theory*, McGraw Hill, New York, 1968.

[3] Williams, F. A., *Combustion Theory*, 2nd Ed., Addison-Wesley, New York, 1985.

[4] Sohrab, S. H., A scale-invariant model of statistical mechanics and modified forms of the first and the second laws of thermodynamics. *Rev. Gén. Therm.* **38**, 845-854 (1999).

[5] Sohrab, S. H., Transport phenomena and conservation equations for multi-component chemically-reactive ideal gas mixtures. *Proceeding of the 31st ASME National Heat Transfer Conference,*HTD-Vol. **328**, 37-60 (1996).

[6] Sohrab, S. H., WSEAS *Transactions on Mathemathics*, Issue **4**, Vol.3, 755 (2004).

[7] Hill, M. J. M., *Phil. Trans. Roy. Soc.* A **185**, 213 (1894).

[8] Panton, R. L., *Incompressible Flow*, Wiley, New York, 1996.

[9] Wolfram, S., and Beck, G., *Mathematica, The Student Book*. Addison Wesley, New York, 1994.

[10] Sohrab, S. H., Reactive Hydrodynamics in Rotating Spherical and Cylindrical Flows *Fourth International Microgravity Combustion Workshop*, NASA, May 19-21, 1997, Cleveland, Ohio.

[11] Sohrab, S. H., Hydrodynamics of spherical flows and geometry of premixed flames near the stagnation-point of axisymmetric viscous counterflows *Fifth International Microgravity Combustion Workshop*, NASA, May 18-20, 1999, Cleveland, Ohio.

[12] Sohrab, S., H., *IASME Transactions* **4**, Vol.1, 634 (2004).

[13] Elsasser, W. M., The earth as a dynamo, *Scientific American*, May, (1958).

[14] Oseen, C. W., *Ark. Mat. Astr. Fys.* ***6***, No. 29, (1910).

[15] Kaplun, S., *J. Math. Mech.**6**,* 595 (1957).

[16] Proudman I., and Pearson, J. R. A., *J. Fluid Mech.**2**,* 237 (1957).