Piecewise polynomial approximations for weakly singular integral equations with discontinuous coefficients

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Abstract: – We study the attainable order of a piecewise polynomial collocation method for the numerical solution of linear integral equations with weakly singular or other nonsmooth kernels. In particular, the kernel may have the form $K(t,s) = g(t,s)|t-s|^{-\nu}$, $0 < \nu < 1$, where g is proposed to be smooth only on $[0,b] \times ([0,b] \setminus \{d\})$, $0 < d < b$. We show that the proposed method is of maximal possible order if the grid is chosen appropriately.

Key words: – Fredholm integral equation, weakly singular kernel, collocation method.

1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $\Omega \subset \mathbb{R}^n$, by $C^m(\Omega)$ we denote the set of m times continuously differentiable functions $x : \Omega \to \mathbb{R}$, $C^0(\Omega) = C(\Omega)$. The set $C[a,b]$ of continuous functions $x : [a,b] \to \mathbb{R}$ is a Banach space with respect to the norm $\|x\|_{C[a,b]} = \max_{a \le t \le b} |x(t)|$.

Let us consider an integral equation of the form

$$u(t) - \int_0^b K(t,s)u(s)ds = f(t)\,,\ 0 \le t \le b\,, \quad (1)$$

with $f \in C[0,b]$ and $K(t,s) = g(t,s)|t-s|^{-\nu}$, $0 < \nu < 1$, where g is a sufficiently smooth function on $[0,b] \times [0,b]$. Solutions of integral equations of this type will in general contain singularities in their derivatives at the endpoints of the interval $[0,b]$, even for smooth forcing functions f (see, for example, [1,5,6]). Therefore difficulties in constructing of high order numerical methods for solving (1) arise. To overcome these difficulties, one can thicken near 0 and b, the grid which is used to built approximate solution [1,5,6].

In the present paper we study the case if g is proposed to be smooth only on $[0,b] \times ([0,b] \setminus \{d\})$, with $d \in (0,b)$. In this case the derivatives of the solution $u(s)$ of equation (1) may have singularities at $s = d$, also [4,5]. Therefore, to get numerical algorithms of higher order for solving (1), we shall thicken the grid near $s = d$, too. In fact, we shall construct a piecewise polynomial collocation method for the numerical solution of a wide class of weakly singular integral equations and show that it is of maximal possible order if the grid is chosen appropriately.

2 Smoothness of the solution

We consider a kernel K in the form

$$K(t,s) = g(t,s)\kappa(t,s) \quad (2)$$

with g and κ satisfying the following assumptions (A1) and (A2), respectively.

(A1) The function $g = g(t,s)$ is m times ($m \ge 1$) continuously differentiable with respect to t and s for $t \in [0,b]$, $s \in [0,b] \setminus \{d\}$, $0 < d < b$, and its derivatives are bounded in the regions $[0,b] \times [0,d)$ and $[0,b] \times (d,b]$. Let p ($0 \le p \le m$) be an integer defined as follows: $p = 0$ if g may have a discontinuity across the line $s = d$; $p \ge 1$ if $g \in C^{p-1}([0,b] \times [0,b])$.

(A2) The function $\kappa = \kappa(t,s)$ is m times (m is fixed in the assumption (A1)) continuously differentiable with respect to t and s for $t, s \in [0,b]$, $t \ne s$, and there exists a real number ν, $-\infty < \nu < 1$, such that the estimate

$$\left|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j \kappa(t,s)\right| \leq$$
$$c \begin{cases} 1\,, & \text{if} \quad \nu + i < 0\,, \\ 1 + |\ln|t-s||\,, & \text{if} \quad \nu + i = 0\,, \\ |t-s|^{-\nu-i}, & \text{if} \quad \nu + i > 0\,, \end{cases} \tag{3}$$

holds with a positive constant c for all $t, s \in [0,b]$, $t \neq s$ and for all $i, j \in \mathbb{N}_0$, $i + j \leq m$.

For $i = j = 0$, condition (3) yields

$$|\kappa(t,s)| \leq c \begin{cases} 1, & \text{if} \quad \nu < 0, \\ 1 + |\ln|t-s||, & \text{if} \quad \nu = 0, \\ |t-s|^{-\nu}, & \text{if} \quad \nu > 0. \end{cases}$$

Thus, a kernel (2) is at most weakly singular for $0 \leq \nu < 1$. For $\nu < 0$, the kernel (2) is bounded but its derivatives may have diagonal singularities. Most important examples of kernels of type (2) are given by

$$K(t,s) = g(t,s)|t-s|^{-\nu}, \quad 0 < \nu < 1,$$
$$K(t,s) = g(t,s) \ln|t-s|,$$

where g is a function which satisfies the condition (A1).

For equations (1) with smooth kernels, the smoothness of the kernel K and the forcing function f determines the smoothness of the solution u on the closed interval $[0,b]$. If we allow weakly singular kernels of type (2), with smooth coefficient functions g: $[0,b] \times [0,b] \to \mathbb{R}$, then the resulting solutions are typically nonsmooth at the endpoints of the interval of integration $[0,b]$, where their derivatives become unbounded. If g is proposed to be smooth only on $[0,b] \times ([0,b] \setminus \{d\})$, where $0 < d < b$, then the derivatives of the solution $u(t)$ of equation (1) may have singularities at $t = d$, also (see Lemma 1 below). In order to characterize those singularities we introduce a set of functions $C^{m,\nu}_{d,p}[0,b]$.

Let $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, $0 < d < b$, $p \in \mathbb{N}_0$, $p \leq m$. Define $C^{m,\nu}_{d,p}[0,b]$ as the collection of continuous functions $u : [0,b] \to \mathbb{R}$ which are m times continuously differentiable in $(0,b) \setminus \{d\}$ and such that the estimate

$$\left|u^{(j)}(t)\right| \leq$$
$$c \begin{cases} 1, & \text{if} \quad j < 1-\nu,\ p \in \{0,1,\ldots,m\}\,; \\ 1 + |\ln t| + |\ln(b-t)|, & \text{if} \quad j = 1-\nu,\ p \in \{1,\ldots,m\}\,; \\ 1 + |\ln t| + |\ln|d-t|| + |\ln(b-t)|, & \text{if} \quad j = 1-\nu,\ p = 0\,; \\ t^{1-\nu-j} + (b-t)^{1-\nu-j}, & \text{if } 1-\nu < j < 1-\nu+p,\ p \in \{1,\ldots,m\}\,; \\ t^{1-\nu-j} + |\ln|d-t|| + (b-t)^{1-\nu-j}, & \text{if } j = 1-\nu+p,\ p \in \{1,\ldots,m-1\}\,; \\ t^{1-\nu-j} + |d-t|^{1-\nu-j+p} + (b-t)^{1-\nu-j}, & \text{if } j > 1-\nu+p,\ p \in \{0,\ldots,m-1\}\,, \end{cases} \tag{4}$$

holds with a positive constant $c = c(u)$ for every $t \in (0,b) \setminus \{d\}$ and $j = 1, \ldots, m$.

The following result characterizes the regularity properties of solutions to equation (1), see [4,5].

Lemma 1. *Let the conditions (A1) and (A2) about the kernel* (2) *be fulfilled. Let $f \in C^{m,\nu}_{d,p}[0,b]$, with m, ν, d, p, fixed in the assumptions (A1) and (A2). If the integral equation* (1) *has an integrable solution $u \in L^1(0,b)$ then $u \in C^{m,\nu}_{d,p}[0,b]$.*

3 Piecewise polynomial interpolation

For given $N = 4n$, $n \in \mathbb{N}$, $b, d, r, r_d \in \mathbb{R}$, $0 < d < b$, $r, r_d \geq 1$, let

$$\Delta_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = b\}$$

be a partition (a grid) for the interval $[0,b]$ with the following nodes $t_0, \ldots, t_N$:

$$\begin{aligned} t_j &= \frac{d}{2}\left(\frac{j}{n}\right)^r, \ j = 0,1,\ldots,n\,; \\ t_{n+j} &= d - \frac{d}{2}\left(\frac{n-j}{n}\right)^{r_d}, \ j = 1,\ldots,n\,; \\ t_{2n+j} &= d + \frac{b-d}{2}\left(\frac{j}{n}\right)^{r_d}, \ j = 1,\ldots,n\,; \\ t_{3n+j} &= b - \frac{b-d}{2}\left(\frac{n-j}{n}\right)^r, \ j = 1,\ldots,n\,. \end{aligned} \tag{5}$$

Then Δ_N is called a graded grid for $[0,b]$. In the present context the so-called grading exponents r, r_d will always satisfy $r \geq 1$ and $r_d \geq 1$. These parameters characterize the accumulation of nodes

$t_0, t_1, \dots t_N$ near the points of possible unboundedness of the derivatives of the solution u of equation (1) (see Lemma 1). For larger r and r_d the grid Δ_N is thicker near 0, d and b. We use two different parameters r and r_d because the order of singularity of the solution u can be different at points 0, b and d. If $r = r_d = 1$ then the grid points (5) are uniformly located in the intervals $[0, d]$ and $[d, b]$.

It follows from (5) that an estimate

$$h_N \equiv \max_{j=1,\dots,N} (t_j - t_{j-1}) \leq cN^{-1} \tag{6}$$

holds with a positive constant c which is independent of N.

For $m \in \mathbb{N}_0$, let $S_m^{(0)}(\Delta_N)$ and $S_m^{(-1)}(\Delta_N)$ be the spline spaces of piecewise polynomial functions on the grid Δ_N:

$$\begin{aligned} S_m^{(0)}(\Delta_N) &= \left\{ u \in C[0,b] : u\big|_{\sigma_j} \in \pi_m, j=1,\dots,N \right\}, \\ S_m^{(-1)}(\Delta_N) &= \left\{ u : u\big|_{\sigma_j} \in \pi_m\,,\ j = 1,\dots,N \right\}. \end{aligned} \tag{7}$$

In (7) π_m denotes the set of polynomials of degree not exceeding m and $u\big|_{\sigma_j}$ is the restriction of u to the subinterval $\sigma_j = [t_{j-1}, t_j]$ $(j = 1,\dots,N)$. Note that the elements of $S_m^{(-1)}(\Delta_N)$ may have jump discontinuities at the interior grid points $t_1, \dots, t_{N-1}$.

In every subinterval $[t_{j-1}, t_j]$, $j = 1,\dots,N$ we define $m \in \mathbb{N}$ interpolation points

$$\xi_{j,q} = t_{j-1} + \frac{\eta_q + 1}{2}(t_j - t_{j-1})\,, \quad q = 1,\dots,m\,;\ j = 1,\dots,N\,, \tag{8}$$

where

$$-1 \leq \eta_1 < \ldots < \eta_m \leq 1 \tag{9}$$

is some fixed system of m parameters on the interval $[-1, 1]$, which is the same for every j and N.

To a given continuous function $u : [0, b] \to \mathbb{R}$ we assign a piecewise polynomial interpolation function $P_N u = P_{N,m-1} u \in S_{m-1}^{(-1)}(\Delta_N)$ which interpolates u at the nodes (8). Let $P_N = P_{N,m-1} : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ be an interpolation operator which assigns to every continuous function $u : [0, b] \to \mathbb{R}$ its piecewise interpolation function $P_N u$:

$$\begin{aligned} &P_N u \in S_{m-1}^{(-1)}(\Delta_N),\ u \in C[0,b], \\ &(P_N u)(\xi_{j,q}) = u(\xi_{j,q}),\ q=1,\dots,m;\ j=1,\dots,N. \end{aligned} \tag{10}$$

Thus, $(P_N u)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$ $(j = 1,\dots,N)$ and may be discontinuous at $t = t_j$, $j = 1,\dots,N-1$; we can treat $P_N u$ as a two-valued function at these points. If $\eta_1 = -1$, $\eta_m = 1$ then $P_N u$ is a continuous function on the interval $[0, b]$.

Let E and F be Banach spaces. By $\mathcal{L}(E, F)$ we denote the Banach space of all linear bounded operators $A : E \to F$ with the norm $\|A\|_{\mathcal{L}(E,F)} = \sup_{x \in E, \|x\|_E \leq 1} \|Ax\|_F$. It follows from [5] that $P_N \in \mathcal{L}(C[t_{j-1}, t_j], C[t_{j-1}, t_j])$ $(j = 1,\dots,N)$ and $P_N \in \mathcal{L}(C[0,b],\ L^\infty(0,b))$. Moreover, the norms of these operators are uniformly bounded in N:

$$\max_{j=1,\dots,N} \big\|P_N\big\|_{\mathcal{L}(C[t_{j-1},t_j], C[t_{j-1},t_j])} \leq c\,, \quad N \in \mathbb{N}\,,$$

$$\big\|P_N\big\|_{\mathcal{L}(C[0,b], L^\infty(0,b))} \leq c\,, \quad N \in \mathbb{N}\,. \tag{11}$$

Here c is a positive constant which is independent of j and N. On the basis of (11) we obtain that

$$\|u - P_N u\|_{L^\infty(0,b)} \to 0 \text{ as } N \to \infty \tag{12}$$

for every $u \in C[0, b]$. A consequence of this is

Lemma 2. *Let $S : L^\infty(0,b) \to C[0,b]$ be a linear compact operator. Then*

$$\|S - P_N S\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \ \textit{as}\ N \to \infty.$$

In the following we present a result about the rate of the error $\|u - P_N u\|_{L^\infty(0,b)}$.

Lemma 3. *Let $u \in C_{d,p}^{m,\nu}[0,b]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $p \in \{0, 1, \dots, m\}$. Let the node points* (8) *with grid points* (5) *and parameters* (9) *be used. Let $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$ be determined by the conditions* (10).

Then

$$\big\|u - P_N u\big\|_{L^\infty(0,b)} \leq c\varepsilon_N\,, \tag{13}$$

where c is a positive constant not depending on N and $\varepsilon_N = \varepsilon_N(m, \nu, p, r, r_d)$ is defined as follows:

$$\varepsilon_N = N^{-m}, \tag{14}$$

for

$$m<1-\nu,\ p\geq 0,\ r\geq 1,\ r_d\geq 1;$$
$$m=1-\nu,\ p=0,\ r>1,\ r_d>1;$$
$$m=1-\nu,\ p>0,\ r>1,\ r_d\geq 1;$$
$$1-\nu<m<1-\nu+p,\ p>0,\quad r\geq\frac{m}{1-\nu},\ r_d\geq 1;$$
$$m=1-\nu+p,\ p>0,\ r\geq\frac{m}{1-\nu},\ r_d>1;$$
$$m>1-\nu+p,\ p\geq 0,\quad r\geq\frac{m}{1-\nu},\ r_d\geq\frac{m}{1-\nu+p};$$

$$\varepsilon_N=N^{-m}\ln N \tag{15}$$

for

$$m=1-\nu,\ p=0,\ r=1,\ r_d\geq 1;$$
$$m=1-\nu,\ p=0,\ r\geq 1,\ r_d=1;$$
$$m=1-\nu,\ p>0,\ r=1,\ r_d\geq 1;$$
$$m=1-\nu+p,\ p>0,\ r\geq\frac{m}{1-\nu},\ r_d=1;$$

$$\varepsilon_N=N^{-r(1-\nu)} \tag{16}$$

for

$$1-\nu<m<1-\nu+p,\ p>0,\quad 1\leq r<\frac{m}{1-\nu},\ r_d\geq 1;$$
$$m>1-\nu+p,\ p\geq 0,\quad 1\leq r<\frac{m}{1-\nu},\ r_d\geq\frac{m}{1-\nu+p};$$

$$\varepsilon_N=N^{-\min\{r(1-\nu),r_d(1-\nu+p)\}} \tag{17}$$

for

$$m>1-\nu+p,\ p\geq 0,\quad 1\leq r<\frac{m}{1-\nu},\ 1\leq r_d<\frac{m}{1-\nu+p};$$

$$\varepsilon_N=N^{-r_d(1-\nu+p)} \tag{18}$$

for

$$m>1-\nu+p,\ p\geq 0,\quad r\geq\frac{m}{1-\nu},\ 1\leq r_d<\frac{m}{1-\nu+p}.$$

Proof. We follow the approach and techniques of [5]. It follows from (11) that

$$\left\|u-P_Nu\right\|_{L^\infty(0,b)}\leq c\max_{j=1,\dots,N}\max_{t_{j-1}\leq t\leq t_j}|u(t)-v(t)|,$$

where c is a positive constant not depending on N and v is an arbitrary element of the space $S_{m-1}^{(-1)}(\Delta_N)$. Thus, in order to study the rate of the error $\|u-P_Nu\|_{L^\infty(0,b)}$, we have to estimate $|u(t)-v(t)|$ for a suitable $v(t)$ on every subinterval $[t_{j-1},t_j]$, $j=1,\dots,N=4n$. In particular, taking

$$v(t)=u(t_j)+u'(t_j)(t-t_j)+\frac{1}{2!}u''(t_j)(t-t_j)^2+\dots+\frac{1}{(m-1)!}u^{(m-1)}(t_j)(t-t_j)^{m-1},$$

where $t\in[t_{j-1},t_j]$, $j=1,\dots,n$, and using (4) for the derivatives of $u\in C_{d,p}^{m,\nu}[0,b]$, we can estimate $u(t)-v(t)$ on the subinterval $[t_{j-1},t_j]\subset[0,\frac{d}{2}]$, $j=1,\dots,n$; in a similar way we can derive the estimates for $u-v$ (with a suitable $v\in S_{m-1}^{(-1)}(\Delta_N)$) on other subintervals $[t_{j-1},t_j]\subset[\frac{d}{2},b]$, $j=n+1,\dots,4n$; see [2] for a detailed proof.

4 Collocation method

We look for an approximation u_N to the solution u of equation (1) determing u_N from the following conditions:

$$\Big[u_N(t)-\int_0^b K(t,s)u_N(s)ds-f(t)\Big]_{t=\xi_{i,p}}=0,$$
$$u_N\in S_{m-1}^{(-1)}(\Delta_N),\quad m\geq 1,$$
$$p=1,\dots,m;\ i=1,\dots,N, \tag{19}$$

with $\{\xi_{i,p}\}$, given by (8).

Theorem 1. *Let the following conditions be fulfilled:*

1) $K\in C_{d,0}^{1,\nu}[0,b]$, $\nu<1$, $0<d<b$;

2) $f\in C[0,b]$;

3) the homogeneous integral equation

$$u(t)=\int_0^b K(t,s)u(s)ds,\quad 0\leq t\leq b, \tag{20}$$

has only the trivial solution $u=0$;

4) the collocation points (8) with grid points (5) and parameters (9) are used.

Then equation (1) *has a unique solution* $u^* \in C[0,b]$. *For all sufficiently large* N, *say* $N \geq N_0$, *the collocation conditions* (19) *determine for every choice of parameters* $-1 \leq \eta_1 < \ldots < \eta_m \leq 1$ *a unique approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^* *and*

$$\sup_{t\in[0,b]} \left|u_N^*(t) - u^*(t)\right| \to 0 \quad as \quad N \to \infty\,. \quad (21)$$

Proof. We consider equation (1) as the equation

$$u = Tu + f \quad (22)$$

in the Banach space $L^\infty(0,b)$, with the operator T, defined by $(Tu)(t) = \int_0^b K(t,s)u(s)ds$. It follows from the assumtion 1) that T is compact as an operator from $L^\infty(0,b)$ to $C[0,b]$ and from $L^\infty(0,b)$ to $L^\infty(0,b)$, also. Since equation $u = Tu$ has only the trivial solution $u = 0$, then there exists the inverse operator $(I-T)^{-1} \in \mathcal{L}(L^\infty(0,b), L^\infty(0,b))$ and equation (22) has a unique solution $u^* = (I-T)^{-1}f \in L^\infty(0,b)$. Since $f \in C[0,b]$ and $T \in \mathcal{L}(L^\infty(0,b), C[0,b])$, then $u^* \in C[0,b]$.

The collocation conditions (19) can be written in the form

$$u_N = P_N T u_N + P_N f\,, \quad (23)$$

with $P_N : C[0,b] \to S_{m-1}^{(-1)}(\Delta_N)$, defined in Sec. 3. By Lemma 2,

$$\left\|T - P_N T\right\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \to 0 \quad \text{for} \quad N \to \infty\,. \quad (24)$$

Using (24) we obtain that $(I - P_N T)$ is invertible for all sufficiently large N, say for $N \geq N_0$, and

$$\left\|(I - P_N T)^{-1}\right\|_{\mathcal{L}(L^\infty(0,b),L^\infty(0,b))} \leq c\,, \quad N \geq N_0\,, \quad (25)$$

where c is a positive constant which is independent of N. This shows that for $N \geq N_0$ equation (23) has a unique solution $u_N^* = (I - P_N T)^{-1} P_N f$. We have for it and u^*, the solution of equation (22),

$$(I - P_N T)(u^* - u_N^*) =$$
$$(I-P_N T)u^* - (I-P_N T)u_N^* = u^* - P_N T u^* - P_N f$$
$$= u^* - P_N f - (P_N u^* - P_N f) = u^* - P_N u^*\,.$$

Therefore,

$$u^* - u_N^* = (I - P_N T)^{-1}(u^* - P_N u^*)\,.$$

Taking the norms and using (25), we have

$$\left\|u^* - u_N^*\right\|_{L^\infty(0,b)} \leq c \left\|u^* - P_N u^*\right\|_{L^\infty(0,b)}, \quad N \geq N_0\,, \quad (26)$$

where c is a constant which is independent of N. Since $u^* \in C[0,b]$, the convergence (21) follows from (12) and (26).

Theorem 2. *Let the following conditions be fulfilled:*

1) *$K(t,s) = g(t,s)\kappa(t,s)$ is subject to the conditions, stated in the assumptions (A1) and (A2) (see Sec. 2);*
2) *$f \in C_{d,p}^{m,\nu}[0,b]$, with m, ν, d, p, fixed in (A1) and (A2);*
3) *equation* (20) *has only the trivial solution $u = 0$;*
4) *the collocation points (8) with grid points (5) and parameters (9) are used.*

Then for all sufficiently large N, *say* $N \geq N_0$, *the collocation conditions* (19) *determine for every choice of parameters* $-1 \leq \eta_1 < \ldots < \eta_m \leq 1$ *a unique approximation* $u_N^* \in S_{m-1}^{(-1)}(\Delta_N)$ *to* u^*, *the exact solution of equation* (1). *The following error estimate holds:*

$$\sup_{0\leq t\leq b} \left|u^*(t) - u_N^*(t)\right| \leq c\varepsilon_N\,, \quad N \geq N_0\,, \quad (27)$$

where c is a positive constant not dependending on N and ε_N is defined by the formulas (14)–(18).

Proof. Due to Theorem 1 we have to prove only the estimate (27). By Lemma 1, $u^* \in C_{d,p}^{m,\nu}[0,b]$. Now the estimate (27) follows from Lemma 3 and the inequality (26).

5 Superconvergence phenomenon

Theorem 2 suggests that by using a collocation method based on piecewise polynomials of degree $m-1$ ($m \geq 1$) and graded grids of type (5), one can reach a convergence order

$$\sup_{0\leq t\leq b} \left|u^*(t) - u_N^*(t)\right| \leq cN^{-m}\,, \quad N \geq N_0\,, \quad (28)$$

for sufficiently large values of grid parameters r and r_d, see (14)–(18) and (27).

In (28) the order m cannot be improved, whereas piecewise polynomials of the order $m-1$ are used for the approximation. Nevertheless, as it will be seen from Theorem 3 below, the convergence order at the collocation points will be higher than $O(N^{-m})$ for a special choice of collocation parameters (9). Actually, we shall assume that the points (9) are the nodes of a quadrature formula

$$\int_{-1}^{1} g(s)ds = \sum_{k=1}^{m} w_k g(\eta_k) + R_m(g)\,, \qquad (29)$$

$$-1 \le \eta_1 < \ldots < \eta_m \le 1\,,$$

which is exact for all polynomials of degree m.

Note that the weights w_k $(k = 1, \ldots, m)$ will not be used in our algorithms. The existence of a quadrature formula (29) which is exact for polynomials of degree m is used in the proof of the following

Theorem 3. *Let $\nu \in \mathbb{R}$, $\nu < 1$, $m \in \mathbb{N}$, $0 < d < b$, $p \in \{0, 1, \ldots, m+1\}$. Assume that the following conditions are fulfilled.*

(i) The kernel $K(t,s) = g(t,s)\kappa(t,s)$ in equation (1) *satisfies the conditions (A1) and (A2) with $m+1$ instead of m.*

(ii) $f \in C^{m+1,\nu}_{d,p}[0,b]$.

(iii) The integral equation (20) *has only the trivial solution $u = 0$.*

(iv) The collocation points (8) with grid points (5) and parameters (9) are used, where r and r_d are chosen so that
if $m < 1-\nu$, $p \ge 0$, then $r \ge 1$, $r_d \ge 1$;
if $m = 1-\nu$, $p = 0$, then $r > 1$, $r_d > 1$;
if $m = 1-\nu$, $p \ge 1$, then $r > 1$, $r_d \ge 1$;
if $1-\nu+p > m > 1-\nu$, $p \ge 1$, then $r \ge \dfrac{m}{1-\nu}$, $r_d \ge 1$;
if $m = 1-\nu+p$, $p \ge 1$, then $r \ge \dfrac{m}{1-\nu}$, $r_d > 1$;
if $m > 1-\nu+p$, $p \ge 0$, then $r \ge \dfrac{m}{1-\nu}$, $r_d \ge \dfrac{m}{1-\nu+p}$.

(v) The quadrature formula (29) is exact for all polynomials of degree m.

Then for all sufficiently large N, say $N \ge N_0$, the collocation conditions (19) *determine a unique approximation $u_N^* \in S^{(-1)}_{m-1}(\Delta_N)$ to $u^* \in C[0,b]$, the exact solution of equation (1). For $N \ge N_0$ the following error estimate holds:*

$$\max_{q=1,\ldots,m;\,j=1,\ldots,N} \left| u_N^*(\xi_{j,q}) - u^*(\xi_{j,q}) \right| \le$$

$$cN^{-m} \begin{cases} N^{-1}, & \text{if} \quad \nu < 0, \\ N^{-1}\ln N, & \text{if} \quad \nu = 0, \\ N^{-(1-\nu)}, & \text{if} \quad \nu > 0. \end{cases}$$

Here c is a positive constant which is independent of N.

Proof. See [2,3].

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