Choice of the regularization parameter in ill-posed problems with rough estimate of the noise level of data

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Abstract: – We consider linear ill-posed problems $Au = f$ in Hilbert spaces. Regularized approximations u_r to solutions u_* of problem $Au = f$ are obtained by a general regularization scheme, including the Tikhonov method, iterative and other methods. We assume that instead of $f \in \mathcal{R}(A)$ noisy data $\tilde{f}$ are available with the approximately given noise level δ: it holds $\|\tilde{f} - f\|/\delta \le C$ for $\delta \to 0$, but $C = \text{const}$ is unknown. We propose a new a-posteriori rule for the choice of the regularization parameter $r = r(\delta)$ guaranteeing $u_{r(\delta)} \to u_*$ for $\delta \to 0$. Note that such convergence is not guaranteed for the parameter choice by the discrepancy principle $\|Au_r - \tilde{f}\| = b\delta$ with $b < C$ and for parameter choice rules which do not use the noise level (quasioptimality criterion, Wahba's generalized cross-validation rule, Hansen's L-curve rule). We give error estimates which in case $\|\tilde{f} - f\| \le \delta$ are quasioptimal and order-optimal.

Key words: – ill-posed problem, noise level, regularization, parameter choice, convergence, discrepancy principle, L-curve.

1 Introduction

We consider an operator equation

$$Au = f\,, \quad f \in \mathcal{R}(A)\,, \tag{1}$$

where $A \in \mathcal{L}(H, F)$ is the linear continuous operator between real Hilbert spaces H and F. In general our problem is ill-posed (see [16,18]): the range $\mathcal{R}(A)$ may be non-closed, the kernel $\mathcal{N}(A)$ may be non-trivial. We suppose that instead of the exact data f we have only an approximation $\tilde{f} \in H$ with noise $\tilde{f} - f$.

The approximate solution u_r of the ill-posed problem $Au = f$ is found by some regularization method and depends on the regularization parameter r. The important problem is how to choose the proper regularization parameter r. If there is some information about the noise level of the data, this information should be used for the choice of r. Consider now the choice of r in situations with a different amount of information about $\|\tilde{f} - f\|$.

Case 1. Full information about the noise level is known: the exact noise level δ with $\|\tilde{f} - f\| \le \delta$ is given. Then the proper parameter choice $r = r(\delta)$ guarantees $u_{r(\delta)} \to u_*$ for $\delta \to 0$, where u_* is the solution of $Au = f$, the nearest to the initial approximation u_0 (see Section 2; often $u_0 = 0$). In this situation proper rules for the choice of r are the discrepancy principle [9,17,18] and its modification [10] (the Raus-Gfrerer rule [2,11] in case of non-selfadjoint problem) and the monotone error rule [6,14].

Case 2. There is no information about noise level. In this case parameter r may be chosen by the quasioptimality criterion [15,16], by the GCV-rule [3,19], by the L-curve rule [8] or by rule of [7]. The serious disadvantage of these rules is that convergence $u_{r(\delta)} \to u_*$ for $\delta \to 0$ is not guaranteed (see [1]).

In applied inverse and ill-posed problems the situation is often between extreme cases 1, 2: some approximate δ is known, but it is unknown, if the inequality $\|\tilde{f} - f\| \le \delta$ holds or not. In this paper we are interested in the case of approximately given noise level δ: instead of the inequality $\|\tilde{f} - f\| \le \delta$ we assume that $\|\tilde{f} - f\|/\delta \le C$ for $\delta \to 0$, where C is an unknown constant. We give a rule for the parameter choice $r = r(\delta)$ guaranteeing $u_{r(\delta)} \to u_*$ for $\delta \to 0$. For self-adjoint problems this rule was lately proposed in [4,5], where convergence is also proven.

2 Regularization methods

We consider the regularization methods in the general form (see [17,18]), using in case $F = H$, $A = A^* \geq 0$ (referred later as the selfadjoint case) the approximation

$$u_r = (I - Ag_r(A))u_0 + g_r(A)\tilde{f}\,, \qquad (2)$$

in general case (in non-selfadjoint case) the approximation

$$u_r = (I - A^*Ag_r(A^*A))u_0 + g_r(A^*A)A^*\tilde{f}\,. \quad (3)$$

Here u_0 is the initial approximation, I is the identity operator and the function $g_r(\lambda)$ satisfies the conditions (4)–(6):

$$\sup_{0\leq\lambda\leq a} \left|g_r(\lambda)\right| \leq \gamma r\,, \quad r \geq 0\,, \qquad (4)$$

$$\sup_{0\leq\lambda\leq a} \lambda^p\left|1-\lambda g_r(\lambda)\right| \leq \gamma_p r^{-p},\ r\geq 0\,, 0\leq p\leq p_0, \qquad (5)$$

$$\sup_{0\leq\lambda\leq a} \sqrt{\lambda}|g_r(\lambda)| \leq \gamma_*\sqrt{r},\ r\geq 0. \qquad (6)$$

Here p_0, γ, γ_p and γ_* are positive constants, $a \geq \|A\|$ for the approximation (2) and $a \geq \|A^*A\|$ for the approximation (3), $\gamma_0 \leq 1$ and the greatest value of p_0, for which the inequality (5) holds is called the qualification of method.

The following pairs of regularization methods are special cases of general methods (2), (3) for problems with $H = F$, $A = A^* \geq 0$ and for general problems respectively.

M1 The Lavrentiev method $u_\alpha = (\alpha I + A)^{-1}\tilde{f}$ and the Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^*f$. Here $u_0 = 0$, $r = \alpha^{-1}$, $g_r(\lambda) = (\lambda+r^{-1})^{-1}$, $p_0 = 1$, $\gamma = 1$, $\gamma_p = p^p(1-p)^{1-p}$, $\gamma_* = 1/2$.

M2 The iterative variants of the Lavrentiev method and of the Tikhonov method. Let $m \in \mathbb{N}$, $m \geq 1$, $u_0 = u_{0,\alpha} \in H$ – initial approximation and $u_{n,\alpha} = (\alpha I + A)^{-1}(\alpha u_{n-1,\alpha} + \tilde{f})$ $(n = 1,\ldots,m)$ (method (2)), $u_{n,\alpha} = (\alpha I + A^*A)^{-1}(\alpha u_{n-1,\alpha} + A^*\tilde{f})$ $(n = 1,\ldots,m)$ (method (3)). Here $r = \alpha^{-1}$, $g_r(\lambda) = \frac{1}{\lambda}\left(1-\left(\frac{1}{1+r\lambda}\right)^m\right)$, $p_0 = m$, $\gamma = m$, $\gamma_p = (p/m)^p(1-p/m)^{m-p}$, $\gamma_* = \sqrt{m}$.

M3 Explicit iteration scheme (the Landweber's method). Let $u_n = u_{n-1}-\mu(Au_{n-1}-\tilde{f})$, $\mu \in (0, 1/\|A\|)$, $n = 1,2,\ldots$ (method (2)), $u_n = u_{n-1} - \mu A^*(Au_{n-1} - \tilde{f})$, $\mu \in (0, 1/\|A^*A\|)$, $n = 1,2,\ldots$ (method (3)). Here $r = n$, $g_r(\lambda) = \frac{1}{\lambda}\left(1-(1-\mu\lambda)^r\right)$, $p_0 = \infty$, $\gamma = \mu$, $\gamma_p = (p/(\mu e))^p$, $\gamma_* = \sqrt{\mu}$.

M4 Implicit iteration scheme. Let $\alpha > 0$ be a constant and $\alpha u_n + Au_n = \alpha u_{n-1} + \tilde{f}$, $n = 1,2,\ldots$ (method (2)), $\alpha u_n + A^*Au_n = \alpha u_{n-1} + A^*\tilde{f}$, $n = 1,2,\ldots$ (method (3)). Here $r = n$, $g_r(\lambda) = \frac{1}{\lambda}\left(1-\left(\frac{\alpha}{\alpha+\lambda}\right)^r\right)$, $p_0 = \infty$, $\gamma = 1/\alpha$, $\gamma_p = (\alpha p)^p$, $\gamma_* = b_0/\sqrt{\alpha}$, where $b_0 = \sup\limits_{0<\lambda<\infty} \lambda^{-1/2}(1-e^{-\lambda}) \approx 0.6382$.

M5 The method of the Cauchy problem: approximation u_r solves the Cauchy problem $u'(r) + Au(r) = \tilde{f}$, $u(0) = u_0$ (method (2)), $u'(r) + A^*Au(r) = A^*\tilde{f}$ (method (3)). Here $g_r(\lambda) = \frac{1}{\lambda}\left(1-e^{-r\lambda}\right)$, $p_0 = \infty$, $\gamma = 1$, $\gamma_p = (p/e)^p$, $\gamma_* = b_0$.

3 Parameter choice for exactly given noise level of data

In regularization methods (2), (3) the error $u_r - u_*$ depends crucially on the choice of a regularization parameter r. If r is too small, the approximation error is large and if r is too large, the error is large due to noise.

At first we consider the choice of r in the case when the exact noise level δ with $\|\tilde{f} - f\| \leq \delta$ is known. Then the most prominent rule for the Tikhonov method and for methods M2–M5 is the discrepancy principle [9,17,18], where the regularization parameter $r = r_D$ is chosen as the solution of the equation $\left\|Au_r - \tilde{f}\right\| \approx b\delta$ with $b = \text{const} > 1$. The second rule in the case of known δ is the modification of the discrepancy principle [10] (the Raus-Gfrerer rule [2,11] in non-selfadjoint case). In this rule the regularization parameter $r = r_{MD}$ is chosen as the solution of the equation $\left\|B_r\left(Au_r - \tilde{f}\right)\right\| \approx b\delta$ with $b = \text{const} > 1$,

$$B_r = \begin{cases} I\,, & \text{if } p_0 = \infty, \\ (K_r(A))^{1/p_0} \text{ for appr. (2)}, & \text{if } p_0 < \infty, \\ (K_r(AA^*))^{1/(2p_0)} \text{ for appr. (3)}, & \text{if } p_0 < \infty, \end{cases}$$

where $K_r(A) = I - Ag_r(A)$.

In the Tikhonov method and in the iterated Tikhonov method the Raus-Gfrerer rule and the monotone error rule [6,14] choose the regularization parameters α_{RG} and α_{ME} as the solutions of the equations $(Au_{m,\alpha} - \tilde{f}, Au_{m+1,\alpha} - \tilde{f}) = b\delta$, $b \geq 1$, $(Au_{m,\alpha} - \tilde{f}, Au_{m+1,\alpha} - \tilde{f})/\|Au_{m+1,\alpha} - \tilde{f}\| = \delta$ respectively. Note that in these methods always $\alpha_{\mathrm{ME}} \leq \alpha_{\mathrm{RG}}$, $\|u_{\alpha_{\mathrm{ME}}} - u_*\| \leq \|u_{\alpha_{\mathrm{RG}}} - u_*\|$.

All rules what we considered guarantee convergence $\|u_r - u_*\| \to 0$ for $\delta \to 0$ and order-optimality: if $u_0 - u_* = (A^*A)^{p/2}v$, $v \in H$, $\|v\| \leq \varrho$, $p > 0$, then $\|u_r - u_*\| \leq C_p \varrho^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}$, where in self-adjoint case $p \in (0, p_0 - 1]$ for $r = r_D$ and $p \in (0, p_0]$ for $r = r_{\mathrm{MD}}$, in non-selfadjoint case $p \in (0, 2p_0 - 1]$ for $r = r_D$ and $p \in (0, 2p_0]$ for $r = r_{\mathrm{RG}} = 1/\alpha_{\mathrm{RG}}$ and for $r = r_{\mathrm{ME}} = 1/\alpha_{\mathrm{ME}}$. The errors of the approximations (2), (3) have corresponding forms

$$u_r - u_* = K_r(A)(u_0 - u_*) + g_r(A)(\tilde{f} - f),$$
$$u_r - u_* = K_r(A^*A)(u_0 - u_*) + g_r(A^*A)A^*(\tilde{f} - f) \tag{7}$$

and in the case $\|\tilde{f} - f\| \leq \delta$ relations (4), (6) yield corresponding estimates

$$\|u_r - u_*\| \leq \|K_r(A)(u_0 - u_*)\| + \gamma r\delta \ (\forall\, r \geq 0),$$
$$\|u_r - u_*\| \leq \|K_r(A^*A)(u_0 - u_*)\| + \gamma_* \sqrt{r}\delta \ (\forall\, r \geq 0). \tag{8}$$

If some rule for choice of the regularization parameter gives parameter $r(\delta)$, which nearly minimizes the corresponding estimate, i.e.

$$\|u_{r(\delta)} - u_*\| \leq \mathrm{const} \inf_{r \geq 0} \{\|K_r(A)(u_0 - u_*)\| + \gamma r\delta\},$$
$$\|u_{r(\delta)} - u_*\| \leq \mathrm{const} \inf_{r \geq 0} \{\|K_r(A^*A)(u_0 - u_*)\| + \gamma_* \sqrt{r}\delta\},$$

then this rule is called quasioptimal. Quasioptimal are the MD-rule, the RG-rule and the monotone error rule. The discrepancy principle is not quasioptimal for methods with finite qualification ($p_0 < \infty$).

It is obvious that if a rule is quasioptimal for method (2) or (3), then this rule is order-optimal for all $p \in (0, p_0]$ or for all $p \in (0, 2p_0]$ respectively.

All these rules are unstable in this sense that if the norm of the actual noise in data is only slightly larger than $b\delta$, then the error of the approximate solution may be arbitrarily large, irrespective of the value of the ratio of the actual and supposed noise level.

There are also heuristic parameter choice rules which do not use the noise level δ: the quasioptimality criterion [15,16], the Wahba's generalized cross-validation rule [3,19], the Hansen's L-curve rule [8] and the rules of [7].

Heuristic rules often work well, but as shown by Bakushinskii [1], one cannot prove the convergence of the approximate solution.

4 Parameter choice for roughly given noise level

In applied ill-posed problems the exact noise level is often unknown. Therefore in the following we assume that only rough supposed error level $\delta > 0$ is given, but we do not know exactly, if $\|\tilde{f} - f\| \leq \delta$ holds or not. We give the rule for the stable parameter choice which guarantees the convergence of the approximate solution to the exact solution if only the ratio $\|\tilde{f} - f\|/\delta$ is bounded in the process $\delta \to 0$:

$$\|\tilde{f} - f\|/\delta \leq C = \mathrm{const}\ (\delta \to 0). \tag{9}$$

Let us introduce the function

$$\varphi(r) = \begin{cases} \sqrt{r}\|A^{1/2}B_r^{3/2}(Au_r - \tilde{f})\| & \text{for appr. (2)}, \\ \sqrt{r}\|A^*B_r^2(Au_r - \tilde{f})\| & \text{for appr. (3)}. \end{cases}$$

We introduce also the constant $\tilde{\gamma}$ as follows: if the qualification of the method is $p_0 = \infty$, then $\gamma = \gamma_{1/2}$; if $p_0 < \infty$, then $\tilde{\gamma} = [\gamma_{p_0/(3+2p_0)}]^{1+3/(2p_0)}$ for the approximation (2) and $\tilde{\gamma} = [\gamma_{p_0/(2+2p_0)}]^{1+1/p_0}$ for the approximation (3).

Note that in m times iterated Tikhonov method $\varphi(r) = \varphi(\alpha^{-1}) = \frac{1}{\sqrt{\alpha}}\|A^*(Au_{m+1,\alpha} - \tilde{f})\|$.

Rule R. Let $b_2 \geq b_1 > \tilde{\gamma}$ and $s \in [0, 1]$ for the approximation (2), $s \in [0, 1/2]$ for the approximation (3). If $\varphi(1) \leq b_2\delta$ then choose $r(\delta) = 1$. In the contrary case we find at first $r_2(\delta) > 1$ such that

$$\varphi(r_2(\delta)) \leq b_2\delta\,, \tag{10}$$

$$\varphi(r) \geq b_1\delta \qquad \forall\, r \in [1, r_2(\delta)]\,. \tag{11}$$

For the regularization parameter $r(\delta)$ we choose the parameter r, for which the function $t(r) = r^s\|B_r(Au_r - \tilde{f})\|$ has the global minimum on the interval $[1, r_2(\delta)]$.

Let us reformulate the rule R for the choice of the stopping index $n(\delta)$ as the parameter r in iterative methods. For this rule R' the analogous results hold as for the rule R.

Rule R'. Let $s \in [0,1]$, $s \in [0,1/2]$ for approximations (2), (3) respectively. Let b be the constant such that $b > \tilde{\gamma}$. Find $n_2(\delta)$ as the first $n = 1, 2, \ldots$, for which $\varphi(n) \leq b\delta$. For the regularization parameter $n(\delta)$ we choose $n \in \mathbb{N}$, for which the function $t(n) = n^s \|Au_n - \tilde{f}\|$ has the global minimum on the interval $[1, n_2(\delta)]$.

Rule R is similar to the rules in [12,13,15,16]. In [12,13] for the regularization parameter the parameter $r_2(\delta)$ was taken. Rule R can be considered as the generalization of rules [12,13], since in case $s = 0$ these rules coincide, while the function $\|B_r(Au_r - \tilde{f})\|$ is monotonically decreasing with respect to r. For non-selfadjoint problems the regularization parameter is chosen in the quasioptimality criterion [15,16] as the global minimizer of the function $r\|B_r(Au_r - \tilde{f})\|$, in rule R as the minimizer of the function $r^s\|B_r(Au_r - \tilde{f})\|$ with $s \in [0, 1/2]$ on the interval $[1, r_2(\delta)]$.

In [12,13] for methods M1–M5 the following results are proven: for each $\tilde{f} \in F$ we have $\lim\limits_{r\to\infty} \varphi(r) = 0$; if $\frac{\|\tilde{f}-f\|}{\delta} \leq$ const for $\delta \to 0$ then $\|u_{r_2(\delta)} - u_*\| \to 0$ for $\delta \to 0$. The first result and the continuity of the function $\varphi(r)$ guarantee that the choice of finite parameters $r_2(\delta)$ and $r(\delta) \leq r_2(\delta)$ according to Rule R is possible. Note that the function $\varphi(r)$ may be nonmonotone and therefore in Rule R we must use the conditions (10)–(11) instead of the inequalities $b_1\delta \leq \varphi(r) \leq b_2\delta$.

The following convergence result is proven in [4,5] for the approximation (2) and in this paper we prove it for the approximation (3).

Theorem 1. *If* $\frac{\|\tilde{f}-f\|}{\delta} \leq$ const *in the process* $\delta \to 0$, *then in methods M1–M5 rule R guarantees convergence*

$$\left\|u_{r(\delta)} - u_*\right\| \to 0 \quad \textit{for} \quad \delta \to 0\,.$$

In the following theorem we give the error estimate, using notation

$$\psi(r) := \|K_r(A^*A)(u_0 - u_*)\| + \gamma_*\sqrt{r}\max\{\delta, \|\tilde{f}-f\|\}$$

(compare (8).

Theorem 2. *Let* $A \in \mathcal{L}(H,F)$, $f \in \mathcal{R}(A)$. *Let the parameter* $r(\delta)$ *in approximation (3) be chosen according to Rule R with* $s \in (0, 1/2)$. *Then for methods M1–M5 the following error estimates hold*

1. If $\|\tilde{f} - f\| \leq \max\{\delta, \delta_0\}$, *where* $\delta_0 := \frac{1}{2}\|B_{r(\delta)}(Au_{r(\delta)} - \tilde{f})\|$, *then*

$$\|u_{r(\delta)} - u_*\| \leq C(b_1, b_*, d_*)\frac{1}{1-2s}\inf_{r\geq 0}\psi(r). \quad (12)$$

Here

$$d_* = \max_{r,r',r(\delta)\leq r\leq r'\leq r_2(\delta)+1} \frac{(r/\varrho)^s\|B_{r/\varrho}(Au_{r/\varrho} - \tilde{f}\|}{(\varrho r')^s\|B_{\varrho r'}(Au_{\varrho r'} - \tilde{f})\|},$$

$b_* = \max\limits_{r(\delta)\leq r\leq R(\delta)} \varphi(r)/\delta \geq b_2$, *$R(\delta)$ is the greatest parameter for which* $\varphi(r) = b_2\delta$ *and* $\varrho = 1,\ 1+(2m/(2m+1))^{2m+1}/8,\ 1+b_0^2/2e,\ 1+b_0^2/2,\ 1+b_0^2/2e$ *for methods M1–M5 respectively,* $b_0 \equiv \sup\limits_{0<\lambda<\infty} \lambda^{-1/2}(1-e^{-\alpha}) \approx 0.6382.$

2. If $\max\{\delta, \delta_0\} < \|\tilde{f}-f\| \leq \frac{1}{2}\|B_1(Au_1 - \tilde{f})\|$, *then*

$$\|u_{r(\delta)} - u_*\| \leq C\Big(\frac{\|\tilde{f}-f\|}{\delta_0}\Big)^{1/2s}\inf_{r\geq 0}\psi(r). \quad (13)$$

The proof of Theorem 2 will be presented in a forthcoming paper.

Proof of Theorem 1. We have from(7) due to (6) and (9) that

$$\|u_{r(\delta)} - u_*\| \leq \|K_{r(\delta)}(A^*A)(u_0 - u_*)\| + C\gamma_*\sqrt{r(\delta)}\delta \quad (14)$$

(compare (8). To prove the theorem, it suffices to show the convergence of the right-hand side of (14). In [13] is proved that

$$\sqrt{r_2(\delta)}\delta \to 0 \quad \text{if} \quad \delta \to 0. \quad (15)$$

From (15) and from the inequality $r(\delta) \leq r_2(\delta)$, follows the convergence of the second term of (14).

To show the convergence of the first term of (14) we consider separately the cases a) $r(\delta) \to \infty$ ($\delta \to \infty$), b) $r(\delta) \leq \overline{r} =$ const ($\delta \to 0$). If $r(\delta) \to \infty$ in process $\delta \to 0$ then using the Banach-Steinhaus theorem it is easy to show that $\|K_{r(\delta)}(A^*A)(u_0 - u_*)\| \to 0$. Consider now the case b) $r(\delta) \leq \overline{r} =$ const ($\delta \to 0$). Then we prove at first that

$$r_2^s(\delta)\|B_{r_2(\delta)}(Au_{r_2(\delta)} - \tilde{f})\| \to 0, \text{ if } \delta \to 0. \quad (16)$$

We have

$$Au_r - \tilde{f} = AK_r(A^*A)(u_0 - u_*) - K_r(AA^*)(\tilde{f} - f), \quad (17)$$

from which with regard the inequality $\|B_r K_r(AA^*)(\tilde{f} - f)\| \leq \|\tilde{f} - f\| \leq C\delta$ follows that

$$r_2^s(\delta)\|B_{r_2(\delta)}(Au_{r_2(\delta)} - \tilde{f})\| \leq$$
$$r_2^s(\delta)\|B_{r_2(\delta)}AK_{r_2(\delta)}(A^*A)(u_0 - u_*)\| + r_2^s(\delta)C\delta. \quad (18)$$

To show the convergence

$$r_2^s(\delta)\|B_{r_2(\delta)}AK_{r_2(\delta)}(A^*A)(u_0 - u_*)\| \to 0 \ (\delta \to 0) \quad (19)$$

we consider separately the cases a) $r_2(\delta) \to \infty$ $(\delta \to 0)$, b) $r_2(\delta) \leq \overline{r} = \text{const}$ $(\delta \to 0)$. If $r_2(\delta) \to \infty$ in the process $\delta \to 0$ then using the Banach-Steinhaus theorem we can prove similarly as in [18] (p.45) that $r^p\|B_r AK_r(A^*A)(u_0 - u_*)\| \to 0$ if $r \to \infty$ $(0 \leq p \leq 1/2)$. Now consider the case $r_2(\delta) \leq \overline{r} = \text{const}$. Using (17), (5) we get

$$r_2(\delta)^{1/2}\|A^*B^2_{r_2(\delta)}AK_{r_2(\delta)}(A^*A)(u_0 - u_*)\| \leq$$
$$r_2(\delta)^{1/2}\|A^*B^2_{r_2(\delta)}(Au_{r_2(\delta)} - \tilde{f})\| +$$
$$r_2(\delta)^{1/2}\|A^*B^2_{r_2(\delta)}K_{r_2(\delta)}(AA^*)(\tilde{f} - f\|)\| \leq$$
$$b_2\delta + \gamma_{1/2}\|\tilde{f} - f\| \leq (b_2 + C\gamma_{1/2})\delta$$

from which follows that

$$\|A^*B^2_{r_2(\delta)}AK_{r_2(\delta)}(A^*A)(u_0 - u_*)\| \to 0 \text{ if } \delta \to 0.$$

In [18] (p.66) the implication

$$AK_{r_n}(A^*A)(u_0 - u_*) \to 0 \ (n \to \infty) \Rightarrow$$
$$K_{r_n}(A^*A)(u_0 - u_*) \to 0 \ (n \to \infty) \quad (20)$$

is proven. Similarly we can show that if $A^*B^2_{r_n}AK_{r_n}(A^*A)(u_0 - u_*) \to 0$ $(n \to \infty)$, then $B^2_{r_n}AK_{r_n}(A^*A)(u_0 - u_*) \to 0$ $(n \to \infty)$. From the last convergence and from the inequality of moments

$$\|B_r AK_r(A^*A)(u_0 - u_*)\| \leq$$
$$\|B_r^2 AK_r(A^*A)(u_0 - u_*)\|^{\frac{1}{2}}\|AK_r(A^*A)(u_0 - u_*)\|^{\frac{1}{2}}$$
$$\leq \|B_r^2 AK_r(A^*A)(u_0 - u_*)\|^{\frac{1}{2}}(\gamma_{\frac{1}{2}} r^{-1/2}\|u_0 - u_*)\|)^{\frac{1}{2}}$$

follows the convergence (19) in case $r_2(\delta) \leq \text{const}$. Now the convergence (16) follows from (18), (19) and (15).

Taking into account the fact that the parameter $r(\delta)$ is the global minimum point of the function $t(r) = r^s\|B_r(Au_r - \tilde{f})\|$ in $[1, r_2(\delta)]$, from (16) follows the convergence

$$r^s(\delta)\|B_{r(\delta)}(Au_{r(\delta)} - \tilde{f})\| \to 0, \text{ if } \delta \to 0.$$

Using (17) we get

$$r^s(\delta)\|B_{r(\delta)}AK_{r(\delta)}(A^*A)(u_0 - u_*)\| \leq$$
$$r^s(\delta)\|B_{r(\delta)}(Au_{r(\delta)} - \tilde{f})\| + r^s(\delta)C\delta \to 0, \text{ if } \delta \to 0.$$

From this relation with implication of type (20) and with use of the inequality of moments we get the convergence $\|K_{r(\delta)}(A^*A)(u_0 - u_*)\| \to 0$ for $\delta \to 0$ which with (14) proves the theorem.

In the following Remarks 1–3 we consider rule R for the approximation (2).

Remark 1. Note that for approximation (2) the analogue of Theorem 2 holds, where in the estimates (12), (13) $2s$ is replaced by s and in the definition of d_* ratio r/ϱ is replaced by r.

Remark 2. If the function $t(r) = r^s\|B_r(Au_r - \tilde{f})\|$ is monotonously increasing on the interval $[r(\delta), \varrho r_2(\delta) + 1]$, then $d_* \leq 1/\varrho^s$. In most of numerical examples we had $d_* \leq 1$.

Remark 3. One can show that in methods M1, M2, M3 and M5 coefficient $c(b_1, b_*, d_*) \leq 2.5$, if $b_1 = b_2 = 1.5\tilde{\gamma}$, $b_* = b_2$, $d_* \leq 1/\varrho^s$.

5 Conclusion

For the choice of the regularization parameter r it is recommendable to use the noise level, while heuristic rules as the L-curve rule, the GCV-rule etc do not guarantee the convergence of the approximations. If the noise level is given only approximately and inequality $\|\tilde{f} - f\| \leq \delta$ is not guaranteed, the discrepancy principle and its modification are unstable. If δ with $\|\tilde{f} - f\|/\delta \leq \text{const}$ for $\delta \to 0$ is given, we recommend to use our rules R and R', guaranteeing convergence and in case $\|\tilde{f} - f\| \leq \delta$ also quasioptimal error estimates.

Note that for increasing parameter $s \in (0, 1/2)$ the error estimate (12) increases and estimate (13) decreases. Therefore, if we are almost sure in inequality $\|\tilde{f} - f\| \leq \delta$, smaller values of s are recommended.

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