Approximation of Cauchy problems for elliptic equations using the method of lines

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Abstract: - In this contribution we deal with the development, theoretical examination and numerical examples of a method of lines approximation for the Cauchy problem for elliptic partial differential equations. We restrict ourselves to the Laplace equation. A more general elliptic equation containing a diffusion coefficient will be considered in a forthcoming paper. Our main results are the regularization of the illposed Cauchy problem and the proof of error estimates leading to convergence results for the method of lines. These results are based on the conditional stability of the continuous Cauchy problem and an approximation by appropriately chosen finite-dimensional spaces, onto which the possibly perturbed Cauchy data are projected. At the end of this paper we present and discuss results of some of our numerical computations. There are multiple applications in material sciences, thermodynamics, medicine etc.; related problems are shape optimization problems which are important for nondestructive testing, e.g. for crack location, thermal tomography and other applications.

Key-Words: - Cauchy problem, ellipitc partial differential equation, illposed problem, method of lines

1 The Cauchy problem for Poisson's equation

We consider the following Cauchy problem for Poisson's equation on a rectangle

$$\Delta u = f \text{ in } \Omega = (0,1) \times (0,L) \tag{1}$$

with given boundary conditions

$$u = f_i \text{ on } \Sigma_i,\ i = 1,2,3,\ \frac{\partial u}{\partial y} = \phi_1 \text{ on } \Sigma_1, \tag{2}$$

where

$$\begin{aligned}
\Sigma_1 &= \big\{(x,0) \in \mathbb{R}^2 \,|\, 0 \le x \le 1\big\} \\
\Sigma_2 &= \big\{(0,y) \in \mathbb{R}^2 \,|\, 0 \le y \le L\big\} \\
\Sigma_3 &= \big\{(1,y) \in \mathbb{R}^2 \,|\, 0 \le y \le L\big\} \\
\Sigma_4 &= \big\{(x,L) \in \mathbb{R}^2 \,|\, 0 \le x \le 1\big\}.
\end{aligned}$$

Here, one tries to identify u and $\partial u/\partial y$ on Σ_4. The functions f_1, ϕ_1 are the given *Cauchy data.* This is a well-known improperly posed problem. In 1923 J. Hadamard [6] has given a classical example showing that the solution of the problem is not continuously dependent on the Cauchy data. Without loss of generality, we can set $f = 0, f_2 = 0, f_3 = 0$. Otherwise, one has to solve a direct problem for Poisson's equation beforehand and add its solution to the solution of the Cauchy problem with the vanishing $f's$.

It is impossible to solve this improperly posed problem by the classical theory of partial differential equations and, therefore, it has required the attention of many mathematicians in the last 50 years. M. M. Lavrent'ev [11] has discussed bounded solutions of the Laplace equation with the Cauchy data in a special two-dimensional domain where the bounded solutions depend continuously on the Cauchy data. Fursikov [5] has extended this approach later to domains in $\mathbb{R}^n$ proving an optimal stability estimate with respect to the H^0-norm. The latter is analogous to Hadamard's classical estimate for analytic functions which forms the content of the three-circles theorem. L. E. Payne [12], [13] studied solutions of more general second-order elliptic equations which are continuously dependent on the Cauchy data under some restrictions on the domains and on the solutions. In 1975, L. E. Payne outlined this problem in [14]. H. Han has considered the problem (1), (2) in [7] and gave an $H^0(\Omega)$-stability estimate.

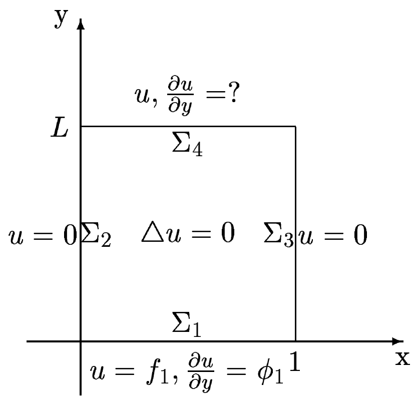

Figure 1: The Cauchy problem for the Laplace equation

R. S. Falk [2] presented a three-lines theorem for the two-dimensional Laplace equation with the Cauchy data, for which a certain stability estimate is given. M. Kubo [10] obtained an H^0-stability estimate for the Cauchy problem for the Laplace equation on a doubly connected bounded domain. K. S. Fayazov and M. M. Lavrent'ev [4] studied the Cauchy problem for elliptic equations with operator coefficients in space. Using the method of logarithmic convexity they proved uniqueness and H^0-stability estimate. In 1995, S. I. Kabanikhin and A. L. Karchevsky [9] presented an optimization method for solving the Cauchy problem for an elliptic equation numerically.

In the paper of H. Han and H.-J. Reinhardt [8] a series of stability estimates for the problem (1), (2) in Sobolev spaces are given, from which several regularization methods can be proposed for computing numerical approximations (s. the paper of H.-J. Reinhardt, H. Han and Hào [16]).

2 Method of lines approximation

It is well-known that elliptic equations can be approximated by using the method of lines. One has two choices namely lines parallel to the x- or y-axis. Our approach requires that the lines should be chosen parallel to the y-axis. With mesh points $x_i = ih, i = 0, \ldots, N, h = 1/N$, we approximate $\frac{\partial^2}{\partial x^2}$ in the Laplace operator in (1) by the central difference quotient of 2^{nd} order. Therefore, approximations $u_i(y)$ for the solution $u(x_i, y)$ of (1), (2) with $f_2 = 0, f_3 = 0, f = 0$ (s. Fig. 1) can be obtained by the solution of the following system of ordinary differential equations, $u_0 = u_N = 0$,

$$u_i'' + \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = 0 \tag{3}$$

with boundary conditions

$$u_i(0) = f_1(x_i), u_i'(0) = \phi_1(x_i) \tag{4}$$

for $i = 1, \ldots N-1$. This system can be decoupled using the eigenvalues and eigenvectors of the $\mathbb{R}^{N-1,N-1}$-matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \\ 1 & -2 & 1 & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 1 & -2 & 1 \\ & 0 & 1 & -2 \end{pmatrix}$$

which are, in this simple model problem, explicitly known,

$$\begin{aligned} \lambda_j &= -\frac{4}{h^2}\sin^2\left(jh\frac{\pi}{2}\right), \\ w^{(j)} &= (\sin jkh\pi)^\top_{k=1,\ldots,N-1}, \end{aligned}$$

$j = 1, \ldots, N-1$. With the orthogonal matrix $W = \left(\widehat{w}^{(1)}|\ldots|\widehat{w}^{(N-1)}\right)$ consisting of the normalized[1] eigenvektors $\widehat{w}^{(j)} = w^{(j)}/|w^{(j)}|_2$ as column vectors and the diagonal matrix $D = \text{diag}(\lambda_j)_{j=1,\ldots,N-1}$, the system (3) is equivalent to

$$V'' + DV = 0 \tag{5}$$

[1] w.r.t. $|.|_2$ =Euclidean norm

where $V = WU$, $U = (u_1, \dots, u_{N-1})$. Explicit solutions of such a system are well-known and can be written as

$$\begin{aligned} v_i(y) &= \xi_i \exp(\sqrt{-\lambda_i} y) + \eta_i \exp(-\sqrt{-\lambda_i} y), \\ & \qquad i = 1, \dots, N-1 \,. \end{aligned}$$

The boundary conditions at $y = 0$ determine the coefficients ξ_i, η_i,

$$\begin{aligned} \xi_i &= \sqrt{\tfrac{h}{2}} \sum_{j=1}^{N-1} \Big(\sin(ijh\pi) f_1(x_j) \\ & \quad + \tfrac{h}{2} \tfrac{\sin(ihj\pi)}{\sin\left(ih\frac{\pi}{2}\right)} \phi_1(x_j) \Big) \\ \eta_i &= \sqrt{\tfrac{h}{2}} \sum_{j=1}^{N-1} \Big(\sin(ijh\pi) f_1(x_j) \\ & \quad - \tfrac{h}{2} \tfrac{\sin(ihj\pi)}{\sin\left(ih\frac{\pi}{2}\right)} \phi_1(x_j) \Big) \end{aligned}$$

This yields the following explicit representation of the solution u_i on the $i-$th line,

$$\begin{aligned} & u_i(y) = (WV)_i(y) \\ = \; & 2h \cdot \sum_{k=1}^{N-1} \Big(\sin(ikh\pi) \big(\cosh(\sqrt{-\lambda_k} y) \cdot \\ & \sum_{j=1}^{N-1} \sin(kjh\pi) f_1(x_j) + \frac{h}{2 \sin\left(kh\frac{\pi}{2}\right)} \cdot \\ & \sinh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi) \phi_1(x_j) \big) \Big) \end{aligned}$$

In the special case $f_1 = 0$, the solution can be written as

$$u_i(y) = \sum_{k=1}^{N-1} h \frac{(\widehat{w}^{(k)}, \Phi_1)_2}{\sqrt{-\lambda_k}} \widehat{w}_k^{(i)} \sinh(\sqrt{-\lambda_k} y) \quad (6)$$

where $(\cdot,\cdot)_2$ denotes the Euclidean scalar product, $\Phi_1 = \big(\phi_1(x_1), \dots, \phi_1(x_{N-1})\big)^\top$ and $\widehat{w}_k^{(i)}$ are the components of $\widehat{w}^{(i)}$. If one chooses $\phi_1(x) = \sin(m\pi x)/m\pi$ for some $m < N$, one ends up with Hadamard's classical example (s. [6]). In this case, the convergence $u_i(y) \to u(x_i, y) (h \to 0)$ can be shown with the error estimate (cf. Charton [1], 3.1)

$$\begin{aligned} & |u(x_i, y) - u_i(y)| \\ \leq \; & |\sin(m\pi x_i)| \frac{m\pi y}{24} \exp(m\pi y) h^2, \quad (7) \\ & i = 1, \dots, n-1 \,. \end{aligned}$$

This example is a special case of the situation when the data functions f_1, ϕ_1 have truncated Fourier series, i.e. $f_1, \phi_1 \in D_M$ for some M where

$$\begin{aligned} D_M = \big\{ & \phi \in C^1(0,1) | \phi(0) = \phi(1) = 0, \\ & \int_0^1 \sin(k\pi s) \phi(s) \, ds = 0, k > M \big\}. \end{aligned}$$

For such f_1, ϕ_1 the problem is conditionally well-posed and the solution of the original problem is obtained by the formula

$$\begin{aligned} & u(x,y) = \\ & \sum_{k=1}^{M} \Big(2 \sin(k\pi x) \big((f_1(.), \sin(k\pi.))_{L_2} \cosh(k\pi y) \\ & + \frac{(\phi_1(.), \sin(k\pi.))_{L_2}}{k\pi} \sinh(k\pi y) \big) \Big) \qquad (8) \end{aligned}$$

For technical reasons, $N > M$ should be assumed. We do not deal with this situation further but mention that convergence with order $O(h^2)$ – as in (7) – can be shown. All details can be found in Charton [1], Chapter 3.

3 Conditional well-posedness and convergence under certain boundedness condition

Instead of data with truncated Fourier series, we now allow data such that the solution of the Cauchy problem (1), (2) remains bounded on Σ_4. This is the widely used condition to stabilize the problem. Using the technique of logarithmic convexity one can show that, with $f_1 = 0, f_2 = 0, f_3 = 0, f = 0$ under the assumption

$$\|u\|_{L_2(\Sigma_4)} \leq E \qquad (9)$$

one obtains the following stability estimate for the solution of (1), (2),

$$\|u(\cdot, y)\|_{L_2} \leq \max(L, 1) \|\phi_1\|_{L_2}^{1-y/L} E^{y/L} . \quad (10)$$

In this case the solution has the form (cf. (8))

$$u(x,y) = 2 \sum_{k=1}^{\infty} \frac{a_k}{k\pi} \sin(k\pi x) \sinh(k\pi y) \quad (11)$$

with $a_k = \left(\phi_1, \sin(k\pi)\right)_{L_2}, k \in \mathbb{N}$. For (9) it is required that ϕ_1 is such that the above series converges for all $(x,y) \in [0,1] \times [0,L]$. Based on the assumption of the convergence of $\sum_{k=1}^{\infty} a_k^2 \exp(2k\pi L)$ the logarithmic convexity of the square of the L_2-Norm of (11) can be proved (see [1] Chapter 4.1).

The convergence for $h \to 0$ of the method of lines approximation is assured by the following steps. First, the data function ϕ_1 – note, that we consider the case $f_1 = 0$ – is projected into the space D_M of functions of truncated Fourier sine series. We even allow perturbed data functions ϕ_1^ε such that $\|\phi_1 - \phi_1^\varepsilon\|_{L_2} \leq \varepsilon$. In this situation it is clear that, in general, $\phi_1^\varepsilon \notin D_M$ even if $\phi_1 \in D_M$. One has to estimate the projection error of the projected data and then the error between the true solution and the method of lines approximation with projected data in D_M. For the convergence, the magnitude of pertubations should depend on the discretization parameter by $h = O(\sqrt{\varepsilon})$ and the dimension of D_M has to be chosen in an optimal way. For the orthogonal projection $P_M : D \to D_M$ w.r.t. the L_2-scalar product, $D = \left\{\phi \in C^1(0,1) | \phi(0) = \phi(1) = 0\right\}$, one has the following estimate provided that (9) holds,

$$\begin{aligned} &\|\phi_1 - \phi_1^*\|_{L_2(\Sigma_1)} \qquad (12) \\ \leq\; &\frac{E}{L^2(1-\exp(-4\pi L))} \cdot \frac{M}{\exp(M\pi L)} . \end{aligned}$$

(Proof see [1], Chapter 4.2)

Besides u the solution of (1), (2), with unperturbed data ϕ_1, let further denote

u^* = solution of (1), (2) with $\phi_1^* = P_M\phi_1$

u_ε = solution of (1), (2) with ϕ^ε

u_ε^* = solution of (1), (2) with $(\phi_1^\varepsilon)^* = P_M\phi_1^\varepsilon$

$u_{i,\varepsilon}^*$ = solution of line method approximation on i−th line with data $((\Phi_1^\varepsilon)^*)_i = (\phi_1^\varepsilon)^*(x_i), i = 1,\ldots,N-1$.

$\overline{(u_\varepsilon^*)_h}$ = continuation of $u_{i,\varepsilon}^*(y)_{i=1,\ldots,N-1}^\top$ in D_M.

Using $u_{i,\varepsilon}^*$, the latter function is given by

$$\begin{aligned} &\overline{(u_\varepsilon^*)_h}(x,y) \\ =\; &\sum_{k=1}^{M} \left(2h \sum_{j=1}^{N-1} \sin(k\pi jh) u_{j,\varepsilon}^*(y)\right) \sin(k\pi x) \end{aligned}$$

The total error essentially consists of three parts which have to be estimated separately,

$$u - \overline{(u_\varepsilon^*)_h} = (u-u^*) + (u^* - u_\varepsilon^*) + (u_\varepsilon^* - \overline{(u_\varepsilon^*)_h})$$

Using (12) and the stability estimate (10), we obtain an estimate for the first part

$$\|(u-u^*)(.,y)\|_{L_2} \leq C_2(L) E \left(\frac{M}{\exp(M\pi L)}\right)^{1-\frac{y}{L}} \qquad (13)$$

In our case of Laplace's equation, the constant C_2 is given by

$$C_2(L) = \max(1,L) \max\left(1, C_1(L)\right) ,$$

where

$$C_1(L) \leq \frac{2}{L^2(1-\exp(-4\pi L))} .$$

For the second part one can show, that

$$\|(u^* - u_\varepsilon^*)(.,y)\|_{L_2} \leq C(y) \frac{\sqrt{8}}{\pi} \frac{\sinh(M\pi y)}{M} \varepsilon \qquad (14)$$

with $C^2(y) \geq (1-\exp(-\pi y))^{-1}$.

Finally the third part fulfills the estimate

$$\begin{aligned} &\|u_\varepsilon^* - \overline{(u_\varepsilon^*)_h}\|_{L_2} \qquad (15) \\ \leq\; &\frac{M^4\pi^3 y}{12} \exp(M\pi y) \|(\phi_1^\varepsilon)^*\|_{L_1} h^2 . \end{aligned}$$

All estimates hold for arbitrary $y \in [0,L]$ (Proof see [1], Chapter 4.2).

The sum of the three contributions on the right-hand sides of (13),(14),(15) now estimate the total error. It is now our aim to choose M and h such that, with $\varepsilon \to 0$, the total error converges to zero. In addition one tries to make the error bound as small as possible. The following theorem summarizes these efforts.

Theorem: *For the solution of (1),(2) let (9) be satisfied and let the series in (11) converge pointwise for every $(x,y) \in [0,1] \times [0,L]$ and uniformly in x. If one chooses*

$$M = \left\lceil \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\pi L} \right\rceil \quad , \quad h \leq \sqrt{\varepsilon}$$

then for every $y \in [0, L)$ the solutions $\overline{(u_\varepsilon^*)_h}$ of the line method approximations converge to u as $\varepsilon \to 0$ with the error estimate

$$\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2} \leq$$

$$C_2(L)E \cdot \left(\frac{\varepsilon \cdot \ln\left(\frac{1}{\varepsilon}\right)}{\pi L} + \varepsilon \right)^{1-\frac{y}{L}}$$

$$+\sqrt{8}C(y)\exp(\pi y)L\frac{\varepsilon^{1-\frac{y}{L}}}{\ln\left(\frac{1}{\varepsilon}\right)}$$

$$+\frac{y}{12\pi}\|(\phi_1)_\varepsilon^*\|_{L_1}\frac{\left(\ln\left(\frac{1}{\varepsilon}\right)+\pi L\right)^4\exp(\pi y)\varepsilon^{1-\frac{y}{L}}}{L^4} \quad .$$

The results can be generalized to the situation when a positive diffusion coefficient $a = a(x)$ multiplies the Laplace operator. The method of lines approximation can then be defined as above. However, for the stability and convergence analysis, the eigenvectors and eigenvalues are not explicitly known and have to be approximated and computed by some numerical method. The convergence of the eigenvalues and eigenvectors for $h \to 0$ requires the analysis of discrete Sturm-Liouville eigenvalue problems. Finally, similar results as above can be proved where the discrete L_2-scalar product has to be replaced by a weighted one using the coefficient $a(x)$.

4 Numerical results

Besides the theoretical analysis we also performed various numerical experiments for several examples. Our computations confirm the theoretical results, in particular the choice of the optimal parameter M. One of our examples is the classical Hadamard example

$$u(x,y) = \frac{\sin(m\pi x)\sinh(m\pi y)}{(m\pi)^2}, \; m \in \mathbb{N}$$

with corresponding exact boundary data

$$\begin{aligned} u(0,y) = u(1,y) &= 0 \quad \forall y \in [0,1] \\ u(x,0) &= 0 \quad \forall x \in (0,1) \\ \frac{\partial u}{\partial y}(x,0) &= \frac{\sin(m\pi x)}{m\pi} =: \phi_1 \quad . \end{aligned}$$

We added a random perturbation of maximum absolute value ε to the data ϕ_1. In the next step one has to dicretize the perturbed data and project it onto the space D_M^h, where we used different values for h and M. The solution of the method of lines can be computed in a very simple way by just putting the perturbed and then projected data into the explicit representation of the solution of the method of lines we presented above (see (6)). Note that now the sum consists of M terms only because of the preceding projection. In all our numerical experiments we could observe, that the error behaved as we could expect in connection with the respective theoretical work. Especially the three parts influencing the total error, which we mentioned in the above theorem, came out very clearly. In the scope of this paper we can only present one picture, which shows the typical kind of results we get with the method of lines (see Figure 2).

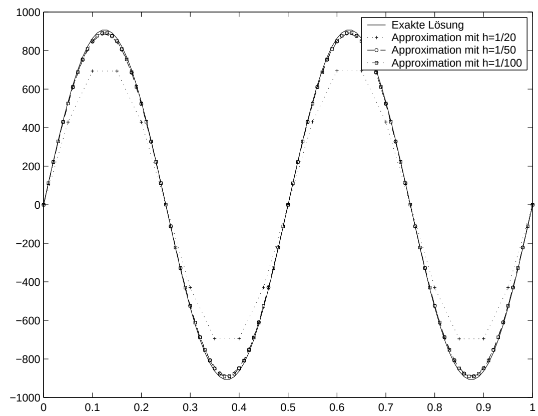

Figure 2: Exact solution u und approximations with the method of lines at $y = 1$ for $m = 4, M = 4, \varepsilon = 10^{-2}, h = \frac{1}{20}, \frac{1}{50}, \frac{1}{100}$

5 Conclusion

The method of lines in connection with the data projection described above is a well fitting regularization scheme for Cauchy problems for elliptical partial differential equations. Theoretical results such as error estimates and a convergence theorem as well as an easy and fast computable numerical scheme give us some important tools to deal with such problems. The basic theorems can be extended to more general elliptic equations, which is carried out in detail in [1].

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