Suboptimal Filter for Dynamic Systems with Different Types of Observations

VLADIMIR SHIN, and RASHID MINHAS
Department of Mechatronics,
Gwangju Institute of Science and Technology
1 Oryong-Dong, Buk-Gu, Gwangju, 500-712
SOUTH KOREA

Abstract: - In [1], [2] we have proposed an optimal mean square combination of an arbitrary number of estimates. In particular, for two estimates, the combination represents the Millman and the Bar-Shalom-Campo formulae for fusion of uncorrelated and correlated estimates respectively. In this paper, we present an application of this combination to the filtering problem. New suboptimal reduced-order robust filter for dynamic systems with different types of observations is proposed. Example demonstrates the effect of the common process noise on the fusion of the state estimates based on observations through several sensors.

Key-Words: - Dynamic system, Kalman filter, suboptimal filter, data fusion, decomposition, minimum mean-square error, Millman's formula, multisensor

1 Introduction

The integration and fusion of information, from a combination of different types of observed instruments (sensors), is often used in the design of high-accuracy control systems. Typical applications that can benefit, the use of multiple sensors, are industrial tasks, military command, mobile robot navigation, multi-target tracking, and aircraft navigation (see [3], [4] and references therein). In recent years, there has been growing interest to fuse multisensor data to increase the accuracy of estimation parameters and system states. This interest is motivated by the availability of different types of local sensors having different spectrum characteristics. The observations, used in the estimation process, are assigned to a common target through association process. If it is decided that all local sensors observe the same target, then the next problem is how to combine (fusion) the correspondence local estimates?

In [1] and [2], we have derived the fusion formula (FF) which represents an optimal mean square linear combination of local estimates with weights depending on cross covariances of estimation errors. The main purpose of this paper is to show how to apply the FF in the filtering problems with different types of observations for more accurate overall state estimate.

This paper is organized as follows. In Section 2, we present the statement of filtering problem with different types of observations. In Section 3, we propose new suboptimal filter which is derived by using the FF. The parallel structure of filter allows fast processing of observations. The obtained filtering algorithm reduces the computational cost and real-time processing requirements. We also demonstrate the relationship between the FF and well-known, the Millman and the Bar-Shalom-Campo formulae. In Section 4, the suboptimal filter is numerically tested. Examples demonstrate the high-accuracy of the proposed filter. Finally, Section 5 is the conclusion.

2 Statement of Filtering Problem with Different Types of Observations

For simplicity, consider a discrete-time linear dynamic system with additive white Gaussian noise,

$$x_{k+1} = F_k x_k + G_k v_k, \quad k = 0,1,\dots \qquad (1)$$

where $x_k \in \mathbf{R}^n$ is state vector, and $v_k \in \mathbf{R}^r$ is a Gaussian random noise, $v_k \sim N(0, Q_k)$.

Suppose that *overall* observation vector $Y_k \in \mathbf{R}^m$ is composed of N different types of observation subvectors (local sensors) $y_k^{(1)}, \ldots, y_k^{(N)}$,

$$Y_k = \begin{bmatrix} y_k^{(1)} \\ \vdots \\ y_k^{(N)} \end{bmatrix}, \tag{2}$$

where $y_k^{(i)}$, $i = 1, \ldots, N$ are determined by the equations

$$\begin{aligned} y_k^{(1)} &= H_k^{(1)} x_k + w_k^{(1)}, \quad y_k^{(1)} \in \mathbf{R}^{m_1}, \\ &\vdots \\ y_k^{(N)} &= H_k^{(N)} x_k + w_k^{(N)}, \quad y_k^{(N)} \in \mathbf{R}^{m_N}, \end{aligned} \tag{3}$$

with $\{w_k^{(1)}\}, \ldots, \{w_k^{(N)}\}$ are zero-mean, white Gaussian observation noises, $w_k^{(i)} \sim \mathbf{N}(0, R_k^{(i)})$. $i = 1, \ldots, N$, $m_1 + \cdots + m_N = m$. The initial state is modeled as a Gaussian random vector, $x_0 \sim N(\bar{x}_0, P_0)$. The N+1 noises $\{v_k\}$, $\{w_k^{(i)}\}$, $i = 1, \ldots, N$, and the initial state x_0 are mutually independent.

It is required to estimate the state x_k of the system given by *overall* observations Y_k.

If we rewrite the observation model (2),(3) in equivalent form

$$Y_k = \begin{bmatrix} H_k^{(1)} \\ \vdots \\ H_k^{(N)} \end{bmatrix} x_k + \begin{bmatrix} w_k^{(1)} \\ \vdots \\ w_k^{(N)} \end{bmatrix} \tag{4}$$

and apply the Kalman filter (KF) equations to the model (1), (4). We can produce the optimal estimate $\hat{x}_k^{opt}$ of the state x_k, based on *overall* observations $Y_k \in \mathbf{R}^m$. However, there are several limitations for the KF practical implementation, such as computational cost and capacity of data transmission. Also numerical errors of the KF design are drastically increased with the state and observation dimensions, for instance, in multisensor intelligent systems [3]. Hence, the KF may be impractical to implementation. So reduced-order suboptimal filters are preferable as there is no need to estimate states by using *overall* measurements Y_k simultaneously. In this paper, we show that the FF may serve as an alternative to solve this filtering problem.

3 Suboptimal Linear Filter

The derivation of new suboptimal reduced-order filter is based on the assumption that the *overall* measurement vector Y_k consists of the combination of the different subvectors $y_k^{(1)}, \ldots, y_k^{(N)}$, which can be processed separately. According to (1) and (3), we have N unconnected dynamic subsystems ($i = 1, \ldots, N$) with state vector $x_k \in \mathbf{R}^n$ and observation subvector (local sensor) $y_k^{(i)} \in \mathbf{R}^{m_i}$:

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k v_k, \\ y_k^{(i)} &= H_k^{(i)} x_k + w_k^{(i)} \end{aligned} \tag{5}$$

Where **i** is the fixed-number of subsystem. Next, let us denote the local estimate of the state x_k based on the local observation $y_k^{(i)}$ by $\hat{x}_{k|k}^{(i)}$. To find $\hat{x}_{k|k}^{(i)}$ we apply the KF to the subsystem (5) [5], [6]. We have

$$\begin{aligned} \hat{x}_k^{(i)} &= F_{k-1}\hat{x}_{k-1}^{(i)} + K_k^{(i)}\left[y_k^{(i)} - H_k^{(i)} F_{k-1}\hat{x}_{k-1}^{(i)}\right], \\ M_k^{(ii)} &= F_{k-1} P_{k-1}^{(ii)} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T, \\ K_k^{(i)} &= M_k^{(ii)} \left(H_k^{(i)}\right)^T \left[\left(H_k^{(i)}\right)^T M_k^{(ii)} H_k^{(i)} + R_k^{(i)}\right]^{-1}, \\ P_k^{(ii)} &= \left[I_n - K_k^{(i)} H_k^{(i)}\right] M_k^{(ii)}, \end{aligned} \tag{6}$$

Where $i = 1, \ldots, N$, and $P_k^{(ii)}$ is the filtering error covariance, i.e.

$$P_k^{(ii)} = \mathrm{cov}\{\tilde{x}_k^{(i)}, \tilde{x}_k^{(i)}\}, \quad \tilde{x}_k^{(i)} = x_k - \hat{x}_k^{(i)}. \tag{7}$$

Thus using the KF matched to (5) at fixed "**i**", we have N local Kalman estimates

$$\hat{x}_k^{(1)}, \quad \ldots, \quad \hat{x}_k^{(N)} \tag{8}$$

based on the observations $y_k^{(1)}$, …, $y_k^{(N)}$, respectively, and corresponding local Kalman error covariances

$$P_k^{(11)}, \quad \ldots, \quad P_k^{(NN)}. \tag{9}$$

The new suboptimal estimate $\hat{x}_k^{sub}$ of the state vector x_k based on the overall measurements Y_k, (2) or (3), is constructed from the local estimates (8) by using the FF [1], [2]:

$$\hat{x}_k^{sub} = \sum_{i=1}^{N} c_k^{(i)} \hat{x}_k^{(i)}, \qquad \sum_{i=1}^{N} c_k^{(i)} = I_n, \tag{10}$$

where I_n is the $n \times n$ unit matrix, and $c_k^{(1)}, \ldots, c_k^{(N)}$ are $n \times n$ the time-varying weighting matrices determined from the mean-square criterion,

$$J_k = E\left(\left\| x_k - \sum_{i=1}^{N} c_k^{(i)} \hat{x}_k^{(i)} \right\|^2\right) \to \min_{c_k^{(i)}}. \tag{11}$$

The following theorem completely defines the suboptimal overall estimate $\hat{x}_k^{sub}$ and its overall error covariance

$$P_k^{sub} = \mathrm{cov}\left(\tilde{x}_k^{sub}, \tilde{x}_k^{sub}\right), \ \tilde{x}_k^{sub} = x_k - \hat{x}_k^{sub}. \tag{12}$$

Theorem 1 (Fusion Formula): [1],[2]. *Let* $\hat{x}_k^{(1)}, \ldots, \hat{x}_k^{(N)}$ are the local Kalman estimates (8) *of* an unknown state - x_k and the weighting matrices $c_k^{(1)}, \ldots, c_k^{(N)}$ are given by

$$\sum_{i=1}^{N} c_k^{(i)} \left[P_k^{(ij)} - P_k^{(iN)}\right] = 0, \quad \sum_{i=1}^{N} c_k^{(i)} = I_n, \tag{13}$$

where $P_k^{(ii)}$, the local Kalman error covariance (9) is determined by the KF (6), and $P_k^{(ij)} = \left(P_k^{(ji)}\right)^T$, $i \neq j$ is cross-covariance,

$$P_k^{(ij)} = \mathrm{cov}\left\{\tilde{x}_k^{(i)}, \tilde{x}_k^{(j)}\right\}, \quad i \neq j. \tag{14}$$

Theorem 2: The cross-covariance $P_k^{(ij)}$ satisfies the following recursion:

$$P_k^{(ij)} = \left[I_n - K_k^{(i)} H_k^{(i)}\right]\left[F_{k-1} P_{k-1}^{(ij)} F_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T\right] \times \left[I_n - K_k^{(j)} H_k^{(j)}\right]^T, \quad P_0^{(ij)} = P_0, \quad i, j = 1, \ldots, N, \tag{15}$$

where the gain $K_k^{(i)}$ is determined by the KF (6).

The formula (10) is called fusion formula *(FF).*

Corollary 1: If $\hat{x}_k^{(1)}$, …, $\hat{x}_k^{(N)}$ are unbiased local Kalman estimates then the suboptimal estimate $\hat{x}_k^{sub}$ in (10) is unbiased.

Corollary 2:- The overall error covariance P_k^{sub} is given by

$$P_k^{sub} = \sum_{i,j=1}^{N} c_k^{(i)} P_k^{(ij)} \left(c_k^{(j)}\right)^T. \tag{16}$$

Thus the local Kalman filters (6), the FF (10) and (13) and the recursive (15) completely define the new suboptimal filter.

In particular case at $N = 2$, the FF (10) and (13) reduces to the Bar-Shalom-Campo formula [7]:

$$\hat{x}_k^{sub} = c_k^{(1)} \hat{x}_k^{(1)} + c_k^{(2)} \hat{x}_k^{(2)},$$

$$c_k^{(1)} = \left[P_k^{(22)} - P_k^{(21)}\right]\left[P_k^{(11)} + P_k^{(22)} - P_k^{(12)} - P_k^{(12)}\right]^{-1},$$

$$c_k^{(1)} = \left[P_k^{(11)} - P_k^{(12)}\right]\left[P_k^{(11)} + P_k^{(22)} - P_k^{(12)} - P_k^{(12)}\right]^{-1}. \tag{17}$$

If the two estimates $\hat{x}_k^{(1)}$ and $\hat{x}_k^{(2)}$ are uncorrelated, i.e. $P_k^{(12)} = P_k^{(21)} = 0$ in Eq. (17), then we have the Millman formulae for the weights [5, 6]:

$$c_k^{(1)} = P_k^{(22)} \left[P_k^{(11)} + P_k^{(22)}\right]^{-1},$$

$$c_k^{(2)} = P_k^{(11)} \left[P_k^{(11)} + P_k^{(22)}\right]^{-1}, \tag{18}$$

Remark 1: The local Kalman estimates $\hat{x}_k^{(1)}, \ldots,$ $\hat{x}_k^{(N)}$ are separated for different types of sensors.

Therefore, they can be implemented in parallel for various types of observations $y_k^{(i)}$, i = 1 , …. N.

The proposed filter is also robust, since it can be corrected even if one of the parallel local Kalman estimate $\hat{x}_k^{(i)}$ diverges. In this case, the corresponding weight matrix $c_k^{(i)}$ in the weighting sum , FF, will tend to zero, thereby indicating that the diverging estimate $\hat{x}_k^{(i)}$ will be discarded in the weighting sum of the FF.

Remark 2: We may note, that the local Kalman filter gains $K_k^{(i)}$, the error covariances $P_k^{(ij)}$, and the weights $c_k^{(i)}$ may be precomputed, since they do not depend on the present observations Y_k. But only on the noises statistics Q_k and $R_k^{(i)}$, and the system matrices $F_k, G_k, H_k^{(i)}$, which are the part of system model (1), (3). Thus, once the observation schedule has been settled, the real-time implementation of the suboptimal filter requires only the computation of the local Kalman estimates $\hat{x}_k^{(1)}, \dots, \hat{x}_k^{(N)}$ and the final fusion estimate $\hat{x}_k^{sub}$.

4 Examples

4.1 Identification of a Scalar Unknown

To estimate the value of a scalar unknown θ from two types of observations the system and observation models are

$$x_{k+1} = x_k, \quad x_k \equiv \theta, \tag{19}$$

$$y_k^{(1)} = x_k + w_k^{(1)}, \quad y_k^{(2)} = x_k + w_k^{(2)}, \tag{20}$$

where $w_k^{(i)} \sim \mathbf{N}(0, r_i), \ i = 1,2; \ x_0 \sim \mathbf{N}(\bar{\theta}, \sigma_\theta^2)$.

The KF gives the optimal mean-square estimate, $\hat{x}_k^{opt}$, of an unknown $x_k \equiv \theta$ based on *overall* observations $Y_k = \left[y_k^{(1)} \quad y_k^{(2)}\right]^T$. We have

$$\hat{x}_k^{opt} = \hat{x}_{k-1}^{opt} + K_k^{opt}\begin{bmatrix} y_k^{(1)} - \hat{x}_{k-1}^{opt} \\ y_k^{(2)} - \hat{x}_{k-1}^{opt} \end{bmatrix}, \quad \hat{x}_0^{opt} = \bar{\theta},$$

$$P_k^{opt} = \frac{r_1 r_2 P_{k-1}^{opt}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{opt}}, \quad P_0^{opt} = \sigma_\theta^2,$$

$$K_k^{opt} = \begin{bmatrix} \dfrac{r_2 P_{k-1}^{opt}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{opt}} \\ \dfrac{r_1 P_{k-1}^{KF}}{r_1 r_2 + (r_1 + r_2) P_{k-1}^{opt}} \end{bmatrix}^T, \tag{21}$$

Using the "step-by-step" induction, we obtain the exact formula for the optimal mean square error (MSE)

$$P_k^{opt} = E\left(\theta - \hat{x}_k^{opt}\right)^2 = \frac{\sigma_\theta^2}{1 + k r_{12} \sigma_\theta^2},$$

$$r_{12} = \frac{r_1 + r_2}{r_1 r_2}. \tag{22}$$

Together with the KF (21), we apply the suboptimal filter. Let denote the local Kalman estimates of the unknown $x_k \equiv \theta$ based on the single observations $y_k^{(1)}$ and $y_k^{(2)}$ by $\hat{x}_k^{(1)}$ and $\hat{x}_k^{(2)}$, respectively. Using the system model with state x_k and single observation $y_k^{(i)}$,

$$x_{k+1} = x_k, \quad y_k^{(i)} = x_k + w_k^{(i)} \tag{23}$$

we obtain the equations for $\hat{x}_k^{(i)}, \quad i = 1,2$,

$$\hat{x}_k^{(i)} = \hat{x}_{k-1}^{(i)} + K_k^{(i)}\left[y_k^{(i)} - \hat{x}_{k-1}^{(i)}\right], \quad \hat{x}_0^{(i)} = \bar{\theta}.$$

$$K_k^{(i)} = \frac{P_{k-1}^{(ii)}}{r_i + P_{k-1}^{(ii)}}, \quad P_k^{(ii)} = \left[1 - K_k^{(i)}\right] P_{k-1}^{(ii)},$$

$$P_0^{(ii)} = \sigma_\theta^2, \ i = 1,2. \tag{24}$$

Next, using the FF (17), one can obtain

$$\hat{x}_k^{sub} = c_k^{(1)} \hat{x}_k^{(1)} + c_k^{(2)} \hat{x}_k^{(2)},$$

$$c_k^{(1)} = \frac{P_k^{(2)} - P_k^{(12)}}{P_k^{(1)} - 2P_k^{(12)} + P_k^{(2)}},$$

$$c_k^{(2)} = \frac{P_k^{(1)} - P_k^{(12)}}{P_k^{(1)} - 2P_k^{(12)} + P_k^{(2)}}, \tag{25}$$

where the local mean square error, $P_k^{(11)}$ and $P_k^{(22)}$ are determined by (24), and the cross-covariance $P_k^{(12)}$, according to Eq. (15) is determined by

$$P_k^{(12)} = \left[1 - K_k^{(1)}\right]\left[1 - K_k^{(2)}\right]P_{k-1}^{(12)}, \quad P_0^{(12)} = \sigma_\theta^2. \qquad (26)$$

Using (24)-(26), one can obtain the exact expressions for $c_k^{(1)}, c_k^{(2)}$ and $P_k^{(12)}$, respectively,

$$c_k^{(1)} = \frac{r_2}{r_1 + r_2}, \quad c_k^{(2)} = \frac{r_1}{r_1 + r_2},$$

$$P_k^{(12)} = \frac{r_1 r_2 \sigma_\theta^2}{\left(r_1 + k\sigma_\theta^2\right)\left(r_2 + k\sigma_\theta^2\right)}. \qquad (27)$$

And finally, using (16), the *overall* MSE takes the form

$$P_k^{sub} = E\left(\theta - \hat{x}_k^{sub}\right)^2 = \frac{r_1 r_2 \sigma_\theta^2}{(r_1 + r_2)^2}\left[\frac{r_2}{r_1 + k\sigma_\theta^2} + \frac{2 r_1 r_2}{\left(r + k\sigma_\theta^2\right)\left(r + k\sigma_\theta^2\right)} + \frac{r_1}{r_2 + k\sigma_\theta^2}\right]. \qquad (28)$$

It is seen from Fig.1, for the case $r_1 = 1$, $r_2 = 0.2$ and $\sigma_\theta^2 = 1$ that the difference between P_k^{opt} and P_k^{sub} is negligible.

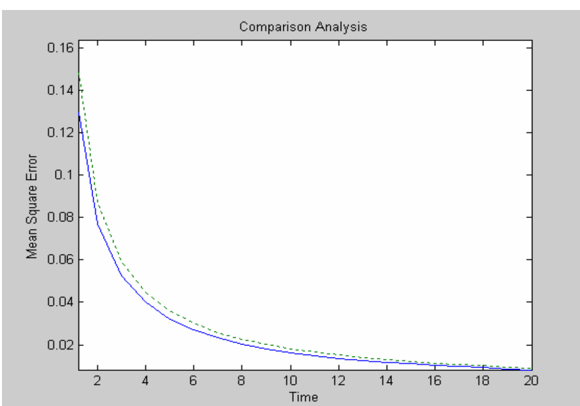

Fig.1. Illustration of the optimal and suboptimal mean square errors: P_k^{opt} (solid line) and P_k^{sub} (dotted line).

4.2 Data Fusion of Multisensor's Estimates

Consider a scalar system described by

$$x_{k+1} = a x_k + v_k, \quad k = 0,1,\ldots,T, \qquad (29)$$

$$y_k^{(i)} = x_k + w_k^{(i)}, \quad i = 1,2,\ldots,N, \qquad (30)$$

where

$$v_k \sim \mathbf{N}(0,q), \; x_0 \sim \mathbf{N}\left(\bar{\theta}, \sigma_\theta^2\right),$$
$$w_k^{(i)} \sim \mathbf{N}(0, r_i), \quad i = 1,\ldots,N.$$

This represents the model which takes N sensor modes. The parameters are subject to a = 0.9, $T = 20$, q = 0.01, $\bar{\theta}$ = 0.5, σ_θ^2 = 1, and $N = 1,2,3,4$. An optimal Kalman filter ($\hat{x}_k^{opt}$) and a suboptimal filter ($\hat{x}_k^{sub}$), consisting of N local Kalman filters $\hat{x}_k^{(1)}$, …, $\hat{x}_k^{(N)}$ as presented in Section 3, were used to estimate x_k. The observation noise variances were set as follows: r_1 = 0.2, r_2 = 0.1, r_3 = 0.06 and r_4 = 0.04. Fig.2 shows the time histories of the optimal P_k^{opt} and suboptimal P_k^{sub} mean square errors as a function of number of sensors $N = 2,3,4$. Note that in the case of single sensor ($N = 1$), the optimal Kalman filter and the suboptimal filter are identical. From the results in Fig.2, it can be seen that mean square errors, P_k^{opt} and P_k^{sub}, are reduced depending upon number of sensors. This means that larger the number of sensors, higher is the estimation accuracy. And also as in example 4.1 the differences between P_k^{opt} and P_k^{sub} are negligible for any number of sensors $N = 2,3,4$.

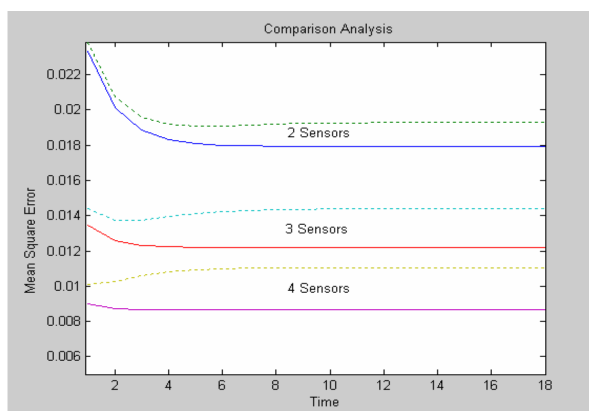

Fig.2. The optimal P_k^{opt} (solid line) and suboptimal P_k^{sub} (dotted line) mean square errors as a function of number of sensors N.

5 Conclusion

In this paper, we present new suboptimal filter for discrete-time linear systems with different types of observations. This filter represents the optimal linear combination of arbitrary number of local Kalman filters. Each local Kalman filter is fused by the minimum mean square error criterion. The new filter has parallel structure and is very suitable for parallel processing of observations. The examples demonstrate the efficiency and high-accuracy of the proposed filter.

The filter can be widely used in the different areas of applications: industrial, military, space, communication, target tracking, inertial navigation and others.

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