Non-linear boundary conditions at the interface of two fluids

IVAN V. KAZACHKOV AND OLGA M. KAZACHKOVA
Department of Energy Technology
Royal Institute of Technology
Brinellvägen 60, HPT/EGI, 10044, Stockholm
SWEDEN
http://www.energy.kth.se/

Abstract: - The boundary conditions at the interface between two immiscible fluids were derived for the general case of large-amplitude perturbations. The interface was modeled as perturbed free boundary that evolves in time, and the non-linear description was performed and analyzed in a wide range of physical situations. The differential equations of the interfacial motion thus obtained might be useful in the non-linear development of classical hydrodynamic instabilities. They should play an important role in the understanding of hydrodynamic phenomena associated with flows involving complex interface evolution including parametrical control (electromagnetic field, vibration, etc.).

Key-Words: - interface of two fluids, boundary conditions, non-linear waves, capillary forces

1 Introduction

The modeling of fluid interfaces (boundaries between different immiscible fluids, fluid and gas, etc.) presents tremendous challenges, which are caused by the interplay between interface dynamics and the fluid flow in contacting regions. Fluid interfacial motion induced by surface tension plays an important role in diverse industrial and natural processes [1-12]. Examples of such phenomena are studied in touch with capillarity [1,2,7], low gravity [3], cavitation [4], hydrodynamic and hydromagnetic stability [5,7,9,10], reactive flows [6], crystallization [8], interfacial transport [11,12], etc.

The current studies of fluid flows with dynamic interfaces between two (or more) layers are devoted to the following problems:

- surface instabilities,
- parametric control of the interfaces with electromagnetic fields, vibrations, etc.,
- governing equations for the interfaces in diverse physical situations: boiling, evaporation, crystallization, etc.

In this paper, a “macroscopic” description was introduced when two fluids were separated by so called interface, which means a surface with zero thickness. And the local dynamic equilibrium conditions of a two fluid system (fluid 1 and fluid 2) on the moving surface of any shape and amplitude was discussed. This situation is a key point for a lot of problems for the flows with free surfaces and interfaces [8-15]. Thus, the boundary conditions for the dynamic interface between two fluids were formulated and analyzed here in detail.The paper was organized as follows. In Section 2 some preliminaries concerning the boundary conditions were stated and the problems were discussed. Section 3 was denoted to analysis of kinematic boundary condition. Then (Section 4) the formulation of non-linear dynamical equilibrium of the interface between two fluids was considered and different cases were analyzed in touch with possible external forces. In Section 5 a derivation of a non-linear dynamical condition was given. Section 6 dealed with some limit cases including linear boundary condition for the small-amplitude perturbations. In Section 7 the conclusion was given as concern to practical application of the results obtained and some planning for further developments was done.

Parametric excitation of oscillations is done in some system by temporal variation of one or several parameters of a system (mass, momentum of inertia, temperature, stiffness coefficient; for the fluids: pressure, viscosity, etc.). Thus, parametric oscillations are excited and maintained by parametric excitation. Examples of parametric oscillations are oscillations of a stiffness coefficient due to a temperature variation in a loaded elastic solid, which are able to evoke later on its vibrations. The oscillations of temperature (pressure) in fluid (gas) flow are able to evoke oscillations of its pressure (temperature) or (and) viscosity with a consequitive oscillations of other flow parameters. Electric and magnetic fields may cause the oscillations in flow of conductive fluid producing the oscillations of other parameters, etc.

2 Preliminaries

First some basic definitions, notations and experimental knowledge were put together and discussed in touch with mathematical simulation of the wave film flow and other hydrodynamic problems with evolutionary boundary interfaces. For example, plane film flow had some kind of uncertainty as concern to boundary conditions. It was assumed that liquid film was spreading out on the rigid surface having the upper free surface or some interface with the other fluid as it was shown in the Figure:

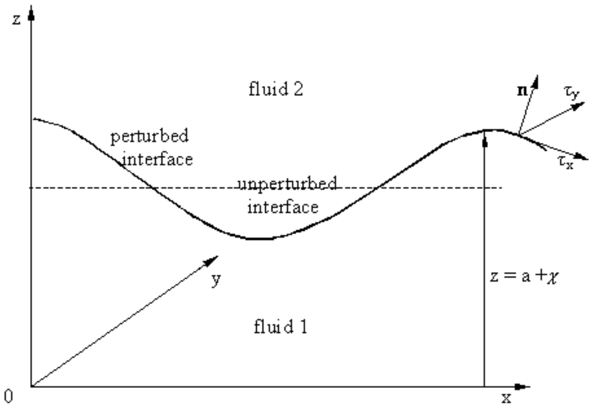

Figure. The evolution of the interface of two fluids.

An orientation of the one normal and two tangential vectors on a perturbed interface shown in Figure evidenced that boundary conditions have dynamically depended on these vectors' orientation at the each and every point of a domain.

If a film thickness was big enough, then Van-der-Waals forces were neglected. By solution of a film stability problem in a linear approach, one did not need to state the initial conditions. Then considering a parametric excitation or suppression of a perturbation of free boundary, normally the solution was sought in the same form using the linear superposition principle. Therefore a fully determined boundary problem required only the statement of the boundary conditions on a rigid surface and at the interface. The classic boundary condition on the body (rigid surface) is zero velocity. Though nowadays there are many evidences that tangential velocity may be non-zero. The question is stated by a lot of researchers starting from Stokes (1845), Lamb (1947), Zhukovskii (1948). Later on, it is shown in the review [16] that on the unmoisten body surface the remarkable slip of fluids is possible. Happel and Brenner [17] consider as mostly reliable hypothesis that tangential velocity of a fluid at each point of a rigid surface has to be counted as proportional to a shear stress at the local point, with so called coefficient of a slip friction β. They assume that β depends only on the fluid and on the body surface properties. The conditions of a slip are also analyzed in some other papers.

3 Kinematic condition

On a deformable free surface $z=a+\chi(x,y,t)$, or interface of two fluids, a kinematic boundary condition was considered in the form (continuity of normal velocity across the interface):

$$z=a+\chi\,,\quad w_1=w_2=\frac{\partial\chi}{\partial t}+u_j\frac{\partial\chi}{\partial x}+v_j\frac{\partial\chi}{\partial y}\,,\qquad (1)$$

where $\{u,v,w\}$ is velocity vector, j=1,2 ("fluid 1" and "fluid 2", respectively), $\chi(x,y,t)$ is perturbation of the interface. In case of a free surface, the second fluid is absent and indexes j are omitted.

The conditions (1) was scrutinized transforming them into the following form ($z=a+\chi(x,y,t)$):

$$w_1=\frac{\partial\chi}{\partial t}+u_j\frac{\partial\chi}{\partial x}+v_j\frac{\partial\chi}{\partial y}\,,\ (u_1-u_2)\frac{\partial\chi}{\partial x}+(v_1-v_2)\frac{\partial\chi}{\partial y}=0,\ (2)$$

where the second equation was obtained by subtraction of the corresponding two equations (1). From the second equation (2) followed that the tangential velocities in both directions were the same for "fluid 1" and "fluid 2" at the interface (continuity of tangential velocity across the interface):

$$z=a+\chi\,,\qquad u_1=u_2\,,\qquad v_1=v_2\,.\qquad (3)$$

Otherwise, if supposed the slipping of fluids at the interface, then $u_1\neq u_2$, $v_1\neq v_2$, where from with account of the second equation in the equation array (2) yields

$$z=a+\chi\,,\qquad \frac{\partial\chi}{\partial x}=-\frac{(v_1-v_2)}{(u_1-u_2)}\cdot\frac{\partial\chi}{\partial y}\,.\qquad (4)$$

So that in case (3) there are available the arbitrary perturbations by x and y directions. But in case of a slip at the interface, the equation (4) must be satisfied as a relationship between perturbations by coordinates x and y. The result was quite unexpected: slip of the phases at the interface requested some relation between the perturbations while, if the fluids are not slipping at the interface, the boundary perturbations might be arbitrary. What was more, in case of a slip of fluids at the interface, there were impossible any plane perturbations of the interface, because if $\partial\chi/\partial y=0$, then $\partial\chi/\partial x=0$.

4 Dynamic conditions

The dynamic conditions at the interface consist of the balance of shear and normal stresses. Therefore on a perturbed interface, in case of immiscible fluids, the capillary forces should be also taken into account. They are big enough by the remarkable perturbations. The capillary force was expressed in the form:

$$p_{\sigma} = \sigma K \quad , \qquad (5)$$

where σ is the surface tension coefficient, K is the average curvature of a perturbed interface between two fluids. According to the differential geometry, K is expressed in an arbitrary point of the interface in the following form:

$$K = \frac{\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial x^2}\left(\frac{\partial \chi}{\partial y}\right)^2 - 2\frac{\partial \chi}{\partial x}\frac{\partial \chi}{\partial y}\frac{\partial^2 \chi}{\partial x \partial y} + \frac{\partial^2 \chi}{\partial y^2}\left(\frac{\partial \chi}{\partial x}\right)^2}{\left[1 + \left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2\right]^{3/2}} \qquad (6)$$

In a linear approach, in case of small perturbations of the interface, from an equilibrium state, the correlation (6) is the well-known Landau formula [7], which is obtained as solution of the variational problem for the minimum of a full free surface energy. It is easily examined that this expression contains in its expansion by the small-amplitude perturbations only the odd-order terms. This is why the linear approximation is successfully applied even in case of small-amplitude non-linear perturbations of a free surface (interface). It is exact solution with accuracy up to second-order terms by perturbations.

Normally, in formulation of the dynamic equilibrium condition at an interface, in a linear approach, all forces are considered by $z = a$. So that only a projection on an unperturbed surface is considered. But in case of large-amplitude perturbations, the problem is substantially non-linear and requires considering the curved perturbed surface at each its point. The normal and tangential vectors, at each point of the interface, may deviate substantially from the stable equilibrium state. Therefore all forces should be projected onto the normal and tangential vectors at each point of the curved surface.

5 Derivation of non-linear conditions

The unit vectors (one normal and two tangential) at each and every point of the deformable interface can be presented in the form:

$$\mathbf{n} = \frac{\left\{-\frac{\partial \chi}{\partial x}, -\frac{\partial \chi}{\partial y}, 1\right\}}{\sqrt{1 + \left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2}}, \quad \tau_x = \frac{\left\{1, 0, \frac{\partial \chi}{\partial x}\right\}}{\sqrt{1 + \left(\frac{\partial \chi}{\partial x}\right)^2}}, \quad \tau_y = \frac{\left\{0, 1, \frac{\partial \chi}{\partial y}\right\}}{\sqrt{1 + \left(\frac{\partial \chi}{\partial y}\right)^2}}. \qquad (7)$$

Using (7) one can determine the stresses on the elementary plane having the normal unit vector **n**. For this purpose, first the following expressions of the hydrodynamic stresses are represented as [4]:

$$\begin{aligned} p_{nn} &= n_x p_{nx} + n_y p_{ny} + n_z p_{nz}, \\ p_{\tau x} &= \tau_{xx} p_{nx} + \tau_{xy} p_{ny} + \tau_{xz} p_{nz}, \\ p_{\tau y} &= \tau_{yx} p_{nx} + \tau_{yy} p_{ny} + \tau_{yz} p_{nz}, \end{aligned} \qquad (8)$$

where are:

$$\begin{aligned} p_{nx} &= n_x p_{xx} + n_y p_{yx} + n_z p_{zx}, \quad && \boldsymbol{n} = \{n_x, n_y, n_z\}, \\ p_{ny} &= n_x p_{xy} + n_y p_{yy} + n_z p_{zy}, \quad && \boldsymbol{\tau}_x = \{\tau_{xx}, \tau_{xy}, \tau_{xz}\}, \\ p_{nz} &= n_x p_{xz} + n_y p_{yz} + n_z p_{zz}, \quad && \boldsymbol{\tau}_y = \{\tau_{yx}, \tau_{yy}, \tau_{yz}\}. \end{aligned} \qquad (9)$$

Then the following well-known expressions for the stress tensor were taken [4]:

$$\begin{aligned} p_{xx} &= -p + 2\mu \frac{\partial u}{\partial x}, \quad p_{xy} = p_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ p_{yy} &= -p + 2\mu \frac{\partial v}{\partial y}, \quad p_{yz} = p_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ p_{zz} &= -p + 2\mu \frac{\partial w}{\partial z}, \quad p_{xz} = p_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \end{aligned} \qquad (10)$$

where the indexes j=1,2 were omitted because the expressions for j=1 and j=2 were the same. Here p was the pressure, μ was the dynamic viscosity. Now substituting the equations (9), (10) into (8), with account of (7), yields the following normal and shear stresses at the perturbed non-linear interface:

$$\begin{aligned} p_{nn} = -p + \frac{2\mu}{\sqrt{1 + \left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2}} \Bigg\{ & \left[\left(\frac{\partial \chi}{\partial x}\right)^2 - 1\right]\frac{\partial u}{\partial x} + \left[\left(\frac{\partial \chi}{\partial y}\right)^2 - 1\right]\frac{\partial v}{\partial y} + \\ & + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \chi}{\partial x}\frac{\partial \chi}{\partial y} - \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial \chi}{\partial x} - \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial \chi}{\partial y}\Bigg\}, \end{aligned} \qquad (11)$$

$$\begin{aligned} p_{\tau x} = \frac{\mu}{\sqrt{1 + \left(\frac{\partial \chi}{\partial y}\right)^2}} \frac{1}{\sqrt{1 + \left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2}} \Bigg\{ & \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left[1 - \left(\frac{\partial \chi}{\partial x}\right)^2\right] - \\ & - 2\left(2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial \chi}{\partial x} - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \chi}{\partial y} - \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial \chi}{\partial x}\frac{\partial \chi}{\partial y}\Bigg\}. \end{aligned}$$

The equations for $p_{\tau x}$ and $p_{\tau y}$ are symmetric with regards to the variables x and y. Therefore here only $p_{\tau x}$ was written explicitly. Then the dynamic equilibrium of the interface might be expressed as the following general conditions:

$$[p_{nn}]_2^1 + [\rho]_2^1 g\chi n_z = \sigma K, \quad [\vec{p}_\tau]_2^1 + [\rho]_2^1 g\chi \tau_z = 0, \qquad (12)$$

where $p_\tau = \{p_{\tau x}, p_{\tau y}\}$, $\tau_z = \{\tau_{xz}, \tau_{yz}\}$ are 2-D vectors. The assignment $[\;]_2^1$ means a jump of corresponding parameter of a fluid at interface, e.g. $[p]_2^1 = p_1 - p_2$. Here ρ is a density, g is an acceleration due to the gravity. In equation (12) the only vertical components of the normal and tangential vectors of the tangential plate at each point of the interface were considered because only one mass force (gravitational) acting in a vertical direction was taken into account. The first equation in (12) is the equilibrium of normal stresses while the other two express the equilibrium of shear stresses in two perpendicular directions in the tangential plate. In general, the gravitation might be directed arbitrary, as well as some other volumetrically distributed forces (e.g. electromagnetic) might be presented. Then, instead of (12) yield the following equations:

$$\begin{aligned} &[p_{nn}]_2^1 + [\rho]_2^1 gn\chi + [f]_2^1 n\chi = \sigma K, \\ &[p_{\tau x}]_2^1 + [\rho]_2^1 g\tau_x\chi + [f]_2^1 \tau_x\chi = 0, \\ &[p_{\tau y}]_2^1 + [\rho]_2^1 g\tau_y\chi + [f]_2^1 \tau_y\chi = 0. \end{aligned} \tag{13}$$

For example, if the lower fluid is electroconductive one under an electromagnetic field with vertical component $H_z(x,y,t)$, then the term [14] $0{,}5\mu_m H_z^2 n_z$ appears in the first equation (13) to the left. Here μ_m is the magnetic permeability. If the other fluid is electroconductive while the first one is non-conductive, the inverse situation comes into being and the sign changes. Then, if the liquids are moving on the surface of some vibrating plate, the problem might be considered in the same way supposed that the inertial coordinate system is touched with the vibrating plate. In this case, $g + g_v$ will replace g, where g_v is acceleration due to vibration, e.g. by given vibration amplitude A_g and frequency ω, $g_v = A_g\cos\omega t$, where t is time.

Substitution (6), (9)-(11) into (13) results in the boundary conditions for a general case. For example, for electroconductive fluid below a non-conductive one, under vertical electromagnetic field yields [14]:

- normal to the interface:

$$\begin{aligned} &(p_2 - p_1)\left[1+\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2\right]^{\frac{3}{2}} + 2\left[1+\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2\right]\bullet \\ &\bullet\left\{\frac{1}{2}\left[(\rho_1-\rho_2)g\chi+\frac{1}{2}\mu_m H_z^2\right]\left[\begin{matrix}\mu_1\left(\dfrac{\partial u_1}{\partial x}\dfrac{\partial\chi}{\partial x}-\dfrac{\partial u_1}{\partial z}-\dfrac{\partial w_1}{\partial x}\right)+ \\ \mu_2\left(\dfrac{\partial u_2}{\partial z}+\dfrac{\partial w_2}{\partial x}-\dfrac{\partial u_2}{\partial x}\dfrac{\partial\chi}{\partial x}\right)\end{matrix}\right]\bullet\right. \\ &\bullet\frac{\partial\chi}{\partial x}+\left[\mu_2\left(\frac{\partial v_2}{\partial z}+\frac{\partial w_2}{\partial y}\right)-\mu_1\left(\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial y}\right)\right]\frac{\partial\chi}{\partial y}+\left[\begin{matrix}\mu_1\left(\dfrac{\partial v_1}{\partial x}+\dfrac{\partial u_1}{\partial y}\right) \\ -\mu_1\left(\dfrac{\partial v_2}{\partial x}+\dfrac{\partial u_2}{\partial y}\right)\end{matrix}\right]\bullet \\ &\left.\bullet\frac{\partial\chi}{\partial x}\frac{\partial\chi}{\partial y}+\left(\mu_1\frac{\partial v_1}{\partial y}-\mu_2\frac{\partial v_2}{\partial y}\right)\left(\frac{\partial\chi}{\partial y}\right)^2+\mu_1\frac{\partial w_1}{\partial z}-\mu_2\frac{\partial w_2}{\partial z}\right\} = \\ &\sigma\left[\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}+\frac{\partial^2\chi}{\partial x^2}\left(\frac{\partial\chi}{\partial y}\right)^2-2\frac{\partial\chi}{\partial x}\frac{\partial\chi}{\partial y}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2\chi}{\partial y^2}\left(\frac{\partial\chi}{\partial x}\right)^2\right], \end{aligned} \tag{14}$$

- tangential to the interface in x-direction:

$$\begin{aligned} &\left[1-\left(\frac{\partial\chi}{\partial x}\right)^2\right]\left[\mu_1\left(\frac{\partial w_1}{\partial x}+\frac{\partial u_1}{\partial z}\right)-\mu_2\left(\frac{\partial w_2}{\partial x}+\frac{\partial u_2}{\partial z}\right)\right]+\left\{2\left[\mu_1\left(\frac{\partial w_1}{\partial z}-\frac{\partial u_1}{\partial x}\right)+\right.\right. \\ &\left.\left.+\mu_2\left(\frac{\partial u_2}{\partial x}-\frac{\partial w_2}{\partial z}\right)\right]+\left[\mu_2\left(\frac{\partial w_2}{\partial y}+\frac{\partial v_2}{\partial z}\right)-\mu_1\left(\frac{\partial w_1}{\partial y}+\frac{\partial v_1}{\partial z}\right)\right]\frac{\partial\chi}{\partial y}\right\}\frac{\partial\chi}{\partial x}+ \\ &+\left[\mu_2\left(\frac{\partial v_2}{\partial x}+\frac{\partial u_2}{\partial y}\right)-\mu_1\left(\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y}\right)\right]\frac{\partial\chi}{\partial y}+\left[(\rho_1-\rho_2)g\chi+\frac{1}{2}\mu_m H_z^2\right] \\ &\times\left\{\left[1+\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2\right]\left[1+\left(\frac{\partial\chi}{\partial x}\right)^2\right]\right\}^{\frac{1}{2}}\frac{\partial\chi}{\partial x}=0 \end{aligned} \tag{15}$$

- and tangential to the interface in y-direction:

$$\begin{aligned} &\left[1-\left(\frac{\partial\chi}{\partial y}\right)^2\right]\left[\mu_1\left(\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial y}\right)-\mu_2\left(\frac{\partial v_2}{\partial z}+\frac{\partial w_2}{\partial y}\right)\right]+\left\{2\left[\begin{matrix}\mu_1\left(\dfrac{\partial w_1}{\partial z}-\dfrac{\partial v_1}{\partial y}\right)+ \\ \mu_2\left(\dfrac{\partial v_2}{\partial y}-\dfrac{\partial w_2}{\partial z}\right)\end{matrix}\right]+\right. \\ &\left.\left[\begin{matrix}\mu_2\left(\dfrac{\partial w_2}{\partial x}+\dfrac{\partial u_2}{\partial z}\right)- \\ \mu_1\left(\dfrac{\partial w_1}{\partial x}+\dfrac{\partial u_1}{\partial z}\right)\end{matrix}\right]\frac{\partial\chi}{\partial x}\right\}\frac{\partial\chi}{\partial y}+\left[\mu_2\left(\frac{\partial u_2}{\partial y}+\frac{\partial v_2}{\partial x}\right)-\mu_1\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)\right]\frac{\partial\chi}{\partial x}+ \\ &\left[(\rho_1-\rho_2)g\chi+\frac{1}{2}\mu_m H_z^2\right]\left\{\left[1+\left(\frac{\partial\chi}{\partial x}\right)^2+\left(\frac{\partial\chi}{\partial y}\right)^2\right]\left[1+\left(\frac{\partial\chi}{\partial y}\right)^2\right]\right\}^{\frac{1}{2}}\frac{\partial\chi}{\partial y}=0 \end{aligned} \tag{16}$$

The equations (14)-(16) were substantially simplified for the plane perturbations when $\partial/\partial y = 0$ or $\partial/\partial x = 0$. Normally, study of a wave motion of a perturbed interface is done in assumption that all parameters are represented through the sum of the stable parameters and perturbed values, which are assumed, in most cases, as small comparing to the corresponding unperturbed ones, e.g.: $v = v_0 + v'$, $p = p_0 + p'$, etc. A linear approximation of (14)-(16) might be got taking the following boundary conditions for the unperturbed system (boundary surface: $z=a$=const):

$$\begin{aligned} &z=a, \quad w_{10}=w_{20}=0, \quad \mu_1\left(\frac{\partial u_{10}}{\partial y}+\frac{\partial v_{10}}{\partial x}\right)=\mu_2\left(\frac{\partial u_{20}}{\partial y}+\frac{\partial v_{20}}{\partial x}\right), \\ &\mu_1\frac{\partial u_{10}}{\partial z}=\mu_2\frac{\partial u_{20}}{\partial z}, \quad \mu_1\frac{\partial v_{10}}{\partial z}=\mu_2\frac{\partial v_{20}}{\partial z}, \\ &p_{10}=p_{20}+2\left(\mu_1\frac{\partial w_{10}}{\partial z}-\mu_2\frac{\partial w_{20}}{\partial z}\right). \end{aligned} \tag{17}$$

Strictly speaking, in general case, there should be $u_{10} \neq u_{20}$, $v_{10} \neq v_{20}$ (tangential slip). Because a general case is too complicated, normally different simplifications are used: linear case, non-linear approximation up to the second-order terms, etc. [12,13]. And the correlations similar to (14)-(17) might serve as basic by derivation of simpler boundary conditions based on some additional hypotheses about the physics of the processes.

6 Linear case

For the small-amplitude perturbations, one could derive the following linear approximation from (14)-(17) using asymptotic expansions by x for all functions in the vicinity of the unperturbed surface:

$$z = a, \qquad w_1 = w_2 = \frac{\partial \chi}{\partial t} + u_{j0}\frac{\partial \chi}{\partial x} + v_{j0}\frac{\partial \chi}{\partial y}; \tag{18}$$

$$p_1 = p_2 + (\rho_1 - \rho_2) g\chi + \frac{\mu_m}{2} H_z^2 + 2\left(\mu_1 \frac{\partial w_1}{\partial z} - \mu_2 \frac{\partial w_2}{\partial z}\right) - \\ -\sigma\left(\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2}\right); \tag{19}$$

$$\mu_1\left(\frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z}\right) - \mu_2\left(\frac{\partial w_2}{\partial x} + \frac{\partial u_2}{\partial z}\right) + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial x} + \\ + 2\left[2\left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right) + \left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right)\right] = 0 ,$$

$$\mu_1\left(\frac{\partial w_1}{\partial y} + \frac{\partial v_1}{\partial z}\right) - \mu_2\left(\frac{\partial w_2}{\partial y} + \frac{\partial v_2}{\partial z}\right) + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial y} + \\ + 2\left[2\left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right) + \left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right)\right] = 0 . \tag{20}$$

If subtract the equation (18) for j=1 from the corresponding equation for j=2, might be got

$$(u_{10} - u_{20})\frac{\partial \chi}{\partial x} + (v_{10} - v_{20})\frac{\partial \chi}{\partial y} = 0 . \tag{21}$$

Equation (21) thus obtained must be satisfied for any deformation χ of the interface, and the parameters of the unperturbed system did not depend on χ. Therefore $u_{10} \equiv u_{20}$,, $v_{10} \equiv v_{20}$ should be. Otherwise, by $u_{10} \neq u_{20}$, $v_{10} \neq v_{20}$ yields

$$\frac{\partial \chi}{\partial x} = \frac{v_{10} - v_{20}}{u_{10} - u_{20}}\frac{\partial \chi}{\partial y}. \tag{22}$$

As (22) showed, a slip on the interface did not tolerate arbitrary perturbations, e.g. plane in particular. In general, from (1) similar correlation

$$z = a, \quad (u_1 - u\ )\frac{\partial \chi}{\partial x} + (v_1 - v\ )\frac{\partial \chi}{\partial y} = 0 . \tag{23}$$

Thus, from kinematic condition yielded that 2-D waves were possible if and only if the slip was absent. The correlation (23) had obvious physical explanation. For plane waves, e.g. $\partial\chi/\partial x = 0$, it did not matter, whether or not there was a slip by y-direction and $\partial\chi/\partial x \neq 0$, because u_1=u_2 (see Figure). But on the unperturbed interface, a slip of phases was possible, and the interface according to (19) might keep smooth (without perturbations). If no slip (u_{10}=u_{20}), it might cause a perturbation of the interface $\partial\chi/\partial x \neq 0$, which was clearly understood from the physical point of view as well.

Analysis of the linear boundary conditions (18)-(20) showed that shear stress on the free surface for a liquid moving in a gas might be substantial, in contradiction to what was normally assumed by many researchers. Thus, this question required careful consideration for each specific case. The equations (18)-(20) were simplified accounting the continuity equation at the interface:

$$\frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y} = -\frac{\partial w_j}{\partial z}, \quad \frac{\partial u_{jo}}{\partial x} + \frac{\partial v_{jo}}{\partial y} = -\frac{\partial w_{jo}}{\partial z}. \tag{24}$$

Then (18)-(20) transformed as follows:

$$z = \chi, \quad w_1 = w_2 = \frac{\partial \chi}{\partial t} + u_{10}\frac{\partial \chi}{\partial x} + v_{10}\frac{\partial \chi}{\partial y}; \tag{25}$$

and dynamic conditions as follows:

$$p_1 = p_2 + 2\left[\mu_2\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right) - \mu_1\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)\right] - \\ - \sigma\left(\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2}\right) + (\rho_1 - \rho_2) g\chi + \frac{\mu_m}{2} H_z^2;$$

$$\mu_1\left(\frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z}\right) - \mu_2\left(\frac{\partial w_2}{\partial x} + \frac{\partial u_2}{\partial z}\right) + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial x} + \\ 2\left[2\left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right) + \left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right)\right]\frac{\partial \chi}{\partial x} = 0; \tag{26}$$

$$\mu_1\left(\frac{\partial w_1}{\partial y} + \frac{\partial v_1}{\partial z}\right) - \mu_2\left(\frac{\partial w_2}{\partial y} + \frac{\partial v_2}{\partial z}\right) + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial y} + \\ + 2\left[2\left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right) + \left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right)\right]\frac{\partial \chi}{\partial y} = 0 .$$

Further, because at z=χ in (26) w_1=w_2, the derivatives of w_1, w_2 by x,y should be equal as well. This resulted in the last two equations (26) in:

$$(\mu_1 - \mu_2)\frac{\partial w_1}{\partial x} + \left(\mu_1 \frac{\partial u_1}{\partial z} - \mu_2 \frac{\partial u_2}{\partial z}\right) + 2\left[2\left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right) + \\ + \left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right)\right]\frac{\partial \chi}{\partial x} + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial x} = 0; \tag{27}$$

$$(\mu_1 - \mu_2)\frac{\partial w_1}{\partial y} + \left(\mu_1 \frac{\partial v_1}{\partial z} - \mu_2 \frac{\partial v_2}{\partial z}\right) + 2\left[2\left(\mu_2 \frac{\partial v_{20}}{\partial y} - \mu_1 \frac{\partial v_{10}}{\partial y}\right) + \\ + \left(\mu_2 \frac{\partial u_{20}}{\partial x} - \mu_1 \frac{\partial u_{10}}{\partial x}\right)\right]\frac{\partial \chi}{\partial y} + \frac{\mu_m}{2} H_z^2 \frac{\partial \chi}{\partial y} = 0 .$$

When the slip of liquids at the interface was absent, then u_{10}=u_{20}, v_{10}=v_{20}, therefore the derivatives by x and y equate as well. Thus, the equations (17) resulted

$$\mu_1\left(\frac{\partial u_{10}}{\partial y} + \frac{\partial v_{10}}{\partial x}\right) = \mu_2\left(\frac{\partial u_{10}}{\partial y} + \frac{\partial v_{10}}{\partial x}\right), \quad \mu_1 \frac{\partial u_{10}}{\partial z} = \mu_2 \frac{\partial u_{20}}{\partial z},$$

$$p_{10} = p_{20} + 2\left(\mu_1 \frac{\partial w_{10}}{\partial z} - \mu_2 \frac{\partial w_{20}}{\partial z}\right), \quad \mu_1 \frac{\partial v_{10}}{\partial z} = \mu_2 \frac{\partial v_{20}}{\partial z}. \tag{28}$$

and, with account of the second equation from (24):

$$\frac{\partial w_{10}}{\partial z}=\frac{\partial w_{20}}{\partial z}. \qquad (29)$$

The first equation in (27) is satisfied only in the following two cases:

a) $\mu_1 = \mu_2$, the same viscosity or the same liquids (trivial case), $p_{10} = p_{20}$;

b) $\frac{\partial u_{10}}{\partial y}=-\frac{\partial v_{10}}{\partial x}, \quad \frac{\partial u_{10}}{\partial x}=-\frac{\partial v_{10}}{\partial y}-\frac{\partial w_{10}}{\partial z}. \qquad (30)$

In the plane flow (e.g. ∂/∂y=0), from (30) follows:

$$\frac{\partial w_{10}}{\partial z}=-\frac{\partial u_{10}}{\partial x}, \qquad \frac{\partial w_{20}}{\partial z}=-\frac{\partial u_{10}}{\partial x}, \qquad (31)$$

$$p_{10}=p_{20}+2(\mu_1-\mu_2)\frac{\partial w_{10}}{\partial z}=p_{20}+2(\mu_2-\mu_1)\frac{\partial u_{10}}{\partial x}.$$

And further analysis of the equations (31) showed that by μ_1=μ_2, the pressure in both fluids was the same so that the interface had no reason to perturb. If $\mu_1 \neq \mu_2$, the situation depended on the sign of the viscosity difference μ_2-μ_1 and on the velocity gradient $\frac{\partial u_{10}}{\partial x}\left(or\,\frac{\partial w_{10}}{\partial z}\right)$. If $\left(\frac{\partial u_{10}}{\partial x}<0\right)$ (velocity decreased by *x*), and the other fluid was more viscous, the pressure in a first fluid was something lower than in a second one. Therefore the interface had tendency to penetrate into the first fluid, etc.

The linear boundary conditions (28) thus obtained expressed an equilibrium of the normal and tangential forces on the interface between two fluids. They correspond to a well-known linear conditions for such situations except the normal stresses in *x*- and *y*- directions projected on a perturbed interface. The last ones have a first order by perturbations and should be also taken into account when velocities of the fluids at the interface were different (slip). Otherwise, the above-mentioned terms were omitted, and the boundary conditions at the interface of two fluids corresponded to the classical ones.

7 Conclusion

In this paper, the non-linear boundary conditions at the interface of two fluids (free surface in limit case) were considered. The equations of a non-linear dynamic evolution of the interface have been derived and analyzed. The results might be of interest for theoretical study, as well as practical applications in flows with free boundaries and interfaces between two fluids, which evolve in time under some type of a parametric action, due to Kelvin-Helmholtz, Tonks-Frenkel, Rayleigh-Taylor or some other kind of instability, etc. Further work to be done is extention of the results to analysis of some problems in touch with instability and parametric control of the interfaces accounting real peculiarities such as, for example, phase slip at the interface and its non-linearity.

The derived non-linear boundary conditions were applied to show that the well-known linear boundary conditions got as a limit case from the obtained ones coinside with the known from literature if and only if a slip of fluids at the interface is absent. Otherwise, if velocities of the fluids at the interface differ (slip of phases), the normal stresses by *x* and *y* projected on a perturbed interface have a first order by perturbations and have to be taken into account.

References:

[1] Levich V.G. *Physicochemical Hydrodynamics*. Prentice-Hall, Englewood Cliffs: New York, 1962.

[2] Lamb H. *Hydrodynamics* (6th edn). Cambridge Univ. Press: Cambridge, 1932.

[3] Myshkis A.D., Babskii V.G., Kopachevskii N.D., Slobozhanin L.A., Tyuptsov A.D. *Low-Gravity Fluid Mechanics*. Springer Verlag: New York, 1987.

[4] Batchelor G.K. *An Introduction to Fluid Dynamics*. Cambridge Univ. Press: Cambridge, 1967.

[5] Drazin P.G., Reid W.H. *Hydrodynamic Stability*. Cambridge Univ. Press: Cambridge, 1981.

[6] Oran E.S., Boris J.P. *Numerical Simulation of Reactive Flow*. Elsevier: New York, 1987.

[7] Landau L.D., Lifshitz E.M. *Fluid Mechanics*. Pergamon Press: New York, 1959.

[8] Sethian J.A., Strain J. *J. Comp. Phys*. 1992; **98:** 231.

[9] Crank J. *Free and Moving Boundary Problems*. Oxford Univ. Press: Oxford, 1984.

[10] Chandrasekhar S. *Hydrodynamic and Hydromagnetic Stability*. Oxford Univ. Press: Oxford, 1961.

[11] Steinchen A. (ed.). *Dynamics of Multiphase Flows across Interfaces*. Springer: Berlin-Heidelberg-NewYork-Barcelona-…, 1996.

[12] Shyy W. *Computational Modeling for Fluid Flow and Interfacial Transport*. Elsevier: Amsterdam–London–New York–Tokyo, 1994.

[13] Scriven L.E. *Chem. Eng. Sci*. 1960; **12**: 98.

[14] Kolesnichenko A.F., Kazachkov I.V., Vodyanuk V.O., Lysak N.V. *Capillary MHD flows with free surfaces*. Naukova Dumka: Kiev, 1988 (In Russian).