Modified middle point scheme for the elastodynamic frictional contact problem.

Houari.B. Khenous
Institut National des Sciences Appliques de Toulouse (INSAT)
Département de Génie Mathématique et modélisation
135, avenue de Rangueil 31077 Toulouse cedex

December, 17th, 2004

ABSTRACT
The purpose of this paper is to present the different time integration scheme used in litterature for the elastodynamic friction contact problem. Each method is detailed and treated in term of energy stability (conservation or dissipation). A modified middle point is employed for treating the problem and to proof the conservation of the system energy. The resulting problem is solved with a Newton method.

KEY WORDS
elastodynamic, unilateral contact, Coulomb friction, Signorini problem, Newton method, middle point scheme and the modified one.

1 Introduction

The frictional contact problems in elastodynamics lead to mathematically complex models, the properties of which remain to be fully understood. The analysis of those problems is of great importance in many engineering applications. The volume of literature on mechanical theories of dynamic contact with friction, and particulary on the analytical or numerical solution of problems of this type, is quiet small. Several authors have attempted the numerical solution of dynamic contact problems using finite element methods. This work is one of them. In this article we consider energy consevring time discretization schemes for the elastodynamic frictional contact problem. Conserving schemes are developed in a strong form and independently of any particular spatial discretization.

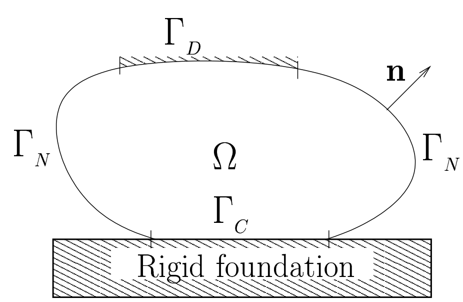

Figure 1: *linearly elastic body Ω in frictional contact with a rigid foundation.*

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain which represents the reference configuration of a linearly elastic body submitted to a Neumann condition on Γ_N, a Dirichlet condition on Γ_D and a unilateral contact with Coulomb friction condition on Γ_C between the body and a flat rigid foundation, where Γ_N, Γ_D and Γ_C are non-overlapping open parts of $\partial\Omega$, the boundary of Ω. We consider the strong formulation of the problem

$$\begin{cases} M\ddot{u} + Ku = f + B_N^*\lambda_N + B_T^*\lambda_T \ \text{ in } V', \\ -\lambda_N \in N_{K_N}(B_N u) \ \text{ on } \Gamma_C, \\ -\lambda_T \in \partial_2 j(\lambda_N, B_T v) \ \text{ on } \Gamma_C, \\ u(0) = u_0, \dot{u}(0) = u_1. \end{cases} \tag{1}$$

witch is equivalent to

$$\begin{cases} M\ddot{u} + Ku = f + B_N^*\lambda_N + B_T^*\lambda_T \ \text{ in } V', \\ \lambda_N = P_{K_N}(\lambda_N - rB_N u) \ \text{ on } \Gamma_C, \\ \lambda_T = P_{\Lambda_T(\mathcal{F}\lambda_N)}(\lambda_T - rB_T v) \ \text{ on } \Gamma_C, \\ u(0) = u_0, \dot{u}(0) = u_1. \end{cases} \tag{2}$$

with

$$V = \{v \in H^1(\Omega;\mathbb{R}^n), v = 0 \ \text{ on } \Gamma_D\},$$
$$X_N = \{v_N|_{\Gamma_C} : v \in V\} \ \text{ and } \ X_T = \{v_T|_{\Gamma_C} : v \in V\},$$

$$K_N = \{v_N \in X_N : v_N \leq 0\},$$
$$N_{\Lambda_N} = N_{K_N^*} = \left(N_{K_N}\right)^{-1},$$
$$j(\lambda_N, v_T) = - < \mu\lambda_N, |v_T| >_{\Gamma_C}$$

$$\begin{aligned} \Lambda_T(\mathcal{F}\lambda_N) &= \{\lambda_T \in X_T' : - < \lambda_T, w_T >_{\Gamma_C} \\ &+ < -\mathcal{F}\lambda_N, \|w_T\| >_{\Gamma_C} \leq 0, \forall w_T \in X_T\}. \end{aligned}$$

where $< .,. >_{\Gamma_C}$ represente the duality product between X_N' and X_N and between X_T' and X_T. and where

$$M : V' \longmapsto V', \qquad K : V \longmapsto V',$$

$\mathcal{F}_N \in V'$, such that $< \mathcal{F}_N, v > = < \lambda_N, v_N >$, $\forall v \in V$,

$\mathcal{F}_T \in V'$, such that $< \mathcal{F}_T, v > = < \lambda_T, v_T >$, $\forall v \in V$,

$$\begin{array}{rcl} B_N^* : X_N' & \longrightarrow & V' \\ \lambda_N & \longmapsto & \mathcal{F}_N, \end{array} \qquad \begin{array}{rcl} B_T^* : X_T' & \longrightarrow & V' \\ \lambda_T & \longmapsto & \mathcal{F}_T, \end{array}$$

$$\begin{array}{rcl} B_N : V & \longrightarrow & X_N \\ u & \longmapsto & u_N, \end{array} \qquad \begin{array}{rcl} B_T : V & \longrightarrow & X_T \\ u & \longmapsto & u_T. \end{array}$$

2 System discretization

In this paragraph, we study different time integration schemes in the sens of stability and energy conservation. We subdevide the time period $[0, T]$ into discrete steps of index n, each encompassing the partition $[t_n, t_{n+1}]$, and we delimit the time increment as $\Delta t = t_{n+1} - t_n$. Tomporally discrete approximations of the state values can be similary indexed, such that $u^n \approx u(t_n)$.

2.1 Energy analysis

We define the system energy by

$$J(u, v) = \frac{1}{2} < Mv, v > + \frac{1}{2} < Ku, u > - < f, u >,$$

Definition 1 *We said the scheme is stable if and only if we have:*

$$\Delta J = J(u^{n+1}, v^{n+1}) - J(u^n, v^n) \leq 0.$$

We choose this definition because it is simple and so easy to manipulate. Of course, it enable us to establish some well known results in the literature and also give a new time integration scheme to have conservation of the system energy.

2.2 Standard middle point(SMP)

The standard middle point scheme reads as

$$u^{n+1} = u^n + \Delta t \, v^{n+\frac{1}{2}}, \qquad u^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2},$$
$$v^{n+1} = v^n + \Delta t \, a^{n+\frac{1}{2}}, \qquad v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}.$$

2.2.1 Method presentation

Introducing the SMP scheme into the system (1), we obtain

$$\begin{cases} u^{n+1} = u^n + \Delta t\, v^{n+\frac{1}{2}}, \\ v^{n+1} = v^n + \Delta t\, a^{n+\frac{1}{2}}, \\ Ma^{n+\frac{1}{2}} + Ku^{n+\frac{1}{2}} = f + B^*\lambda^{n+\frac{1}{2}}, \\ -\lambda_{_N}^{n+\frac{1}{2}} \in N_{K_{_N}}(B_{_N} u^{n+\frac{1}{2}}), \\ -\lambda_{_T}^{n+\frac{1}{2}} \in \partial_2 j(\lambda_{_N}^{n+\frac{1}{2}}, B_{_T} v^{n+\frac{1}{2}}), \\ u(0) = u_{_0}, v(0) = u_{_1}. \end{cases} \tag{3}$$

where $B^*\lambda^{n+\frac{1}{2}} = B_{_N}^*\lambda_{_N}^{n+\frac{1}{2}} + B_{_T}^*\lambda_{_T}^{n+\frac{1}{2}}$. Formulation (3) is equivalent to the following problem

$$\begin{cases} \text{Find } (u^{n+1}, \lambda_{_N}^{n+\frac{1}{2}}, \lambda_{_T}^{n+\frac{1}{2}}) \\ \left(\dfrac{2M}{\Delta t^{\,2}} + \dfrac{K}{2}\right) u^{n+1} = \hat{f} + B^*\lambda^{n+\frac{1}{2}} \\ -2\lambda_{_N}^{n+\frac{1}{2}} \in N_{K_{_N}}(B_{_N} u^{n+\frac{1}{2}}), \\ -\lambda_{_T}^{n+\frac{1}{2}} \in \partial_2 j(\lambda_{_N}^{n+\frac{1}{2}}, \frac{1}{\Delta t} B_{_T} u^{n+\frac{1}{2}}), \\ u(0) = u_{_0}, v(0) = u_{_1}. \end{cases} \tag{4}$$

where

$$\hat{f} = f + \frac{2}{\Delta t^{\,2}} Mu^n + \frac{2}{\Delta t} Mv^n - \frac{1}{2} Ku^n, \tag{5}$$

2.2.2 Stability analysis

We start by studying the system energy.

$$\begin{aligned} \Delta J &= J(u^{n+1}) - J(u^n) \\ &= \frac{1}{2} < M(v^{n+1} - v^n), v^{n+1} + v^n > \\ &+ \frac{1}{2} < K(u^{n+1} - u^n), u^{n+1} + u^n > \\ &- < f, u^{n+1} - u^n >, \\ &= \Delta t\ < Ma^{n+\frac{1}{2}} + Ku^{n+\frac{1}{2}}, v^{n+\frac{1}{2}} > \\ &- < f, u^{n+1} - u^n > \\ &= \Delta t\ < B_{_N}^*\lambda_{_N}^{n+\frac{1}{2}} + B_{_T}^*\lambda_{_T}^{n+\frac{1}{2}}, v^{n+\frac{1}{2}} > \\ &= \Delta t\ < \lambda_{_N}^{n+\frac{1}{2}}, v_{_N}^{n+\frac{1}{2}} > + \Delta t\ < \lambda_{_T}^{n+\frac{1}{2}}, v_{_T}^{n+\frac{1}{2}} > \\ &\le < \lambda_{_N}^{n+\frac{1}{2}}, u_{_N}^{n+1} - u_{_N}^n > \\ &= 2 < \lambda_{_N}^{n+\frac{1}{2}}, u_{_N}^{n+\frac{1}{2}} - u_{_N}^n > \\ &\le -2 < \lambda_{_N}^{n+\frac{1}{2}}, u_{_N}^n > . \end{aligned}$$

If $u_{_N}^n \le 0$ then $\Delta J \le 0$ and so the SMP scheme is stable for the frictional contact problem, else we will have energy dissipation. Hence, we can't conclude on the stability of the scheme in the general case.

2.3 Modified middle point (MMP)

We consider the same SMP scheme and we implicit the contact force to obtain the following MMP scheme:

$$\begin{aligned} u^{n+1} &= u^n + \Delta t\, v^{n+\frac{1}{2}}, \quad u^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2}, \\ v^{n+1} &= v^n + \Delta t\, a^{n+\frac{1}{2}} + \Delta t\, a_{_N}^{n+1} \quad v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}. \end{aligned}$$

This modification allows us to establish the energy conservation of the following discretized system

$$\left\{\begin{array}{l} u^{n+1} = u^n + \Delta t\, v^{n+\frac{1}{2}}, \\ \\ v^{n+1} = v^n + \Delta t\, a^{n+\frac{1}{2}} + \Delta t\, a_{_N}^{n+1}, \\ \\ Ma^{n+\frac{1}{2}} + Ku^{n+\frac{1}{2}} = f + B_{_T}^{*}\lambda_{_T}^{n+\frac{1}{2}} \\ \\ Ma_{_N}^{n+1} = B_{_N}^{*}\lambda_{_N}^{n+1} \\ \\ -\lambda_{_N}^{n+1} \in N_{K_{_N}}(B_{_N}u^{n+1}), \\ \\ -\lambda_{_T}^{n+\frac{1}{2}} \in \partial_2 j(\lambda_{_N}^{n+\frac{1}{2}}, B_{_T}v^{n+\frac{1}{2}}), \\ \\ u(0) = u_{_0}, v(0) = u_{_1}. \end{array}\right. \tag{6}$$

This problem is equivalent to the following problem

$$\left\{\begin{array}{l} \text{Find } (u^{n+1}, \lambda_{_N}^{n+1}, \lambda_{_T}^{n+\frac{1}{2}}) \\ \\ \left(\dfrac{2M}{\Delta t^{\,2}} + \dfrac{K}{2}\right) u^{n+1} = \hat{f} + B^{*}\lambda^{n+\frac{1}{2}}, \\ \\ Ma_{_N}^{n+1} = B_{_N}^{*}\lambda_{_N}^{n+1}, \\ \\ -\lambda_{_N}^{n+1} \in N_{K_{_N}}(B_{_N}u^{n+1}), \\ \\ -\lambda_{_T}^{n+\frac{1}{2}} \in \partial_2 j(\lambda_{_N}^{n+\frac{1}{2}}, \frac{1}{\Delta t}B_{_T}u^{n+\frac{1}{2}}), \\ \\ u(0) = u_{_0}, v(0) = u_{_1}. \end{array}\right. \tag{7}$$

where $\hat{f}$ is already defined and we choose

$$\lambda_{_N}^{n+\frac{1}{2}} = \frac{\lambda_{_N}^{n+1} + \lambda_{_N}^{n}}{2}.$$

Lemme 1 *The system (6) is stable in the sens of the Definition (1).*

Proof 1 *Of course, we have*

$$\begin{array}{rcl} \Delta J & = & J(u^{n+1}, v^{n+1}) - J(u^n, v^n) \\ & = & \dfrac{1}{2} < M(v^{n+1} + v^n), v^{n+1} - v^n > \\ & + & \dfrac{1}{2} < K(u^{n+1} + u^n), u^{n+1} - u^n > \\ & - & < f, u^{n+1} - u^n > \\ & = & \dfrac{\Delta t}{2} < M(v^{n+1} + v^n), a^{n+\frac{1}{2}} + a_{_N}^{n+1} > \\ & + & \dfrac{1}{2} < K(u^{n+1} - u^n), u^{n+1} + u^n > \\ & - & \Delta t\ < f, v^{n+\frac{1}{2}} > \\ & = & \Delta t\ < Ma^{n+\frac{1}{2}} + Ku^{n+\frac{1}{2}} - f, v^{n+\frac{1}{2}} > \\ & + & \Delta t\ < Ma_{_N}^{n+1}, v^{n+\frac{1}{2}} > \\ & = & \Delta t\ < \lambda_{_T}^{n+\frac{1}{2}}, v_{_T}^{n+\frac{1}{2}} > + \Delta t\ < \lambda_{_N}^{n+1}, v_{_N}^{n+\frac{1}{2}} > \\ & \leq & \Delta t\ < \lambda_{_N}^{n+1}, v_{_N}^{n+\frac{1}{2}} > \\ & = & < \lambda_{_N}^{n+1}, u_{_N}^{n+1} - u_{_N}^{n} > \\ & = & - < \lambda_{_N}^{n+1}, u_{_N}^{n} > \\ & \leq & 0. \end{array}$$

So the system (6) is stable. ■

Remark : This result gives a new idea about a new time integration scheme which we are still studying. The trick of the new scheme is to replace the implicit contact force by sort of linear combination of contact forces of two successive time steps $\alpha\lambda_{_N}^{n+1} + (1-\alpha)\lambda_{_N}^{n}$ and this method will introduce a new parameter which will be fixed at the begining or considered as a inconnue and will be computed in order to establish energy conservation.

3 Conclusion

Although the mathematical idealization of full conservation may serve as an aproximation model for actual physical systems, development of algorithmic methods within a conservating framework tends to lend greater insight into the direct effects of numerical discretization on such systems as well as their physically dissipative analogs. In this work, we proof the stability of the elastodynamic problem with frictional contact using an appropriate time integration scheme. This scheme gives also others idea about new time integrations schemes. My hope is to coroborate this results with numerical results. This is the following step for a futur work.

References

[1] O. GONZALEZ, Exact energy and momentum conserving algorithms for general models in nonlinear eleasticity, Comp. Meth. Appl. Mech. Engrg., 2000, **190**, 1763-1783

[2] T.J.R. HUGUES, W.K. LIU & P. CAUCHY, Transient finite element formulations that preserve energy, J. Appl. Mech., 45, pages 366-370, 1978.

[3] H.B. KHENOUS, Y. RENARD & J. POMMIER, Hybrid discretization of the Signorini problem with Coulomb friction. Theoretical aspects and comparison of some numerical solvers, submitted, 2004.

[4] P. LABORDE & Y. RENARD, Fixed point strategies for elastostatic frictional contact problems, submitted, 2003.

[5] T.A. LAURSEN & G.R. LOVE, Improved implicit integrators for transient impact problems-geometric admissibility within the conserving framework, Int. J. Numer. Meth. Engng, 2002, **53**, 245-274

[6] J. POMMIER & Y. RENARD, An Open Source generic C++ library for finite element methods, http://www.gmm.insa-toulouse.fr/getfem.