Motions of Infinite Mass-Spring Systems

KHIEM V. NGO
Department of Aeronautics and Astronautics
Stanford University
Stanford, CA 94305-4035
UNITED STATES OF AMERICA

Abstract: - In the study of physical, mechanical, and electrical systems one often encounters differential-difference equations and recurrence relations. The sources from which these equations arise may be quite different but their mathematical forms are very similar. For example, there is an analogy between mass-spring systems and electrical systems whereby point masses correspond to inductances and springs correspond to capacitances [1]. Another area where differential-difference equations occur is in the numerical solution of the wave equations if the spatial variable is discretized [2]. It's important to understand how these systems behave as time evolves and how changes in the parameters of the model influence this behavior. In this paper, we study systems consisting of point masses joined together by springs. In particular, we present some of the mathematical methods involved and how they are used to solve these practical problems. By obtaining the solution for these simple mass-spring systems, we indirectly obtain solutions for many similar applied problems in mechanics, physics, and engineering.

Key-Words: - Mass-spring system, vibration, wave, separation of variables, Laplace transform, Bessel function

1 Introduction

This paper presents how mathematics comes into play in the modeling process of a simple, yet fundamental, physical system so that one can understand how this type of systems behaves subject to the changes in the parameters. In particular, we are looking at a simple mass-spring system with identical masses and spring constants. We find the solution for this system by two methods: separation of variables and Laplace transform (Section 2). Then, we perturb the system by changing one of the spring constants. This results in the appearance of an eigenvalue and a vibration that has the form of a standing wave (Section 3). Lastly, we consider a rather generic system with various masses and spring stiffness subject to forcing functions $F_j(t)$.

In this case, we obtain the solution by means of modal analysis (Section 4). In most cases, the study of infinite mass-spring systems involves differential-difference equations, which, in turn, result in three term recurrence relations, whose solutions can be quickly obtained. To keep the content focused, lengthy algebras have been eliminated. As such, the audience are encouraged to verify the results presented here. The key part in this work lies in the mathematical modeling of these physical models, which should be interesting and important to those who consider working on engineering and science. In addition, this paper provides a good source for the teaching of mathematics to engineering students.

2 A Simple Mass-Spring System

Let's consider a system with infinitely many objects of the same mass connected by identical springs. In such a system, the displacement of one object depends on the displacement of others. So, by a recurrence relation, if we know the displacement of any two consecutive objects, then we can determine the displacement of the third adjacent object. We assume that the displacement of the object at the zeroth position, i.e. at n = 0, is b and that its initial velocity is zero. The other objects are supposed to have zero initial displacements and zero initial velocity. From the combination of Newton's second law of motion and Hooke's law, the equations of motion for this system read as follows:

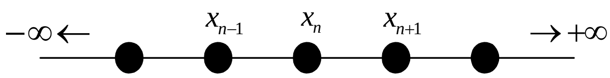

Fig.1 An infinite system

$$\begin{cases} m\ddot{x}_n = k(x_{n+1} - x_n) - k(x_n - x_{n-1}) \\ x_0(0) = b \quad \dot{x}_0(0) = 0 \\ x_n(0) = 0 \quad \dot{x}_n(0) = 0 \qquad n \neq 0 \end{cases} \tag{1}$$

We will solve (1) by two methods: separation of variables and Laplace transform.

2.1 Solution by Separation of Variables

We seek solutions of the form $x_n(t) = c_n e^{i\alpha t}$, where α is a constant independent of *n* and *t*. Substituting into (1) we obtain $kc_{n+1} - (2k - m\alpha^2)c_n + kc_{n-1} = 0$. To solve this recurrence relation, we assume $c_n = \lambda^n$ so that $k\lambda^2 - (2k - m\alpha^2)\lambda + k = 0$ whose solutions are

$$\lambda_\pm = \left(1 - \frac{\alpha^2}{2\omega^2}\right) \pm \frac{\alpha\sqrt{\alpha^2 - 4\omega^2}}{2\omega^2}$$

If $\alpha^2 - 4\omega^2 > 0$, it can be shown that λ_+ is real and $|\lambda_+| < 1$. Consequently, $\lambda_+^n \to \infty$ as $n \to -\infty$, and $\lambda_-^n = \lambda_+^{-n} \to \infty$ as $n \to +\infty$. As the solution of this type is unphysical, we discard it and consider the case when $\alpha^2 - 4\omega^2 < 0$.

$$\lambda_\pm = \left(1 - \frac{\alpha^2}{2\omega^2}\right) \pm i\,\frac{\alpha\sqrt{4\omega^2 - \alpha^2}}{2\omega^2}$$

Since $|\lambda_\pm| = 1$, and hence it suffices to let

$$\cos\varphi = 1 - \frac{\alpha^2}{2\omega^2} \qquad \sin\varphi = \frac{\alpha\sqrt{4\omega^2 - \alpha^2}}{2\omega^2}$$

so that $\lambda_+ = e^{i\varphi}$ and $\lambda_- = e^{-i\varphi}$. It follows from that $\alpha = 2\omega\sin(\varphi/2)$, where $\varphi \in (-\pi, \pi)$. Thus, the general solution of (1) is of the form

$$\begin{aligned} x_n(t) &= \int_{-\pi}^{\pi} e^{i(\alpha t + n\varphi)} f_1(\varphi)d\varphi + \int_{-\pi}^{\pi} e^{i(\alpha t - n\varphi)} f_2(\varphi)d\varphi \\ &= \int_{-\pi}^{\pi} e^{i(\alpha t + n\varphi)} f_1(\varphi)d\varphi + \int_{-\pi}^{\pi} e^{-i(\alpha t - n\varphi)} f_2(-\varphi)d\varphi \end{aligned}$$

Also, for the boundary conditions, we have

$$x_n(0) = \int_{-\pi}^{\pi} e^{in\varphi}(f_1 + g_2)d\varphi = \begin{cases} b & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{2}$$

where $g_2(\varphi) = f_2(-\varphi)$. In view of this equation, we deduce that $f_1 = g_2$. Hence, (2) yields

$$x_n(0) = 2\int_{-\pi}^{\pi} e^{in\varphi} f_1(\varphi)d\varphi = \begin{cases} b & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Expanding $f_1(\varphi)$ in a Fourier series, we have:

$$f_1(\varphi) = \sum_{n=-\infty}^{\infty} c_n e^{in\varphi} \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\varphi)e^{-in\varphi}d\varphi$$

So, $b = x_0(0) = 4\pi c_0$, i.e. $c_0 = b/4\pi$. Also, $c_n = 0$ for $n \neq 0$ because $x_n(0) = 0$. $f_1(\varphi) = g_2(\varphi) = b/4\pi$. It follows that

$$x_n(t) = \frac{b}{4\pi}\int_{-\pi}^{\pi} \{\exp[i(n\varphi + \alpha t)] + \exp[i(n\varphi - \alpha t)]\}d\varphi$$

Therefore, for *n* = 0 (recall that α depends on φ),

$$\begin{aligned} x_0(t) &= \frac{b}{4\pi}\int_{-\pi}^{\pi} (e^{i\alpha t} + e^{-i\alpha t})d\varphi \\ &= \frac{b}{\pi}\int_{-\pi/2}^{\pi/2} \cos(2\omega t \sin\varphi)d\varphi = bJ_0(2\omega t), \end{aligned}$$

where

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin t)dt = \frac{2}{\pi}\int_0^{\pi/2} \cos(z\sin t)dt$$

denotes the Bessel function of order zero [3]. For $n \neq 0$, we use $J_n(z) = \frac{2}{\pi}\int_0^{\pi/2} \cos(z\sin u - nu)du$.

$$\begin{aligned} x_n(t) + x_{-n}(t) &= \frac{b}{2\pi}\int_{-\pi}^{\pi} [\cos(n\varphi + \alpha t) + \cos(n\varphi - \alpha t)]d\varphi \\ &= \frac{b}{\pi}\int_{-\pi/2}^{\pi/2} \cos(2\omega t\sin\varphi + 2n\varphi)d\varphi \\ &\quad + \frac{b}{\pi}\int_{-\pi/2}^{\pi/2} \cos(2\omega t\sin\varphi - 2n\varphi)d\varphi \\ &= b[J_{2n}(2\omega t) + J_{-2n}(2\omega t)] \end{aligned}$$

Similarly,

$$\begin{aligned} x_n(t) - x_{-n}(t) &= \frac{ib}{2\pi}\int_{-\pi}^{\pi} [\sin(n\varphi + \alpha t) + \sin(n\varphi - \alpha t)]d\varphi \\ &= \frac{ib}{\pi}\int_{-\pi}^{\pi} \sin n\varphi \cos\alpha t d\varphi = 0 \end{aligned}$$

Thus, $x_n(t) = x_{-n}(t)$ for all *t* and *n*, and the solution is $x_n(t) = b[J_{2n}(2\omega t)]$, $n = 0, \pm 1, \pm 2, ...$ Below are the plots of the mass-spring motion with spring constant *k* = 1, time step *h* = 1, and $\omega = \sqrt{k/m} = 0.5$.

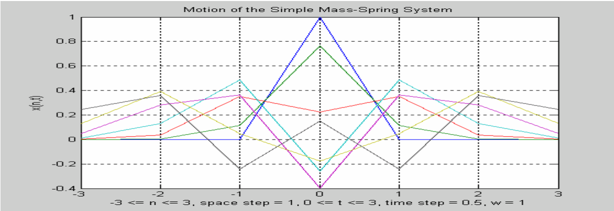

Fig.2

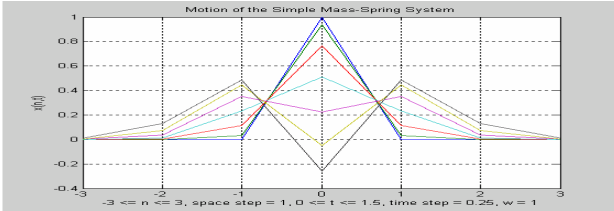

Fig.3

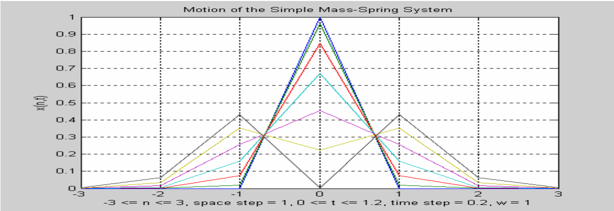

Fig.4

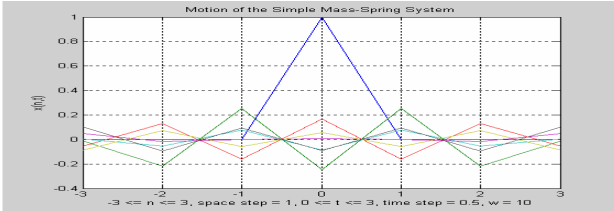

Fig.5

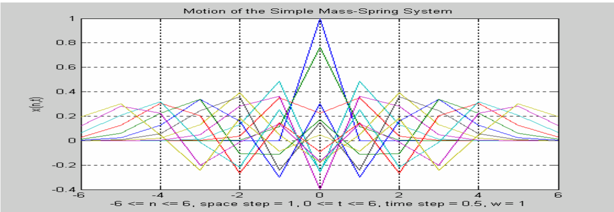

Fig.6

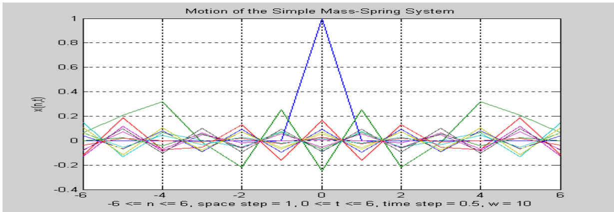

Fig.7

As shown in the analysis, $x_n(t) = x_{-n}(t)$ for all x and t, hence the motion of the spring is symmetric. In plotting, we let the initial state of the mass-spring system at $n = 0$ is $b = 1$. Fig.2 represents the spring motion with the time step of 0.5 and $\omega = 1$. Note that the same motion exists if the time step is decreased by half and ω is double, or if the time step is double and ω is decreased by half. As the time step gets smaller, Fig.3 & 4, the spring motion approaches the initial state, which is $x_0(0) = b = 1$ by assumption. The same behavior also exists when $\omega \to 0$. This is because as ω tends to zero, the masses tend to infinity, and this greatly reduces the vibration of the springs.

2.2 Solution by Laplace Transforms

We start from a system with $2N+1$ objects and then let $N \to +\infty$. We assume that the two ends of the finite chain are free.

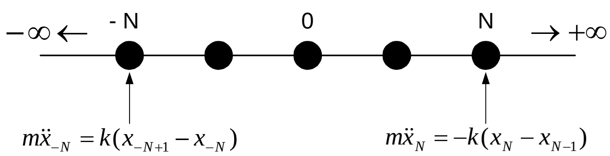

Fig.8

$$
\begin{aligned}
&\ddot{x}_{-N}(t) = \omega^2(x_{-N+1} - x_N) && n = -N \\
&\ddot{x}_n(t) = \omega^2(x_{n+1} - x_n) - \omega^2(x_n - x_{n-1}) && -N < n \le -1 \\
&\ddot{x}_0(t) = \omega^2(x_1 - x_0) - \omega^2(x_0 - x_{-1}) && n = 0 \\
&\ddot{x}_n(t) = \omega^2(x_{n+1} - x_n) - \omega^2(x_n - x_{n-1}) && 1 \le n < N \\
&\ddot{x}_N(t) = \omega^2(x_{N+1} - x_N) && n = N
\end{aligned}
\tag{3}
$$

Applying Laplace transform $X_n(s) = \int_0^\infty e^{-st} x_n(t)dt$ to (3), we have

$$
\begin{aligned}
&s^2 X_{-N} = \omega^2(X_{-N+1} - X_{-N}) && n = -N \\
&s^2 X_n = \omega^2(X_{n+1} - 2X_n + X_{n-1}) && -N < n \le -1 \\
&s^2 X_0 - sb = \omega^2(X_1 - 2X_0 + X_{-1}) && n = 0 \\
&s^2 X_n = \omega^2(X_{n+1} - 2X_n + X_{n-1}) && 1 \le n < N \\
&s^2 X_N = -\omega^2(X_N - X_{N-1}) && n = N
\end{aligned}
\tag{4}
$$

The equations (4a) & (4e) describe the two free ends of the chain. Putting $X_n = \mu^n$, (4b) becomes the quadratic equation $\omega^2\mu^2 - (s^2 + 2\omega^2)\mu + \omega^2 = 0$, whose solutions are

$$\mu_\pm = 1 + (s^2 \pm s\sqrt{s^2 + 4\omega^2})/2\omega^2$$

Note that $\mu_+\mu_- = 1$, $\mu_+ > 1$, *and* $0 < \mu_- < 1$. The solutions of (4) are of the form $X_n = C_1\mu_+^n + C_2\mu_-^n$. It follows that $X_n = X_{-n}$. Hence, it suffices to only consider $n \ge 0$. Letting $\rho = s/\omega$ and $\varepsilon = sb/\omega^2$, we obtain for $n = N$ and $n = 0$, respectively,

$$(1+\rho^2)(C_1\mu_+^N + C_2\mu_-^N) = C_1\mu_+^{N-1} + C_2\mu_-^{N-1} \qquad (5)$$

$$2(C_1\mu_+ + C_2\mu_-) = (2+\rho^2)(C_1 + C_2) - \varepsilon \qquad (6)$$

where in the last equation, we used (4c) and the fact that $x_1 = x_{-1}$. From (5) and (6), we obtain a nonhomogeneous system for C_1 and C_2:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\varepsilon \end{bmatrix}$$

$$a_{11} = \mu_+^N - \mu_+^{N-1} + \rho^2\mu_+^N \qquad a_{12} = \mu_-^N - \mu_-^{N-1} + \rho^2\mu_-^N$$
$$a_{21} = 2\mu_+ - 2 - \rho^2 \qquad a_{22} = 2\mu_- - 2 - \rho^2$$

Using Cramer's rule and the fact that $\mu_+ > 1$ and $0 < \mu_- < 1$, we conclude that

$$C_1 = \frac{\varepsilon a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad C_2 = \frac{-\varepsilon a_{11}}{a_{11}a_{22} - a_{12}a_{21}}$$

We have $C_1 \to 0$ as $N \to +\infty$ and $C_2 \to 0$ as $N \to -\infty$. Thus for fixed *n*, as $N \to +\infty$,

$$X_n = \frac{b}{\sqrt{s^2 + 4\omega^2}}\left(1 + \frac{s^2 - s\sqrt{s^2+4\omega^2}}{2\omega^2}\right)^n$$

Now,

$$(s - \sqrt{s^2 + 4\omega^2})^2 = 2(s^2 - s\sqrt{s^2+4\omega^2} + 2\omega^2)$$

$$1 + \frac{s^2 - s\sqrt{s^2+4\omega^2}}{2\omega^2} = \frac{s^2 - s\sqrt{s^2+4\omega^2} + 2\omega^2}{2\omega^2}$$
$$= \frac{2(s^2 - s\sqrt{s^2+4\omega^2} + 2\omega^2)}{4\omega^2} = \frac{(s - \sqrt{s^2+4\omega^2})^2}{4\omega^2}$$
$$= \frac{4\omega^2}{(s + \sqrt{s^2+4\omega^2})^2}.$$ It follows that

$$X_n = \frac{b2^{2n}\omega^{2n}}{\sqrt{s^2+4\omega^2}\,(s + \sqrt{s^2+4\omega^2})^{2n}}$$

Taking the inverse Laplace transform, the solution is found to be $x_n(t) = bJ_{2n}(2\omega t)$, which is the same result obtained earlier by the separation of variable method. This is a direct follow from the formula $L\{J_\nu(ax)\} = \left[a/(p + \sqrt{p^2 + a^2})\right]^\nu (1/\sqrt{p^2 + a^2})$. It can be found in [5] (formula #39, p.1145).

3 A Perturbed Mass-Spring System

We now consider the same system as in Section 2 except that we replace one of the springs by a spring of variable stiffness *k*'. This system is governed by the equations below:

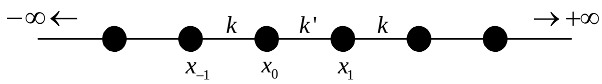

Fig.9 A perturbed system

$$m\ddot{x}_n = \begin{cases} k'(x_1 - x_0) - k(x_0 - x_{-1}) & n = 0 \\ k(x_{n+1} - x_n) - k(x_n - x_{n-1}) & n \le -1 \\ k(x_2 - x_1) - k'(x_1 - x_0) & n = 1 \\ k(x_{n+1} - x_n) - k(x_n - x_{n-1}) & n \ge 2 \end{cases} \qquad (7)$$

Assuming again that the solutions are of the form $x_n(t) = \lambda_n e^{i\alpha t}$, we get $m\ddot{x}_n = -m\alpha^2\lambda_n e^{i\alpha t}$. Setting $\mu = -m\alpha^2$, the system (7) becomes

$$\mu\lambda_n = \begin{cases} k'(\lambda_1 - \lambda_0) - k(\lambda_0 - \lambda_{-1}) & n = 0 \\ k(\lambda_{n+1} - \lambda_n) - k(\lambda_n - \lambda_{n-1}) & n \le -1 \\ k(\lambda_2 - \lambda_1) - k'(\lambda_1 - \lambda_0) & n = 1 \\ k(\lambda_{n+1} - \lambda_n) - k(\lambda_n - \lambda_{n-1}) & n \ge 2 \end{cases} \qquad (8)$$

Using the same analysis as before, we obtain, for $n \ge 2$, $\lambda_n = \lambda_\pm^n$, and $\lambda_\pm = 1 + (\mu \pm \sqrt{4\mu k + \mu^2})/2k$. Proceeding with the same procedures, we obtain

$$x_n(t) = \lambda_n e^{i\alpha t} \quad \text{where} \quad \lambda_n = \begin{cases} \lambda_+^n, & n \ge 1 \\ \eta\lambda_-^n, & n \le 0 \end{cases}$$

are solutions that go to zero as $n \to \pm\infty$. Note that in going through the mathematics, it is found, for $\mu < -4k$, $k' - k \ge 0$. Similar analysis shows that there is no eigenvalue for $\mu > 0$. Note that the spring response is the real part of the solution, which is $\mathrm{Re}[\, x_n(t)\,] = \lambda_n \cos[\alpha t]$. Therefore, the vibration of the spring exhibits a symmetric characteristics.

Obviously, $\lambda(n)$ controls the amplitude of the spring motion, which decreases as $n \to \pm\infty$. Therefore, the vibration dies out at both ends. The same phenomenon happens as the difference in the spring stiffness k' and k increases. This can be seen in Fig.12 and Fig.13 and when compared with Fig.10. On the other hand, as this difference decreases, $k' \to k$, we are coming back to the original simple mass-spring system discussed in Sections 2.1 and 2.2. As such, the spring vibrates with amplitudes, which are becoming constant, Fig.15. In addition, as the masses are double, Fig.11, the motion on the right of n = 0.5 is inverse symmetric to that on the left. Below are the plots of the motion of the mass-spring system in the intervals $-5 \le n \le 5$ and $0 \le t \le 1$.

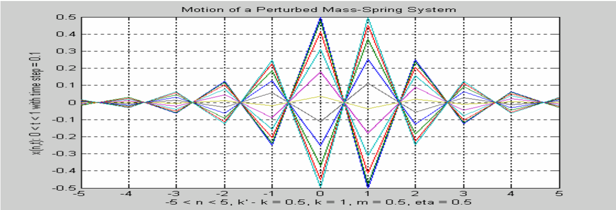

Fig.10

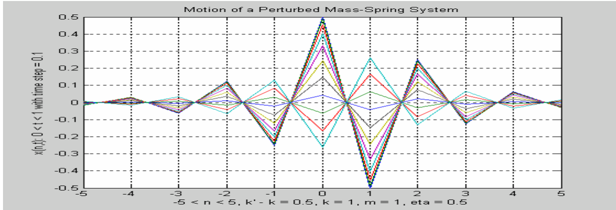

Fig.11

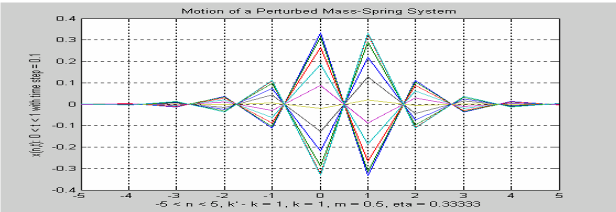

Fig.12

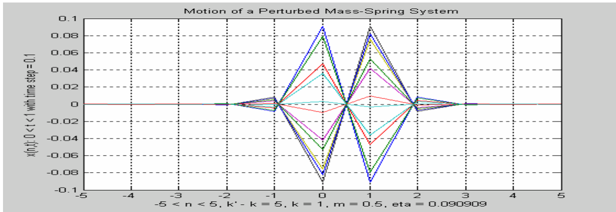

Fig.13

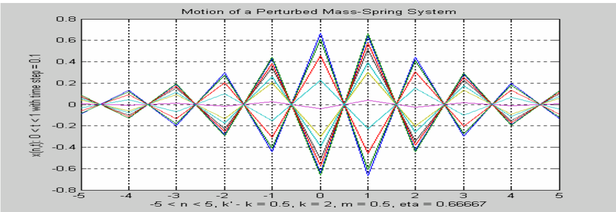

Fig.14

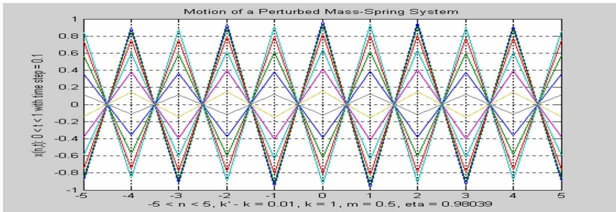

Fig.15

4 Mass-Spring Systems with Various Masses and Spring Stiffness

Suppose now we consider a finite discrete system

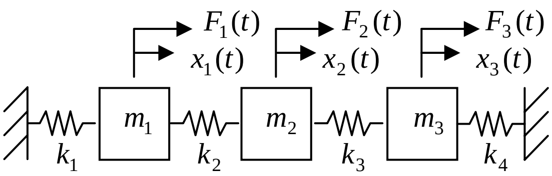

Fig.16 A finite system

To get the infinite mass-spring system, we just need to let the number of masses go to infinity at both ends. In reality, the two ends of either system, finite or infinite, must be connected to something to hold it still before vibrations can exist. Thus it suffices to look at a finite discrete system, Fig.16. We are to seek a solution for this system by the modal analysis method. By Newton's second law of motion,

$$[m]\{\ddot{x}(t)\} + [k]\{x(t)\} = \{F(t)\} \qquad (9)$$

For illustrations, we let $m_1 = m_3 = 2m$, $m_2 = 4m$, and assume all the springs have the same stiffness k. Thus, the mass and stiffness matrices are

$$[m] = \begin{bmatrix} 2m & & \\ & 4m & \\ & & 2m \end{bmatrix}, \quad [k] = \begin{bmatrix} 2k & -k & \\ -k & 2k & -k \\ & -k & 2k \end{bmatrix}$$

where $\{x(t)\} = [x_1(t)\, x_2(t)\, x_3(t)]^T$ is a displacement vector. Using the method of modal analysis, we first solve the free vibration problem or homogeneous system. In doing so, we again assume a solution of the form $\vec{x}(t) = \vec{u}\exp(i\omega t)$ and substitute it into (9), we obtain the eigenvalue problem

$$([k]-\omega^2[m])\{u\}=\{0\}$$

To understand the relationship between the finite and infinite systems, one can observe the behavior of the eigenfrequency ω as the dimensions of $[k]$ and $[m]$ get larger. The above equation has a non-trivial solution if and only if $\det([k]-\omega^2[m])=0$. Upon solving this characteristic equation, we get

$$\begin{cases} \omega_1 = \dfrac{1}{2}\sqrt{3-\sqrt{5}}\,\Omega \\ \omega_2 = \Omega \\ \omega_3 = \dfrac{1}{2}\sqrt{3+\sqrt{5}}\,\Omega \end{cases}, \text{ where } \Omega=\sqrt{k/m}.$$

Consequently, the natural modes of the system are

$$\{u\}_1=\begin{bmatrix} -1+\sqrt{5} \\ 2 \\ 0 \end{bmatrix},\ \{u\}_2=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \{u\}_3=\begin{bmatrix} 0 \\ 2 \\ -(1+\sqrt{5}) \end{bmatrix}$$

In order to exploit the usefulness of the eigenmodes toward obtaining the solution, we normalize them with respect to the mass matrix $[m]$. As such, we assume the normalized modal vectors are of the form $\{\tilde{u}\}_i=c_i\{u\}_i$ for i = 1,2,3. By the orthogonality relation of modal vectors, we have $\{\tilde{u}\}_i^T[m]\{\tilde{u}\}_j=\delta_{ij}$. It follows that

$$\{\tilde{u}\}_1=\frac{0.23}{\sqrt{m}}\{u\}_1,\ \{\tilde{u}\}_2=\frac{0.5}{\sqrt{m}}\{u\}_2,\ \{\tilde{u}\}_3=\frac{0.17}{\sqrt{m}}\{u\}_3$$

$$[\tilde{u}]=\frac{1}{\sqrt{m}}\begin{bmatrix} 0.283 & 0.5 & 0 \\ 0.458 & 0 & 0.329 \\ 0 & 0.5 & -0.532 \end{bmatrix}$$

To allow interaction among the modes, we express the displacement $\vec{x}(t)$ as a linear superposition of the normal modes so that $\{x(t)\}=[\tilde{u}]\{\xi(t)\}$, where $\{\xi(t)\}=\{\xi_1(t)\ \xi_2(t)\ \xi_3(t)\}^T$ for some coefficients ξ_1, ξ_2, ξ_3. Upon substitution into (9), we get

$$[m][\tilde{u}]\{\ddot{\xi}(t)\}+[k][\tilde{u}]\{\xi(t)\}=\{F(t)\}$$

Applying the orthogonality relation of eigenvectors

$$\underbrace{[\tilde{u}]^T[m][\tilde{u}]}_{1}\{\ddot{\xi}(t)\}+\underbrace{[\tilde{u}]^T[k][\tilde{u}]}_{\omega^2}\{\xi(t)\}=[\tilde{u}]^T\{F(t)\}$$

so that $\ddot{\xi}_j(t)+\omega_j^2\xi_j(t)=N_j(t)$ for j = 1, 2, 3, where $N(t)=[\tilde{u}]^T\{F(t)\}$. Using the initial conditions we solve this second order differential equations for ξ_j.

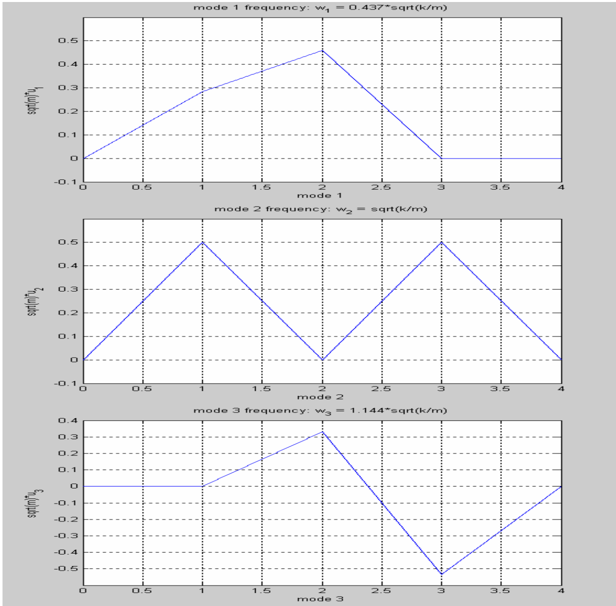

Fig.17 Natural modes of vibration

5 Conclusion

In short, this paper presents how mathematics comes into play in the modeling process of simple, yet fundamental, physical systems so that one can understand how this type of systems behaves subject to the changes in the parameters; masses and spring stiffness. In particular, we look at several simple mass-spring systems and present some of the well known mathematical methods involved in obtaining the solutions for these systems. By obtaining the solution for these simple mass-spring systems, we indirectly obtain solutions for many similar applied problems in mechanics, physics, and engineering.

References:

[1] J. Keener, *Principles of Applied Mathematics, Transform. & Approx.*, A. Wesley, NY, 1988

[2] G. Baldock and T. Bridgeman, *Mathematical Theory of Wave Motion*, J. Wiley & Sons, 1981

[3] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978

[4] L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice Hall, 1996

[5] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 1980